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Some results on the stability of $\sigma_{eap}(\cdot)$ and $\sigma_{e\delta}(\cdot)$ of linear relations

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Abstract. In this research paper, we provide firstly the necessary and sufficient conditions for the algebraic sum to become a closed and closable linear relation. Secondly, we investigate the stability of the essential approximate point spectrum $\sigma_{eap}(.)$ as well as the essential defect spectrum $\sigma_{e\delta}(T)$, in terms of linear relations on Banach spaces, which were introduced by T. Àlvarez and all in [1].

1. Introduction

The multivalued linear operator which was originally introduced into Functional Analysis by J. Von Neumann [19] is an intrinsic factor for the investigation of differential equations [10] controlled by non-densely defined operators. The adjoint of such operators is linear relations. These linear operators are more appropriate for the definition of the closure, the completion and the inverse of linear relations. In fact, exploring some Cauchy problems associated with parabolic type equations in Banach spaces [13] is a good example of the investigations conducted about the multivalued linear operator. It is worth mentioning that there are significant research works dealing with linear relations. Among these works, we can cite that of M. Gromov [15] handling a treatise on partial differential relations, that of A. Favini and A. Yagi [13] addressing the application of multivalued methods to the solution of differential equations. Other research works tackled the development of a fixed point theory for linear relations into mild solutions of quasi-linear differential inclusions of evolution as well as into many problems of fuzzy theory (see for example [3, 7, 14, 18, 20]). There are also many papers on semi-Fredholm linear relations and other classes related to them (see for example [6, 8, 9]).

Throughout this work, except where stated otherwise, X, Y and Z will denote complex normed linear spaces, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $L(X, Y)$ denotes the class of all linear bounded operators on X into Y. A multivalued linear operator (or a linear relation) T from X to Y is a mapping from a subspace

$$\mathcal{D}(T) := \{x \in X : Tx \neq \emptyset\}$$

of X, called the domain of T, into $\mathcal{P}(Y)\backslash\{\emptyset\}$ (collection of non-empty subsets of Y) such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all non-zero scalars $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(T)$. If T maps the points of its domain to

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singletons, then T is said to be a single valued linear operator (or simply an operator).
A linear relation is uniquely determined by its graph $G(T)$, which is defined by

$$G(T) := \{(x, y) \in X \times Y : x \in \mathcal{D}(T) \ \ and \ \ y \in Tx\}.$$

In this notation, $\mathcal{LR}(X, Y)$ denotes the class of all linear relations on X into Y, if $X = Y$ would simply denote $\mathcal{LR}(X, X) := \mathcal{LR}(X)$.
The inverse of T is the linear relation, T^{-1} defined by

$$G(T^{-1}) := \{(y, x) \in Y \times X : (x, y) \in G(T)\}.$$

The subspace $\mathcal{N}(T) := T^{-1}(0)$ is called the null space of T, and T is called injective if $\mathcal{N}(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator. The range of T is the subspace $\mathcal{R}(T) := T(\mathcal{D}(T))$, and T is called surjective if $\mathcal{R}(T) = Y$. When T is injective and surjective, we say that T is bijective. The quantities

$$\alpha(T) := dim\ (\mathcal{N}(T)) \ \ and \ \ \beta(T) := codim(\mathcal{R}(T)) = dim(Y/_{\mathcal{R}(T)})$$

are called the nullity (or the kernel index) and the deficiency of T, respectively. We also write $\overline{\beta}(T) := codim(\overline{\mathcal{R}(T)})$. The index of T is defined by $i(T) := \alpha(T) - \beta(T)$ provided that both $\alpha(T)$ and $\beta(T)$ are not infinite. If $\alpha(T)$ and $\beta(T)$ are infinite, then T is said to have no index.

Let M be a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$ and let $T \in \mathcal{LR}(X, Y)$. Then, the restriction $T_{|M}$, is the linear relation given by

$$G(T_{|M}) := \{(m, y) \in G(T) : m \in M\} = G(T) \cap (M \times Y).$$

For $S, T \in \mathcal{LR}(X, Y)$ and $R \in \mathcal{LR}(Y, Z)$, the sum $S + T$ and the product RS are the linear relations defined by

$$G(T + S) := \{(x, y + z) \in X \times Y : (x, y) \in G(T)\ and\ (x, z) \in G(S)\},\ and$$

$$G(RS) := \{(x, z) \in X \times Z : (x, y) \in G(S), (y, z) \in G(R)\ for\ some\ y \in Y\}$$

respectively, and if $\lambda \in \mathbb{K}$, the λT is defined by

$$G(\lambda T) := \{(x, \lambda y) : (x, y) \in G(T)\}.$$

If $T \in \mathcal{LR}(X)$ and $\lambda \in \mathbb{K}$, then the linear relation $\lambda - T$ is indicated by

$$G(\lambda - T) := \{(x, y - \lambda x) : (x, y) \in G(T)\}.$$

Let $T \in \mathcal{LR}(X, Y)$. We write Q_T for the quotient map from Y into $Y/\overline{T(0)}$. It's clear, therefore, that $Q_T T$ is an operator. For all $x \in \mathcal{D}(T)$, we define $\|Tx\| := \|Q_T Tx\|$, and the norm of T is defined by $\|T\| := \|Q_T T\|$. We note that $\|Tx\|$ and $\|T\|$ are not real norms. In fact, a nonzero relation can have a zero norm. T is said to be closed if its graph $G(T)$ is a closed subspace of $X \times Y$. The closure of T, denoted by $\overline{T}$, is defined in terms of its graph $G(\overline{T}) := \overline{G(T)}$. We denote by $\mathcal{CR}(X, Y)$ the class of all closed linear relations on X into Y, if $X = Y$ would simply denote $\mathcal{CR}(X, X) := \mathcal{CR}(X)$. If $\overline{T}$ is an extension to T (that is, $\overline{T}_{|\mathcal{D}(T)}$), we say that T is closable.
Let $T \in \mathcal{LR}(X, Y)$. We say that T is continuous if for each neighbourhood V in $\mathcal{R}(T)$, the inverse image $T^{-1}(V)$ is a neighbourhood in $\mathcal{D}(T)$ equivalently to $\|T\| < \infty$; open if T^{-1} is continuous, bounded if $\mathcal{D}(T) = X$ and T is continuous, bounded below if it is injective and open, and compact if $\overline{Q_T T(B_{\mathcal{D}(T)})}$ is compact in Y ($B_{\mathcal{D}(T)} := \{x \in \mathcal{D}(T) : \|x\| \leq 1\}$). We denote by $\mathcal{KR}(X, Y)$ the class of all compact linear relations on X into Y, if $X = Y$ would simply denote $\mathcal{KR}(X, X) := \mathcal{KR}(X)$.
We present the following definitions suggegted by R. W. Cross [11]:

Definition 1.1. [11, Definition, IV.3.1] *Let $T \in \mathcal{LR}(X, Y)$, and let X_T denote the vector space $\mathcal{D}(T)$ normed by*

$$\|x\|_T := \|x\| + \|Tx\|, \qquad for\ all\ x \in \mathcal{D}(T).$$

Let $G_T \in \mathcal{LR}(X_T, X)$ be the identity injection of $X_T = (\mathcal{D}(T), \|.\|_T)$ into X, i.e.,

$$\mathcal{D}(G_T) = X_T, \quad G_T(x) = x, \quad for\ all\ x \in X_T.$$

Definition 1.2. [11, Definition, VII.2.1] *Let $S, T \in \mathcal{LR}(X, Y)$. S is said to be T-bounded if $\mathcal{D}(T) \subset \mathcal{D}(S)$, and there exist non-negative constants a, and b, such that*

$$\|Sx\| \leq a\|x\| + b\|Tx\| \quad for\ all\ x \in \mathcal{D}(T). \tag{1}$$

In that case, the infimum of the constant b which satisfies (1) *is called the T-bound of S.*

Definition 1.3. [11, Definition VII.2.1] *Let $T \in \mathcal{LR}(X, Y)$. A relation $S \in \mathcal{LR}(X, Y)$ is said to be T-compact if $\mathcal{D}(T) \subset \mathcal{D}(S)$, and SG_T is compact.*
S is called T-precompact if $\mathcal{D}(T) \subset \mathcal{D}(S)$, and SG_T is precompact.

Remark 1.4. *It is clear that G_T is a bounded operator, and if T is single valued, then X_T is norm isomorphic to the subspace $G(T) \subset X \times Y$.*

If X is a normed linear space, then $X^{'}$ will denote the dual norm of X, i.e.,the space of all continuous linear functionals x' are defined on X, with the norm

$$\|x'\| = \inf\{\lambda : |x'x| \leq \lambda\|x\|\ for\ all\ x \in X\}.$$

If $K \subset X$, and $L \subset X^{'}$, we shall adopt the following notation:

$$\begin{aligned} K^{\perp} &:= \{x' \in X^{'} : x' = 0\ for\ all\ x \in K\}, \\ L^{\top} &:= \{x \in X : x' = 0\ for\ all\ x' \in L\}. \end{aligned}$$

It's obvious then that, $K^{\perp}$ and $L^{\top}$ are closed linear subspaces of $X^{'}$ and X respectively.
The adjoint of T, T', is defined by

$$G(T') = G(-T^{-1})^{\perp} \subset Y^{'} \times X^{'}$$

where $\langle (y, x), (y', x') \rangle := \langle x, x' \rangle + \langle y, y' \rangle$. This implies that

$$(y', x') \in G(T') \text{ if, and only if, } y'y - x'x = 0 \text{ for all } (x, y) \in G(T).$$

Similarly, we have $y'y = x'x$ for all $y \in Tx$, $x \in \mathcal{D}(T)$. Hence, $x' \in T'y$ if, and only if, $y'Tx = x'x$ for all $x \in \mathcal{D}(T)$.

Definition 1.5. [11, Definition, V.1.1] (*i*) *A linear relation $T \in \mathcal{LR}(X, Y)$ is said to be upper semi-Fredholm, and denoted by $T \in \mathcal{F}_+(X, Y)$, if there exists a finite codimensional subspace M of X is for which $T_{|M}$ is injective and open.*
(*ii*) *A linear relation T is said to be lower semi-Fredholm, and is indicated by $T \in \mathcal{F}_-(X, Y)$, if its conjugate T' is upper semi-Fredholm.*

For the case when X and Y are Banach spaces, we extend the class of closed single valued Fredholm type operators given earlier to include closed multivalued operators, and we note that the definitions of the $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$ are consistent with

$$\Phi_+(X, Y) := \{T \in \mathcal{CR}(X, Y) :\ R(T)\ is\ closed,\ and\ \alpha(T) < \infty\},$$

$$\Phi_-(X, Y) := \{T \in \mathcal{CR}(X, Y) :\ R(T)\ is\ closed,\ and\ \beta(T) < \infty\}.$$

If $X = Y$, this would simply denote $\Phi_+(X, Y)$, $\Phi_-(X, Y)$, $\mathcal{F}_+(X, Y)$, and $\mathcal{F}_-(X, Y)$ by respectively $\Phi_+(X)$, $\Phi_-(X)$, $\mathcal{F}_+(X)$, and $\mathcal{F}_-(X)$.

Definition 1.6. [2, Definition, 4.1] *Let $T \in \mathcal{CR}(X)$. The resolvent set of T is defined by*

$$\rho(T) := \{\lambda \in \mathbb{C} : (\lambda - T)^{-1}\ is\ everywhere\ defined,\ and\ single\ valued\}.$$

The spectrum of T is $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Definition 1.7. [1] *Let $T \in \mathcal{CR}(X)$.*
(i) *We define the essential approximate point spectrum of T by:*

$$\sigma_{eap}(T) := \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{ap}(T+K).$$

(ii) *We define the essential defect spectrum of T by:*

$$\sigma_{e\delta}(T) := \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\delta}(T+K),$$

where $\mathcal{K}_T(X) := \{K \in \mathcal{KR}(X) : \mathcal{D}(T) \subset \mathcal{D}(K), K(0) \subset T(0)\}$,

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\},$$

and

$$\sigma_{\delta}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}.$$

In [11], R. W. Cross introduced a concept of essential spectrum of a multivalued linear operator (also called linear relations or subspaces) in a complex normed space in terms of the nullity, and deficiency of its complete closure. He demonstrated its stability under relative compact perturbation with certain additional conditions (see [11, Theorem VII.3.2]). In the thesis [22], D. Wicox identified five distinct essential spectra of linear relations in Banach spaces in terms of semi-Fredholm properties, and highlighted their stability under relative compact perturbation with some additional conditions, and under compact perturbation, separately, in [21] (see Remarks 5.1, and 5.2 for more detailed discussions about their relations with those obtained in the present paper). In [1], the authors characterized such essential spectra in terms of semi-Fredholm linear relations and they also obtained set forward spectral mapping theorems for $\sigma_{eap}(T)$ and $\sigma_{e\delta}(T)$. They established a characterization of $\sigma_{eap}(T)$ and $\sigma_{e\delta}(T)$ by means of upper and lower semi-Fredholm a linear relations.
This paper, is devoted in the first place, to the study of some properties of unbounded linear relations. Among other things, we show that most of the results obtained by A. Jeribi [16, Theorems, 2.1.4 and 2.1.5] remain valid for operators in the context of multivalued linear operators. In this work, we present some sufficient conditions so that if A, and B are two unbounded linear relations, then their algebraic sum $A+B$ is also a closed linear relation. In addition, we focus on the stability of the essential spectra of $A+B$.
In the second place, the purpose of this paper is to investigate perturbation theorems for semi-Fredholm linear relations so as to verify the stability properties of the essential approximate point spectrum $\sigma_{eap}(T)$ and the essential defect spectrum $\sigma_{e\delta}(T)$ of closed and closable linear relations under relatively compact and precompact perturbation on Banach spaces.
The paper is organized as follows. Section 2 displays preliminaries which will be needed in the sequel. In section 3, we establish criteria for closedness and closability of the algebraic sum $A+B$ in case A and B are two unbounded linear relations given that A is B-bounded with A-bound δ (see Theorem 3.7). In section 4, we investigate the stability of the essential approximate point spectrum and the essential defect spectrum of closed and closable linear relations under relatively compact and precompact perturbation on Banach spaces (see Theorem 4.6).

2. Preliminaries

The mains objective of this section is to introduce the basic concepts, notations, and elementary results which are used throughout the work.

Definition 2.1. [11, Definition IV.1.1] *Let $\mathcal{I}(X)$, $\mathcal{C}(X)$, and $\mathcal{P}(X)$ denote respectively the infinite dimensional, finite codimensional, and closed finite codimensional subspaces of a normed linear space X.*
Let $\Upsilon_{XY} := \{\Gamma, \Gamma_0, \overline{\Gamma}_0, \Delta\}$ where $f : LR(X,Y) \to [0, \infty] \in \Upsilon_{XY}$ is defined as follows: If $\dim \mathcal{D}(T) < \infty$, then

$f(T) := 0$ for all $f \in \Upsilon_{XY}$. Otherwise,

$$\begin{aligned}
\Gamma(T) &:= \inf_{M \in \mathcal{I}(\mathcal{D}(T))} \|T_{|M}\|, \\
\Gamma_0(T) &:= \inf_{M \in \mathcal{C}(\mathcal{D}(T))} \|T_{|M}\|, \\
\overline{\Gamma}_0(T) &:= \inf_{M \in \mathcal{P}(\mathcal{D}(T))} \|T_{|M}\|, \\
\Delta(T) &:= \sup_{M \in \mathcal{I}(\mathcal{D}(T))} \Gamma(T_{|M}).
\end{aligned}$$

Hence, the following inequalities hold: $\Gamma(T) \leq \Delta(T) \leq \Gamma_0(T) \leq \overline{\Gamma}_0(T)$.

Definition 2.2. [11, Definition V.1.1] *Let $T \in \mathcal{LR}(X, Y)$. We say that T is precompact if $Q_T T B_{\mathcal{D}(T)}$ is totally bounded in Y, and strictly singular if there is no infinite dimensional subspace M of $\mathcal{D}(T)$ for which $T_{|M}$ is injective, and open.*

Remark 2.3. *Let $S, T \in \mathcal{LR}(X, Y)$.*

(i) *If S is 0-bounded, then S is bounded.*

(ii) *The inequality (1) is equivalent to*

$$\|Sx\|^2 \leq a_1^2\|x\|^2 + b_1^2\|Tx\|^2\ , \quad x \in \mathcal{D}(T), \tag{2}$$

where $a_1 = (a^2 + ab)^{\frac{1}{2}}$, and $b_1 = (b^2 + ab)^{\frac{1}{2}}$.

(iii) *S is T-bounded if, and only if, S is $(\lambda - T)$-bounded for some $\lambda \in \mathbb{C}$.*

(iv) *S is T-bounded if, and only if, $\mathcal{D}(T) \subset \mathcal{D}(S)$, and SG_T is bounded.*

(v) *S is T-bounded with T-bound δ if, and only if, $Q_S S$ is $Q_T T$-bounded with $Q_T T$-bound δ.*

Lemma 2.4. *Let T, S, and $U \in \mathcal{LR}(X, Y)$.*

(i) [2, Lemma 2.5] *If $S(0) \subset T(0)$, and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then $T - S + S = T$.*

(ii) [2, Lemma 2.5] *If $S(0) \subset T(0)$, $\mathcal{D}(T) \subset \mathcal{D}(S)$, and $U \subset T - S$, then $U + S \subset T$.*

(iii) [22, Proposition 2.8.2] *If $T \in \mathcal{CR}(X, Y)$, then X_T and $T(0)$ are complete.*

Lemma 2.5. [5, Lemma 3.1] *Let $S, T \in \mathcal{LR}(X, Y)$. If $S(0) \subset T(0)$, and S is T-bounded with T-bound $\delta < 1$, then S is $(T + S)$-bounded with $(T + S)$-bound $\leq \frac{\delta}{1-\delta}$.*

Proposition 2.6. [11, Proposition II.5.3] *Let $T \in \mathcal{LR}(X, Y)$, the following properties are equivalent:*

(i) *T is closed.*

(ii) *$Q_T T$ is closed, and $T(0)$ is closed.*

Proposition 2.7. [11, Proposition II.5.7] *Let $T \in \mathcal{LR}(X, Y)$, the following properties are equivalent:*

(i) *T is closable.*

(ii) *$Q_T T$ is closable, and $T(0)$ is closed.*

In particular, if T is continuous, and $T(0)$ is closed, then T is closable.

Lemma 2.8. [11, Exercise II.5.18] *Let $S \in \mathcal{CR}(Y, Z)$ and $T \in L(X, Y)$. Then ST is closed.*

Proof. Let $(x_n, y_n) \in G(ST)$ such that (x_n, y_n) converges to (x, y). Then $y_n \in STx_n = S(Tx_n)$. This implies that $(Tx_n, y_n) \in G(S)$. Given that T is a bounded operator, and S is closed, then $(Tx, y) \in G(S)$. From this perspective, $y \in S(Tx) = STx$.

Proposition 2.9. [11, Proposition II.5.8] *Let* $T \in \mathcal{LR}(X, Y)$. *We have* $\|\overline{T}\| \leq \|T\|$ *with equality holding if* $\overline{T}(0) = \overline{T(0)}$.

Lemma 2.10. [1, Lemma 2.3] *Let* X *be complete,* $T \in \mathcal{CR}(X)$, *and* $K \in \mathcal{K}_T(X)$.

(*i*) *If* $T \in \Phi_+(X)$, *then* $T + K \in \Phi_+(X)$ *with* $i(T + K) = i(T)$.

(*ii*) *If* $T \in \Phi_-(X)$, *then* $T + K \in \Phi_-(X)$ *with* $i(T + K) = i(T)$.

Proposition 2.11. *Let* $T \in \mathcal{LR}(X, Y)$

(*i*) [11, Theorem V.2.6] *T is strictly singular if, and only if,* $\Delta(T) = 0$.

(*ii*) [11, Corollary V.7.6] *If* X, Y *are complete and* $T \in \mathcal{CR}(X, Y)$, *then* $T \in \mathcal{F}_+(X, Y)$ *if, and only if,* $T \in \Phi_+(X, Y)$.

(*iii*) [11, Theorem V.2.4] *If* $\dim \mathcal{D}(T) = \infty$, *then* $T \in \mathcal{F}_+(X, Y)$ *if, and only if,* $\Gamma(T) > 0$.

(*iv*) [11, Corollary V.2.5] $T \in \mathcal{F}_+(X, Y)$ *if, and only if,* $TG_T \in \mathcal{F}_+(X_T, Y)$.

(*v*) [11, Theorem V.2.2] *T is precompact if, and only if,* $\overline{\Gamma}_0(T) = 0$.

(*vi*) [11, Corollary V.2.8, and Theorem V.2.2] *If* $\Gamma_0(T) = 0$, *then T is strictly singular.*

(*vii*) [11, Corollary IV.2.14] *T is continuous if, and only if,* $\overline{\Gamma}_0(T) < \infty$.

(*viii*) [11, Corollary V.2.3] *If T is precompact, then T is continuous.*

Theorem 2.12. [11, Theorem V.3.2] *Let* S, $T \in \mathcal{LR}(X, Y)$ *such that* $S(0) \subset \overline{T(0)}$. *If* $\Delta(S) < \Gamma(T)$, *then* $T + S \in \mathcal{F}_+(X, Y)$.

Proposition 2.13. *Let* T, $S \in \mathcal{LR}(X, Y)$.

(*i*) [11, Proposition III.1.5] *Let* $\mathcal{D}(T) \subset \mathcal{D}(S)$. *If S is continuous, then* $(T + S)' = T' + S'$.

(*ii*) [11, Proposition V.5.15] *Let* $T \in \mathcal{CR}(X, Y)$. $T \in \mathcal{KR}(X, Y)$ *if, and only if,* $T' \in \mathcal{KR}(Y', X')$.

(*iii*) [11, Proposition V.7.5] $T \in \mathcal{F}_+(X, Y)$ *if, and only if,* $T' \in \mathcal{F}_-(Y', X')$ *and* $T' \in \mathcal{F}_+(Y', X')$ *if, and only if,* $T \in \mathcal{F}_-(X, Y)$.

(*iv*) [11, Proposition V.7.8] *If* $\dim S(0) < \infty$, *then* $T + S - S \in \mathcal{F}_+(X, Y)$ *if, and only if,* $T \in \mathcal{F}_+(X, Y)$.

(*v*) [11, Proposition V.5.27] *If T is closable, then* $T \in \mathcal{F}_-(X, Y)$ *if, and only if,* $TG_T \in \mathcal{F}_-(X_T, Y)$.

(*vi*)[11, Proposition V.5.12] *Let* $\mathcal{D}(T) \subset \mathcal{D}(S)$, *and let* $T \in \mathcal{F}_-(X, Y)$. *If S is precompact, then* $T + S \in \mathcal{F}_-(X, Y)$.

Proposition 2.14. [9, Theorem 2.17] *Let* $S \in \mathcal{LR}(X, Y)$, $T \in \mathcal{F}_+(X, Y)$ *with* $G(S) \subset G(T)$, *and* $\dim D(S) = \infty$. *Then,* $S \in \mathcal{F}_+(X, Y)$.

Proposition 2.15. [11, Propositions VII.2.2 and VII.2.3] *Let* $T, S \in \mathcal{LR}(X, Y)$.

(*i*) *Let* $\lambda \in \mathbb{C}$, *the norms* $\|.\|_T$ *and* $\|.\|_{\lambda - T}$ *are equivalent.*

(*ii*) *If* $S(0) \subset T(0)$, *and S is T-bounded with T-bound* $\delta < 1$, *then the norms* $\|.\|_T$, *and* $\|.\|_{T+S}$ *are equivalent.*

In [1], the authors displayed the characterisation for the essential approximate point spectrum as well as the essential defect spectrum of T. Hence, we have the following Lemma:

Proposition 2.16. [1, Theorem 3.1] *Let* X *be complete,* $T \in \mathcal{CR}(X)$ *and* $\lambda \in \mathbb{C}$. *Then,*

(*i*) $\lambda \notin \sigma_{eap}(T)$ *if, and only if,* $T - \lambda \in \Phi_+(X)$ *and* $i(T - \lambda) \leq 0$.

(*ii*) $\lambda \notin \sigma_{e\delta}(T)$ *if, and only if,* $T - \lambda \in \Phi_-(X)$ *and* $i(T - \lambda) \geq 0$.

3. Main results

In this section, we establish criteria and we give a necessary and sufficient condition for closedness and closability of two linear relations of which there is one of the two relatively bounded entries.

Lemma 3.1. *Let $S, T \in \mathcal{LR}(X,Y)$ satisfies $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. If S is T-compact, then S is T-bounded.*

Proof. Suppose that S is not bounded. Then, assume without loss of generality that for each positive integer n, there exists an $\{x_n\}_n \in \mathcal{D}(T)$ such that

$$\|x_n\| + \|Tx_n\| = 1, \quad \textit{and} \ \ \|Sx_n\| > n$$

which implies that $\{x_n\}_n$, and $\{Tx_n\}_n$ are bounded, and that S is T-compact. Let $y_n \in Sx_n$. We can extract a convergent subsequence, which is a contradiction.

Lemma 3.2. *Let A, B, and $C \in \mathcal{LR}(X,Y)$.*

(i) *If A is B-bounded with B-bound δ_1, and B is C-bounded with C-bound δ_2, then A is C-bounded with C-bound $\delta_1\delta_2$.*

(ii) *If B is T-bounded with T-bound δ_1, and C is T-bounded with T-bound δ_2, then $A = B \pm C$ is T-bounded with T-bound $(\delta_1 + \delta_2)$.*

Proof. (i) Since A is B-bounded, and B is C-bounded, there exist $a,\ b,\ c,\ d \geq 0$, such that $\|Ax\| \leq a\|x\| + b\|Bx\|$ for all $x \in \mathcal{D}(B)$, and $\|Bx\| \leq c\|x\| + d\|Cx\|$ for all $x \in \mathcal{D}(C)$. It follows that, for all $x \in \mathcal{D}(C)$, $\|Ax\| \leq (a + bc)\|x\| + bd\|Cx\|$, and $\mathcal{D}(C) \subset \mathcal{D}(A)$.

(ii) Since B is T-bounded, and C is T-bounded, there exist $a,\ b,\ c,\ d \geq 0$, such that $\|Bx\| \leq a\|x\| + b\|Tx\|$ for all $x \in \mathcal{D}(T)$, and $\|Cx\| \leq c\|x\| + d\|Tx\|$, for all $x \in \mathcal{D}(T)$. It follows that, for all $x \in \mathcal{D}(T)$, $\|Ax\| = \|(B \pm C)x\| \leq \|Bx\| + \|Cx\| \leq (a + c)\|x\| + (b + d)\|Tx\|$, and $\mathcal{D}(T) \subset \mathcal{D}(A)$.

Lemma 3.3. *Let $T, S \in \mathcal{LR}(X,Y)$ such that $S(0) \subset \overline{T(0)}$, and $\mathcal{D}(T) \subset \mathcal{D}(S)$.*

(i) *Q_TS is single valued.*

(ii) $\|Q_TS\| \leq \|S\|$.

Proof. (i) $Q_TS(0) \subset Q_T\overline{T(0)} = 0$.
(ii) $\|Q_TSx\| = d(\overline{T(0)}, Sx) \leq d(S(0), Sx) = \|Sx\|$.

Lemma 3.4. *Let S, T, A and $B \in \mathcal{LR}(X,Y)$.*

(i) *If S and T are closable, and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then S is T-bounded with T-bound α if, and only if, $\overline{S}$ is $\overline{T}$-bounded with $\overline{T}$-bound α.*

(ii) *If $S = A + B$ is T-bounded with T-bound β, B is A-bounded with A-bound $\delta < 1$, and $B(0) \subset A(0)$, then A is T-bounded with T-bound $\gamma \leq \frac{\beta}{1-\delta}$.*

Proof. (i) Let S be T-bounded with T-bound α. Then, we obtain by Remark 2.3 (v), Q_SS which is Q_TT-bounded with Q_TT-bound α. The latter is equivalent to $\overline{Q_SS}$ is $\overline{Q_TT}$-bounded with $\overline{Q_TT}$-bound α by [12, Lemma, 8.1]. Since $\mathcal{D}(T) = \mathcal{D}(Q_TT)$, and $\overline{Q_TT} = Q_{\overline{T}}\overline{T}$ (by [11, Proposition, II.5.2]), the latter holds if, and only if, $\overline{S}$ is $\overline{T}$-bounded with $\overline{T}$-bound α.

(ii) Since B is A-bounded with A-bound $\delta < 1$, and $B(0) \subset A(0)$, then if we apply Lemma 2.5, we obtain B which is S-bounded with S-bound $\alpha \leq \frac{\delta}{1-\delta}$. Also, S is T-bounded with T-bound β. Hence, by Lemma 3.2 (i), it follows that B is T-bounded with T-bound $\alpha\beta \leq \frac{\delta\beta}{1-\delta}$. So, B is T-bounded with T-bound $\alpha\beta \leq \frac{\delta\beta}{1-\delta}$, and knowing that S is T-bounded with T-bound β and $A = S - B$, then by Lemma 3.2 (ii), we get A which is T-bounded with T-bound $\gamma \leq \frac{\beta}{1-\delta}$.

Remark 3.5. *Lemma* 3.4 *is a generalization of* [12, Lemma, 8.1].

Lemma 3.6. *Let $S, T \in \mathcal{LR}(X,Y)$ such that $S(0) \subset T(0)$. Suppose that S is T-bounded with T-bound δ, and Y is complete.*

(i) *If T is closable, and $\delta < 1$, then $T + S$ is closable.*

(ii) *If $T + S$ is closable, and $\delta < \frac{1}{2}$, then T is closable.*

If in addition we have S is closable, then we get $\overline{S+T} = \overline{S} + \overline{T}$ and $\mathcal{D}(\overline{T+S}) = \mathcal{D}(\overline{T})$.

Proof. (i) Let $(T + S)(0) = T(0)$ and T be closable. Then, by Proposition 2.7, we obtain $(T + S)(0)$ is closed, and $Q_T T$ is closable.
Since S is T-bounded with T-bound $\delta < 1$, then ,by Remark 2.3 (v), we obtain $Q_T S$ is $Q_T T$-bounded with $Q_T T$-bound < 1. So, we get $Q_{S+T}(T + S) = Q_T(T + S) = Q_T T + Q_T S$ is closable by [17, Theorem, IV.1.1]. Applying Proposition 2.7, it follows that $T + S$ is closable.

(ii) Since $T + S$ is closable, S is T-bounded with T-bound $\delta < \frac{1}{2}$, and by Lemma 2.4, we get $T = T + S - S$. Besides, by Lemma 2.5, we get S which is $(T + S)$-bounded with $(T + S)$-bound < 1. Therefore, using (i), it follows that T is closable. Since T and S are closable, and S is T-bounded with T-bound < 1, then, by Lemma 3.4 (i), we obtain $\overline{S}$ is $\overline{T}$-bounded with $\overline{T}$-bound < 1. So, by Lemma 3.7 (i), we get $\overline{S} + \overline{T}$ is closed. Since $T \subset \overline{T}$, and $S + T \subset \overline{S} + \overline{T}$, then $\overline{T+S} \subset \overline{\overline{T} + \overline{S}} = \overline{T} + \overline{S}$ ($\overline{S} + \overline{T}$ is closed).
On the one hand, we obtain by Lemma 2.4, $T = T+S-S$. Thus, $T = T+S-S \subset \overline{T+S} - \overline{S}$. On the other hand, we have S is T-bounded with T-bound $\delta < \frac{1}{2}$. Hence, by Lemma 2.5 S is $(T+S)$-bounded with $(T+S)$-bound < 1. Using Lemma 3.4 (i), we obtain $\overline{S}$ which is $(\overline{T+S})$-bounded with $(\overline{T+S})$-bound < 1. Therefore, by Lemma 3.7, we get $\overline{T+S} - \overline{S}$ is closed. Therefore, $\overline{T} \subset \overline{\overline{T+S} - \overline{S}} = \overline{T+S} - \overline{S}$, and by Lemma 2.4 ($ii$) we obtain $\overline{T} + \overline{S} \subset \overline{T+S}$. As a matter of fact, $\overline{T} + \overline{S} = \overline{T+S}$, and $\mathcal{D}(\overline{T+S}) = \mathcal{D}(\overline{T} + \overline{S}) = \mathcal{D}(\overline{T})$.

Theorem 3.7. *Let $T \in \mathcal{LR}(X,Y)$, and $S \in \mathcal{LR}(X,Y)$ be T-bounded with T-bound δ satisfying $S(0) \subset T(0)$.*

(i) *If T is closed, and $\delta < 1$, then $T + S$ is closed.*

(ii) *If $T + S$ is closed, and $\delta < \frac{1}{2}$, then T is closed.*

Proof. (i) Clearly $(T + S)(0) = T(0)$. Let T be closed. Then, we get, by Proposition 2.6, $(T + S)(0)$ which is closed, and $Q_T T$ which is closed. Since S is T-bounded with T-bound $\delta < 1$, then, by Remark 2.3 (v), we obtain $Q_T S$ which is $Q_T T$-bounded with $Q_T T$-bound < 1. Therefore, $Q_{S+T}(T + S) = Q_T(T + S) = Q_T T + Q_T S$ which is closed by [17, Theorem, IV.1.1 p 190]. Applying Proposition 2.6, it follows that $T + S$ is closed.

(ii) Since $T+S$ is closed, S is T-bounded with T-bound $\delta < \frac{1}{2}$, and by Lemma 2.4, we get $T = T+S-S$. Then, by Lemma 2.5, we obtain S which is $(T + S)$-bounded with $(T + S)$-bound < 1. Applying (i), it follows that T is closed.

Lemma 3.8. *Let $T \in \mathcal{CR}(X,Y)$, and $S \in \mathcal{LR}(X,Y)$. If $\mathcal{D}(T) \subset \mathcal{D}(S)$, and S is closable, then S is T-bounded.*

Proof. Let $S_1 := S_{|\mathcal{D}(T)}$. Since S is closable, then S_1 is closable, and hence $\overline{S_1}$ is a closed extension to S_1. It follows from Lemma 2.8 that $\overline{S_1}G_T$ is closed, and thus by Lemma 2.4 (ii), it follows that $\overline{S_1}G_T$ is bounded recalling that X_T is complete. This implies that $\overline{S_1}$ is T-bounded. Thus, by Lemma 3.4 S_1 is T-bounded i.e., S is T-bounded.

Lemma 3.9. *Let A, B and $C \in \mathcal{LR}(X,Y)$ satisfying $B(0) \cup C(0) \subset A(0)$. Suppose that B is A-bounded with A-bound δ_1, C is A-bounded with A-bound δ_2, and Y is complete.*

(i) *If $\delta_1 + \delta_2 < 1$, and A is closed, then $A + B + C$ is closed.*

(ii) *If $\delta_1 + \delta_2 < \frac{1}{2}$, and $A + B + C$ is closed, then A is closed.*

Proof. (i) Since B is A-bounded with A-bound δ_1, and C is A-bounded with A-bound δ_2, then we get, by Lemma 3.2 (ii), $B + C$ which is A-bounded with A-bound $(\delta_1 + \delta_2) < 1$. Applying Lemma 3.7 ($i$), we obtain $A + B + C$ which is closed.

(*ii*) Since B is A-bounded with A-bound δ_1, and C is A-bounded with A-bound δ_2. Then, for all $x \in \mathcal{D}(A)$, we have

$$\begin{aligned}
\|(B+C)x\| &\leq \|Bx\| + \|Cx\| \\
&\leq (a_1+a_2)\|x\| + (b_1+b_2)\|Ax\| \\
&\leq (a_1+a_2)\|x\| + (b_1+b_2)\|(A+B+C-B-C)x\| \\
&\leq (a_1+a_2)\|x\| + (b_1+b_2)\|(A+B+C)x\| + (b_1+b_2)\|(B+C)x\| \\
&\leq \frac{(a_1+a_2)}{1-(b_1+b_2)}\|x\| + \frac{(b_1+b_2)}{1-(b_1+b_2)}\|(A+B+C)x\|.
\end{aligned}$$

Therefore, we obtain $(B+C)$ which is $(A+B+C)$-bounded with $(A+B+C)$-bound < 1. On the other side, by Lemma 2.4 (*i*), we get $A = A+B+C-B-C$. As a matter of fact, $A+B+C$ is closed, and $(B+C)$ is $(A+B+C)$-bounded with $(A+B+C)$-bound < 1. Finally applying (*i*), it follows that A is closed.

Lemma 3.10. *Let A, B and $C \in \mathcal{LR}(X,Y)$ such that $C(0) \subset B(0) \subset A(0)$. Suppose that B is A-bounded with A-bound $\delta_1 < 1$, C is B-bounded with B-bound δ_2, and Y is complete. Then,*
If $\frac{\delta_1\delta_2}{1-\delta_1} < 1$, and A is closed, then $A+B+C$ is closed.

Proof. Since A is closed and B is A-bounded with A-bound $\delta_1 < 1$, then, by Lemma 3.7 (*i*), we get $A+B$ which is closed.
On the one hand, B is A-bounded with A-bound $\delta_1 < 1$, then by Lemma 2.5, we obtain B which is $(A+B)$-bounded with $(A+B)$-bound $\leq \frac{\delta_1}{1-\delta_1}$. On the other hand, C is B-bounded with B-bound δ_2. Then, by Lemma 3.2, we get C which is $(A+B)$-bounded with $(A+B)$-bound $\leq \frac{\delta_1\delta_2}{1-\delta_1}$.
Since C is $(A+B)$-bounded with $(A+B)$-bound < 1, and $A+B$ is closed. Then, by Lemma 3.7, we obtain $A+B+C$ which is closed.

4. Stability of $\sigma_{eap}(T)$ and $\sigma_{e\delta}(T)$

In this section, we explore the stability of the essential approximate point spectrum and the essential defect spectrum of closed and closable linear relations under relatively compact and precompact perturbation on Banach spaces.

Lemma 4.1. *Let $S \in \mathcal{LR}(X,Y)$ and $T \in \mathcal{F}_+(X,Y)$ with $\dim \mathcal{D}(T) = \infty$.*
If S is precompact, then S is strictly singular, and $\Delta(S) < \Gamma(T)$.
If additionally, $S(0) \subset \overline{T(0)}$, then $T+S \in \mathcal{F}_+(X,Y)$.

Proof. Since $T \in \mathcal{F}_+(X,Y)$, and $\dim \mathcal{D}(T) = \infty$, then $\Gamma(T) > 0$ by Proposition 2.11 (*iii*). Using Proposition 2.11 (*v*), we get $\overline{\Gamma}_0(S) = 0$ (S is precompact). Since $\Delta(S) \leq \overline{\Gamma}_0(S) = 0$ we have $\Delta(S) = 0$. Then, S is strictly singular by Proposition 2.11 (*i*). Thus, by Theorem 2.12, it follows that $T+S \in \mathcal{F}_+(X,Y)$.

Lemma 4.2. *Let $S, T \in \mathcal{LR}(X,Y)$. If S is T-precompact, then S is strictly singular.*

Proof. Since S is T-precompact, then SG_T is precompact. By Proposition 2.11 (*viii*), it follows that SG_T is continuous.
On the one hand, referring to [11, Exercise IV.1.5 p 95], we obtain $0 = \overline{\Gamma}_0(SG_T) = \Gamma_0(SG_T)$. On the other hand, $\Gamma_0(SG_T) = \frac{\Gamma_0(S)}{1+\Gamma_0(T)}$ by Proposition [11, Proposition IV.3.4 p 104]. So, $\Gamma_0(S) = 0$. Therefore S is strictly singular by Proposition 2.11 (*vi*).

Proposition 4.3. *Let $S, T \in \mathcal{LR}(X,Y)$ with $G(S) \subset G(T)$, and $\dim \mathcal{D}(S) = \infty$. Then,*

(*i*) *If $T \in \mathcal{F}_+(X,Y)$, then $T+S \in \mathcal{F}_+(X,Y)$.*

(*ii*) *If $T+S \in \mathcal{F}_+(X,Y)$, and $\dim S(0) < \infty$, then $T \in \mathcal{F}_+(X,Y)$.*

Proof. (*i*) Since we have $G(S) \subset G(T)$, then we get

$$\begin{aligned} G(T+S) &:= \{(x, y+z) \in X \times Y : (x,y) \in G(T) \text{ and } (x,z) \in G(S)\} \\ &\subset \{(x, y+z) \in X \times Y : (x,y) \in G(T) \text{ and } (x,z) \in G(T)\} \\ &\subset G(T). \end{aligned}$$

Now $\dim \mathcal{D}(T+S) = \dim \mathcal{D}(T) \cap \mathcal{D}(S) = \dim \mathcal{D}(S) = \infty$. So, by Proposition 2.14, we have $T+S \in \mathcal{F}_+(X,Y)$.

(*ii*) Since we have $G(S) \subset G(T)$, then we obtain

$$\begin{aligned} G(S) &:= \{(x,y) \in X \times Y : x \in \mathcal{D}(S) \subset \mathcal{D}(T) \text{ and } y \in Sx \subset Tx\} \\ &\subset \{(x,y) \in X \times Y : x \in \mathcal{D}(S) \cap \mathcal{D}(T) \text{ and } y \in Tx + Sx\} \\ &:= \{(x,y) \in X \times Y : x \in \mathcal{D}(S+T) \text{ and } y \in (T+S)x\} \\ &:= G(T+S). \\ G(T+S-S) &:= \{(x, y+z) \in X \times Y : (x,y) \in G(T+S) \text{ and } (x,z) \in G(S)\} \\ &\subset \{(x, y+z) \in X \times Y : (x,y) \in G(T+S) \text{ and } \\ &\quad\ (x,z) \in G(S) \subset G(T+S)\} \\ &:= G(T+S). \end{aligned}$$

$\dim \mathcal{D}(T+S-S) = \dim \mathcal{D}(T) \cap \mathcal{D}(S) = \dim \mathcal{D}(S+T) = \infty$. Therefore, by Proposition 2.14, we have $T+S-S \in \mathcal{F}_+(X,Y)$. Thus, by Proposition 2.13 (*iv*), it follows that $T \in \mathcal{F}_+(X,Y)$.

Corollary 4.4. *Let $S \in \mathcal{LR}(X,Y)$, and $T \in \mathcal{LR}(X,Y)$ be continuous with $G(S') \subset G(T')$, and $\dim \mathcal{D}(S') = \infty$. Then,*

(*i*) *If $T \in \mathcal{F}_-(X,Y)$, then $T+S \in \mathcal{F}_-(X,Y)$.*

(*ii*) *If $T+S \in \mathcal{F}_-(X,Y)$, and $\dim S'(0) < \infty$, then $T \in \mathcal{F}_-(X,Y)$.*

Proof. (*i*) Let $T \in \mathcal{F}_-(X,Y)$. Then, by Proposition 2.13 (*iii*), $T' \in \mathcal{F}_+(Y',X')$. Since $G(S') \subset G(T')$, by Proposition 4.3, we have $T'+S' \in \mathcal{F}_+(Y',X')$. Thus, by Proposition 2.13 (*i*), we have $(T+S)' \in \mathcal{F}_+(Y',X')$. So, $T+S \in \mathcal{F}_-(X,Y)$ by Proposition 2.13 (*iii*).

(*ii*) Let $T+S \in \mathcal{F}_-(X,Y)$, then by Proposition 2.13 (*iii*), and Proposition 2.13 (*i*), we get $T'+S' \in \mathcal{F}_+(Y',X')$. By Proposition 4.3, we obtain $T' \in \mathcal{F}_+(Y',X')$. Thus, by Proposition 2.13 (*iii*) $T \in \mathcal{F}_-(X,Y)$.

Proposition 4.5. *Let X, Y be complete, and let T, $S \in \mathcal{LR}(X,Y)$. If S is T-precompact, and $S(0) \subset T(0)$, then $i(T) = i(T+S)$.*

Proof. Clearly, $\mathcal{R}(TG_T) = \mathcal{R}(T)$, $\mathcal{N}(TG_T) = \mathcal{N}(T)$, $\mathcal{R}((T+S)G_T) = \mathcal{R}(T+S)$, and $\mathcal{N}((T+S)G_T) = \mathcal{N}(T+S)$. Since S is T-precompact, then SG_T is precompact, and Y is complete. By Remark [11, Note V.1 p 134], SG_T is compact. Then, by Lemma 2.10, we get $i(T) = i(TG_T) = i(TG_T + SG_T)$. So, we obtain $i(T) = i(T+S)$.

Theorem 4.6. *Let X be complete, and let $T \in \mathcal{CR}(X)$. Suppose $S \in \mathcal{LR}(X)$ is T-precompact with T-bounded $\delta < 1$, $\overline{\mathcal{D}(T)} \subset \mathcal{D}(S)$ with $\dim \mathcal{D}(T) = \infty$, and $S(0) \subset T(0)$. Then,*

$$\sigma_{eap}(T+S) = \sigma_{eap}(T),$$

and

$$\sigma_{e\delta}(T+S) = \sigma_{e\delta}(T).$$

Proof. Let S be T-precompact, then SG_T is precompact, and X as well as X_T are complete. By Remark [11, Note V.1 p 134], we get SG_T which is compact. By Theorem 3.7, and Lemma 3.1, we obtain $T+S$ is closed. Suppose that $\lambda \notin \sigma_{eap}(T)$. Then, by Proposition 2.16 (*i*) $\lambda - T \in \Phi_+(X)$. By Proposition 2.11 (*iv*), we get $(\lambda - T)G_{\lambda - T} \in \Phi_+(X_T)$, which gives $(\lambda - T)G_T \in \Phi_+(X_T)$ by referring to Proposition 2.15 (*i*). Since SG_T is compact, then using Lemma 4.1, it follows that $(\lambda - T + S)G_T \in \Phi_+(X_T)$. Hence, by Proposition 2.15 (*ii*), we obtain $(\lambda - (T+S))G_{\lambda-(T+S)} \in \Phi_+(X_T)$. Thus, $(\lambda - (T+S)) \in \Phi_+(X)$ by Proposition 2.11 (*iv*). We have $i(\lambda - T) = i(\lambda - (T+S))$ by Proposition 4.5, that is $\lambda \notin \sigma_{eap}(T+S)$ by Proposition 2.16 (*i*). Referring to the above, we infer that

$$\sigma_{eap}(T+S) \subseteq \sigma_{eap}(T).$$

Conversely, let $\lambda \notin \sigma_{eap}(T+S)$. Then, based upon Proposition 2.16 (i), $(\lambda-(T+S)) \in \Phi_+(X)$. Similary, it follows that $(\lambda-(T+S-S)) \in \Phi_+(X)$. If we use Lemma 2.4 ($i$), then $\lambda - T \in \Phi_+(X)$. By Proposition 4.5, we have $i(\lambda-T) = i(\lambda-(T+S))$ that is $\lambda \notin \sigma_{eap}(T)$ relying on Proposition 2.16 (i). Therefore, we infer that

$$\sigma_{eap}(T+S) = \sigma_{eap}(T).$$

Now, suppose that $\lambda \notin \sigma_{e\delta}(T)$. Then, by Proposition 2.16 (ii), $\lambda - T \in \Phi_-(X)$. Applying Proposition 2.13 (v), we obtain $(\lambda-T)G_{\lambda-T} \in \Phi_-(X_T)$. Using Proposition 2.15 (i), we get $(\lambda-T)G_T \in \Phi_-(X_T)$. Since SG_T is precompact, then by Proposition 2.13 (vi), we obtain, $(\lambda-(T+S))G_{(\lambda-T)} \in \Phi_-(X_T)$. Resorting to Proposition 2.15 (i), as well as (ii), we get $(\lambda-(T+S))G_{\lambda-(T+S)} \in \Phi_-(X_T)$. So, applying Proposition 2.13 (v), we get $(\lambda-(T+S)) \in \Phi_-(X)$. By Proposition 4.5, we have $i(\lambda-T) = i(\lambda-(T+S))$, that is $\lambda \notin \sigma_{e\delta}(T+S)$ by Proposition 2.16 (ii). Then,

$$\sigma_{e\delta}(T+S) \subset \sigma_{e\delta}(T).$$

Conversely, let $\lambda \notin \sigma_{e\delta}(T+S)$. Then, by Proposition 2.16 (ii) $\lambda-(T+S) \in \Phi_-(X)$. Using Proposition 2.13 (v), we obtain $(\lambda-(T+S))G_{\lambda-(T+S)} \in \Phi_-(X_T)$. Referring to Proposition 2.15 (i) and (ii), we get $(\lambda-(T+S))G_T \in \Phi_-(X_T)$. The latter holds if, and only if, $((\lambda-(T+S))G_T)' \in \Phi_+(X_T')$ which is maintained by Proposition 2.13 (iii). As a matter of fact, using Proposition 2.11 ($viii$), and Proposition 2.13 (i), we get $((\lambda-T)G_T)' + (SG_T)' \in \Phi_+(X_T')$. Since SG_T is precompact, then by Proposition 2.13 (ii) and arguing as before, we have $((\lambda-T)G_T)' \in \Phi_+(X_T')$. Besides, using Proposition 2.13 (iii), we get $(\lambda-T)G_T \in \Phi_-(X_T)$. Therefore, by Proposition 2.13 (v), $((\lambda-T) \in \Phi_-(X)$. By Proposition 4.5, We have $i(\lambda-T) = i(\lambda-(T+S))$, that is $\lambda \notin \sigma_{e\delta}(T)$, which is confirmed by Proposition 2.16 (ii). From this perspective, we infer that

$$\sigma_{e\delta}(T+S) = \sigma_{e\delta}(T).$$

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