Functional Analysis,
Approximation and
Computation
15 (1) (2023), 1–12

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Common fixed point theorems in partial metric spaces satisfying common $(E.A)$-property and an implicit relation

G. S. Saluja[a]

[a] *H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India*

Abstract. In this article, we prove some common fixed point theorems for pair of mappings in the setting of partial metric spaces satisfying common $(E.A)$-property via an implicit relation. We give some consequences of the established results. Also, we give some examples to demonstrate the validity of the results. Our results extend, generalize and improve several results from the existing literature regarding contraction condition involving rational terms and partial metric spaces.

1. Introduction

The first important result on fixed points for contractive mapping was the well-known Banach contraction principle appeared in explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation [9]. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. In a metric space setting it can be briefly stated as follows.

Theorem 1.1. *([9]) Let $(\mathcal{X}, \rho)$ be a complete metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-map satisfying*

$$\rho(\mathcal{T}(x), \mathcal{T}(y)) \leq k\, \rho(x, y), \quad \text{for all } x, y \in \mathcal{X}, \tag{1}$$

where $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point z in $\mathcal{X}$ and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = \mathcal{T}x_n, \quad n = 0, 1, 2, \ldots \tag{2}$$

converges to z, for any $x_0 \in \mathcal{X}$.

Remark 1.2. *Inequality (1) implies the continuity of $\mathcal{T}$.*

There are many generalizations of this principle. These generalizations are made either by using different contractive conditions or by imposing some additional condition on the ambient spaces. On the other hand, a number of generalizations of metric spaces have been done and one of such generalization is partial metric

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25.
Keywords. Common fixed point; Common $(E.A)$-property; Weakly compatible mapping; Partial metric space.
Received: 6 September 2022; Accepted: 14 September 2022
Communicated by Dragan S. Djordjević
Email address: saluja1963@gmail.com (G. S. Saluja)

space introduced in 1992 by Matthews [24, 25]. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews proved the partial metric version of Banach fixed point theorem ([9]). Then, many authors gave some generalizations of the result of Matthews and proved some fixed point theorem in this space (see, i.e., [3], [4], [5], [18], [19], [20], [21], [28], [46] and many others).

In 1976, Jungck [15] initiated the study of common fixed point for a pair of commuting mappings satisfying contractive type conditions. In 1982, Sessa [43] introduced a weaker concept of commutativity, which is generally known as weak commutativity and proved some interesting results on the existence of common fixed points for a pair of self maps. He also showed that weak commuting mappings are commuting but the converse need not to be true. Later, Jungck [16] generalized the concept of weak commutativity by introducing the notion of compatible mappings which is more general than weakly commuting mappings and showed that weak commuting maps are compatible but converse need not be true. In 1996, Jungck [17] generalized the concept of compatibility by introducing weakly compatible mappings.

The study of common fixed points for non compatible mappings was initiated by Pant [29]. In 2002, Aamri and El Moutawakil [1] introduced a new concept called $(E.A)$-property for pair of mappings which is a generalization of non compatible mappings and they proved some common fixed point theorems. The concept of $(E.A)$-property allows us to replace the completeness requirement of the space by a more general condition of closeness of range. In [31], Pathak et al. established a common fixed point theorem in metric space for an integral type condition and using implicit relation and the $(E.A)$ property. Some authors showed that the notion of weakly compatible mappings and mappings satisfying $(E.A)$-property are independent (see, [30], [32]).

In 2005, Liu *et al.* [23] introduced the notion of common $(E.A)$-property. Many authors established common fixed point theorems by using common $(E.A)$-property in the setting of metric spaces and variants of metric spaces (see, for example, [6], [12], [13], [27]).

Many classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [32], [33] and in some other papers.

This direction of research produced a consistent literature on fixed point, common fixed point and coincidence point theorems in various ambient spaces. For more details see [7, 10, 11, 14, 34–36, 38–42].

In 2016, Tiwari and Gupta [44] proved some common fixed point theorems in metric spaces satisfying an implicit relation involving quadratic terms. In 2019, Neog *et al.* [27] prove common fixed results of set valued maps for A_φ-contraction and generalized ϕ-type weak contraction in the setting of metric spaces.

Recently, Tiwari and Thakur in [45] proved some common fixed point theorems for pair of mappings satisfying common $(E.A)$-property in the setting of complete metric spaces and give application of the established result.

Motivated by the work of [27, 44, 45] and some others, the main purpose of this work is to prove some common fixed point theorems for contractive condition involving rational terms satisfying common $(E.A)$-property and an implicit relation in the framework of partial metric spaces.

2. Preliminaries

In this section, we recall some basic definitions, properties and auxiliary results of partial metric spaces.

Definition 2.1. *([25]) Let $\mathcal{X}$ be a nonempty set and $p\colon \mathcal{X}\times\mathcal{X} \to [0,\infty)$ be such that for all $u, v, w \in \mathcal{X}$ the followings are satisfied:*

$(P1)$ $u = v \Leftrightarrow p(u,u) = p(u,v) = p(v,v)$,

$(P2)$ $p(u,u) \leq p(u,v)$,

$(P3)$ $p(u,v) = p(v,u)$,

$(P4)$ $p(u,v) \leq p(u,w) + p(w,v) - p(w,w)$.

Then p is called partial metric on $\mathcal{X}$ and the pair $(\mathcal{X}, p)$ is called partial metric space (in short PMS).

Remark 2.2. *It is clear that if $p(u,v) = 0$, then $u = v$. But, on the contrary $p(u,u)$ need not be zero.*

Example 2.3. *([8]) Let $\mathcal{X} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0,+\infty)$ and $p\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}^+$ given by $p(u,v) = \max\{u,v\}$ for all $u,v \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.*

Example 2.4. *([8]) Let $\mathcal{X} = \{[a,b] : a,b \in \mathbb{R}, a \le b\}$. Then $p\Big([a,b],[c,d]\Big) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on $\mathcal{X}$.*

Various applications of this space has been extensively investigated by many authors (see, for example, K*ü*nzi [22], Valero [46] for details).

Remark 2.5. *([19]) Let $(\mathcal{X}, p)$ be a partial metric space.*

• *The function $d_p\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}^+$ defined as $d_p(u,v) = 2p(u,v) - p(u,u) - p(v,v)$ is a metric on $\mathcal{X}$ and $(\mathcal{X}, d_p)$ is a metric space.*

• *The function $d_s\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}^+$ defined as $d_s(u,v) = \max\{p(u,v) - p(u,u), p(u,v) - p(v,v)\}$ is a metric on $\mathcal{X}$ and $(\mathcal{X}, d_s)$ is a metric space.*

Note also that each partial metric p on $\mathcal{X}$ generates a T_0 topology τ_p on $\mathcal{X}$, whose base is a family of open p-balls $\{B_p(u,\varepsilon) : u \in \mathcal{X}, \varepsilon > 0\}$ where $B_p(u,\varepsilon) = \{v \in \mathcal{X} : p(u,v) < p(u,u) + \varepsilon\}$ for all $u \in \mathcal{X}$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [24].

Definition 2.6. *([24]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(Γ_1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is said to be convergent to a point $r \in \mathcal{X}$ if and only if $p(r,r) = \lim_{n\to\infty} p(r_n, r)$,*

(Γ_2) *a sequence $\{r_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(r_n, r_m)$ exists and finite,*

(Γ_3) *$(\mathcal{X}, p)$ is said to be complete if every Cauchy sequence $\{r_n\}$ in $\mathcal{X}$ converges to a point $r \in \mathcal{X}$ with respect to τ_p. Furthermore,*

$$\lim_{n,m\to\infty} p(r_n, r_m) = \lim_{n\to\infty} p(r_n, r) = p(r,r).$$

(Γ_4) *A mapping $G\colon \mathcal{X} \to \mathcal{X}$ is said to be continuous at $r_0 \in \mathcal{X}$ if for every $\varepsilon > 0$, there exists $\alpha > 0$ such that $G\Big(B_p(r_0,\alpha)\Big) \subset B_p\Big(G(r_0),\varepsilon\Big)$.*

Definition 2.7. *([26]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(Δ_1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is called 0-Cauchy if $\lim_{n,m\to\infty} p(r_n, r_m) = 0$,*

(Δ_2) *$(\mathcal{X}, p)$ is said to be 0-complete if every 0-Cauchy sequence $\{r_n\}$ in $\mathcal{X}$ converges to a point $r \in \mathcal{X}$, such that $p(r,r) = 0$.*

Definition 2.8. *Let X be a non-empty set and let $P, Q\colon \mathcal{X} \to \mathcal{X}$ be two self mappings of $\mathcal{X}$. Then a point $u \in \mathcal{X}$ is called a*

(Λ_1) *fixed point of operator P if $P(u) = u$;*

(Λ_2) *common fixed point of P and Q if $P(u) = Q(u) = u$.*

Definition 2.9. *([2]) Let A and B be single valued self-mappings on a set $\mathcal{X}$. If $u = Az = Bz$ for some $z \in \mathcal{X}$, then z is called a coincidence point point of A and B, and u is called a point of coincidence of A and B.*

Definition 2.10. *([16]) Let R and T be single valued self-mappings on a set $\mathcal{X}$. Mappings R and T are said to be commuting if $RTw = TRw$ for all $w \in \mathcal{X}$.*

Definition 2.11. *([17]) Let C and D be single valued self-mappings on a set $\mathcal{X}$. Mappings C and D are said to be weakly compatible if they commute at their coincidence points, i.e., if $Cu = Du$ for some $u \in \mathcal{X}$ implies $CDu = DCu$.*

Definition 2.12. *([1]) Let H and K be two self-mappings of a partial metric space $(\mathcal{X}, p)$. We say that H and K satisfy $(E.A)$-property if there exists a sequence $\{r_n\}$ in $\mathcal{X}$ such that*

$$\lim_{n\to\infty} Hr_n = \lim_{n\to\infty} Kr_n = t,$$

for some $t \in \mathcal{X}$.

Definition 2.13. *([23]) Two pairs (A, S) and (B, T) of self-mappings of a partial metric space $(\mathcal{X}, p)$ are said to satisfy common $(E.A)$-property if there exists two sequence $\{r_n\}$ and $\{z_n\}$ in $\mathcal{X}$ such that*

$$\lim_{n\to\infty} Ar_n = \lim_{n\to\infty} Sr_n = \lim_{n\to\infty} Bz_n = \lim_{n\to\infty} Tz_n = t,$$

for some $t \in \mathcal{X}$.

Lemma 2.14. *([24, 25]) Let $(\mathcal{X}, p)$ be a partial metric space. Then*

(Θ_1) *a sequence $\{r_n\}$ in $(\mathcal{X}, p)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{X}, d_p)$,*

(Θ_2) *$(\mathcal{X}, p)$ is complete if and only if the metric space $(\mathcal{X}, d_p)$ is complete,*

(Θ_3) *a subset E of a partial metric space $(\mathcal{X}, p)$ is closed if a sequence $\{r_n\}$ in E such that $\{r_n\}$ converges to some $r \in \mathcal{X}$, then $r \in E$.*

Lemma 2.15. *([4]) Assume that $r_n \to r$ as $n \to \infty$ in a partial metric space $(\mathcal{X}, p)$ such that $p(r, r) = 0$. Then $\lim_{n\to\infty} p(r_n, u) = p(r, u)$ for every $u \in \mathcal{X}$.*

Remark 2.16. *(see [19]) Let $(\mathcal{X}, p)$ be a PMS. Therefore, for all $u, v \in \mathcal{X}$*

(i) if $p(u, v) = 0$, then $u = v$;

(ii) if $u \neq v$, then $p(u, v) > 0$.

Definition 2.17. *([37]) Consider the class of functions $\Phi = \{\phi | \phi\colon [0, \infty) \to [0, \infty)\}$, which satisfy the following assertions:*

(Φ_1) *$t_1 \leq t_2$ implies $\phi(t_1) \leq \phi(t_2)$;*

(Φ_2) *$(\phi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$;*

(Φ_3) *$\sum_{n=1}^{\infty} \phi^n(t)$ is convergent for all $t > 0$.*

If conditions (Φ_1)-(Φ_2) hold, then ϕ is called a comparison function and if the comparison function satisfies (Φ_3), then ϕ is called a strong comparison function.

Remark 2.18. *([37]) If $\phi\colon [0, \infty) \to [0, \infty)$ is a comparison function, then $\phi(t) < t$ for all $t > 0$, $\phi(0) = 0$ and ϕ is right continuous at 0.*

Recently, Tiwari and Tripathi [45] introduced the following notion.

Definition 2.19. *([45]) Let $\mathbb{R}_+$ be the set of all non-negative real numbers and A_φ be the collection of all functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the conditions:*

(1) *α is continuous on $\mathbb{R}_+^4$ (with respect to the Euclidean metric on $\mathbb{R}_+^4$);*

(2) *for all $u, v \in \mathbb{R}_+$, if*

(2_a) *$u \leq \alpha(u, v, v, v)$ or*

(2_b) *$u \leq \alpha(v, u, v, v)$ or*

(2_c) *$u \leq \alpha(v, v, u, v)$, then $u \leq \varphi(v)$, where φ is a strong comparison function. If $\varphi(t) = kt$ for $k \in [0, 1)$ and for all $t > 0$, then we have $\alpha \in A_\varphi$.*

Now, we define the following implicit relation.

Implicit Relation.

Definition 2.20. *Let $\mathbb{R}_+$ be the set of all non-negative real numbers and A_ϕ be the collection of all functions $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy the following conditions:*

(1) *α is continuous on $\mathbb{R}_+^5$ (with respect to the Euclidean metric on $\mathbb{R}_+^5$);*

(2) *for all $x, y \in \mathbb{R}_+$, if*

(α_{2a}) *$x \le \alpha(y, y, x, y, x)$ or*

(α_{2b}) *$x \le \alpha(y, x, y, x, y)$ or*

(α_{2c}) *$x \le \alpha(x, y, y, x, y)$, then $x \le \phi(y)$, where ϕ is a strong comparison function. If $\phi(t) = qt$ for $q \in [0, 1)$ and for all $t > 0$, then we have $\alpha \in A_\phi$.*

3. Main Results

In this section, we shall prove some common fixed point theorems in the setting of partial metric spaces using common $(E.A)$-property and an implicit relation.

Theorem 3.1. *Let $(\mathcal{X}, p)$ be a partial metric space and let $A, B, S, T\colon \mathcal{X} \to \mathcal{X}$ be four self-mappings of $\mathcal{X}$. If there exists some $\alpha \in A_\phi$ such that for all $x, y \in \mathcal{X}$ satisfying the following conditions:*

(i)

$$\begin{aligned} p(Tx, Sy) \quad &\le \quad \alpha\Big(p(Ax, By), p(Ax, Tx), p(Sy, By), p(Tx, By), \\ &\qquad \frac{p(Sy, By)[1 + p(By, Tx)]}{[1 + p(Ax, By)]}\Big), \end{aligned} \tag{3}$$

(ii) *the pairs (A, T) and (B, S) are weakly compatible;*

(iii) *the pairs (A, T) and (B, S) satisfy common $(E.A)$-property;*

(iv) *$A(X) \subseteq S(X)$ and $B(X) \subseteq T(X)$.*

Also, assume that $A(\mathcal{X})$ or $B(\mathcal{X})$ is closed in $\mathcal{X}$. Then A, B, S and T have a unique common fixed point $z \in \mathcal{X}$ with $p(z, z) = 0$.

Proof. Since the pairs (A, T) and (B, S) satisfy the common $(E.A)$-property, then by definition 2.13, there exists two sequence $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = z,$$

for some $z \in \mathcal{X}$. Further, since $B(\mathcal{X})$ is closed subset of $\mathcal{X}$, there exists $u \in \mathcal{X}$ such that $Bu = z$. We claim that $Su = z$. If not, then from equation (3), we have

$$\begin{aligned} p(z, Su) \quad &= \quad p(Tx_n, Su) \\ &\le \quad \alpha\Big(p(Ax_n, Bu), p(Ax_n, Tx_n), p(Su, Bu), p(Tx_n, Bu), \\ &\qquad \frac{p(Su, Bu)[1 + p(Bu, Tx_n)]}{[1 + p(Ax_n, Bu)]}\Big), \end{aligned}$$

taking the limit as $n \to \infty$, using $(P3)$ and by hypothesis $p(z, z) = 0$, we obtain

$$\begin{aligned} p(z, Su) \quad &\le \quad \alpha\Big(p(z, z), p(z, z), p(Su, z), p(z, z), \\ &\qquad \frac{p(Su, z)[1 + p(z, z)]}{[1 + p(z, z)]}\Big) \\ &= \quad \alpha\Big(0, 0, p(z, Su), 0, p(z, Su)\Big), \end{aligned}$$

by Definition of (α_{2a}), we get $p(z, Su) = 0$, that is, $Su = z$. Therefore, $Su = Bu = z$. Hence u is a coincidence point of the pair (B, S). Now, since $A(X) \subseteq S(X)$, there exists $v \in \mathcal{X}$ such that $Av = z$. We claim that $Tv = z$. From equation (3), we have

$$\begin{aligned} p(Tv, z) &= p(Tv, Sy_n) \\ &\leq \alpha\Big(p(Av, By_n), p(Av, Tv), p(Sy_n, By_n), p(Tv, By_n), \\ &\qquad \frac{p(Sy_n, By_n)[1 + p(By_n, Tv)]}{[1 + p(Av, By_n)]}\Big) \\ &= \alpha\Big(p(z, By_n), p(z, Tv), p(Sy_n, By_n), p(Tv, By_n), \\ &\qquad \frac{p(Sy_n, By_n)[1 + p(By_n, Tv)]}{[1 + p(z, By_n)]}\Big), \end{aligned}$$

taking the limit as $n \to \infty$, using $(P3)$ and by hypothesis $p(z, z) = 0$, we obtain

$$\begin{aligned} p(Tv, z) &\leq \alpha\Big(p(z, z), p(z, Tv), p(z, z), p(Tv, z), \\ &\qquad \frac{p(z, z)[1 + p(z, Tv)]}{[1 + p(z, z)]}\Big) \\ &= \alpha\Big(0, p(Tv, z), 0, p(Tv, z), 0\Big), \end{aligned}$$

by Definition of (α_{2b}), we get $p(Tv, z) = 0$, that is, $Tv = z$. Therefore, $Tv = Av = z$. Hence v is a coincidence point of the pair (A, T). Thus $Bu = Su = Av = Tv = z$ and by weak compatibility of the pairs (A, T) and (B, S), we deduce that $Bz = Sz$ and $Az = Tz$. Now, we show that z is a fixed point of T. By equation (3) and using $p(z, z) = 0$ for some $z \in \mathcal{X}$, we have

$$\begin{aligned} p(Tz, z) &= p(Tz, Su) \\ &\leq \alpha\Big(p(Az, Bu), p(Az, Tz), p(Su, Bu), p(Tz, Bu), \\ &\qquad \frac{p(Su, Bu)[1 + p(Bu, Tz)]}{[1 + p(Az, Bu)]}\Big) \\ &= \alpha\Big(p(Tz, z), p(Tz, Tz), p(z, z), p(Tz, z), \\ &\qquad \frac{p(z, z)[1 + p(z, Tz)]}{[1 + p(Tz, z)]}\Big) \\ &= \alpha\Big(p(Tz, z), 0, 0, p(Tz, z), 0\Big), \end{aligned}$$

by Definition of (α_{2c}), we get $p(Tz, z) = 0$, that is, $Tz = z$. Hence z is a fixed point of T. Since $Az = Tz = z$, we conclude that z is a fixed point of A. Now, we show that z is a fixed point of S. For this, using equation (3) and $p(z, z) = 0$ for some $z \in \mathcal{X}$, we have

$$\begin{aligned} p(z, Sz) &= p(Tv, Sz) \\ &\leq \alpha\Big(p(Av, Bz), p(Av, Tv), p(Sz, Bz), p(Tv, Bz), \\ &\qquad \frac{p(Sz, Bz)[1 + p(Bz, Tv)]}{[1 + p(Av, Bz)]}\Big) \\ &= \alpha\Big(p(z, Sz), p(z, z), p(Sz, Sz), p(z, Sz), \\ &\qquad \frac{p(Sz, Sz)[1 + p(Sz, z)]}{[1 + p(z, Sz)]}\Big) \\ &= \alpha\Big(p(z, Sz), 0, 0, p(z, Sz), 0\Big), \end{aligned}$$

by Definition of (α_{2c}), we get $p(z, Sz) = 0$, that is, $z = Sz$. Hence z is a fixed point of S. Since $Bz = Sz = z$, we conclude that z is a fixed point of B. Thus, z is a common fixed point of A, B, S and T.

Now, we show the uniqueness of the common fixed point. For this, we assume that z' is another common common fixed point of A, B, S and T such that $Az' = Bz' = Sz' = Tz' = z'$ with $z' \neq z$. From equation (3) and $p(z, z) = 0$ for some $z \in \mathcal{X}$, we obtain

$$\begin{aligned} p(z, z') &= p(Tz, Sz') \\ &\leq \alpha\Big(p(Az, Bz'), p(Az, Tz), p(Sz', Bz'), p(Tz, Bz'), \\ &\qquad \frac{p(Sz', Bz')[1 + p(Bz', Tz)]}{[1 + p(Az, Bz')]}\Big) \\ &= \alpha\Big(p(z, z'), p(z, z), p(z', z'), p(z, z'), \\ &\qquad \frac{p(z', z')[1 + p(z', z)]}{[1 + p(z, z')]}\Big) \\ &= \alpha\Big(p(z, z'), 0, 0, p(z, z'), 0\Big) \end{aligned} \tag{4}$$

by Definition of (α_{2c}), we get $p(z, z') = 0$, that is, $z = z'$. This shows the uniqueness of the common fixed point of A, B, S and T. This completes the proof. □

If we take $T = S$ and $B = A$ in Theorem 3.1, then we have the following result.

Corollary 3.2. *Let $(\mathcal{X}, p)$ be a partial metric space and let $A, S \colon \mathcal{X} \to \mathcal{X}$ be two self-mappings of $\mathcal{X}$. If there exists some $\alpha \in A_\phi$ such that for all $x, y \in \mathcal{X}$ satisfying the following conditions:*
(i)

$$\begin{aligned} p(Sx, Sy) &\leq \alpha\Big(p(Ax, Ay), p(Ax, Sx), p(Sy, Ay), p(Sx, Ay), \\ &\qquad \frac{p(Sy, Ay)[1 + p(Ay, Sx)]}{[1 + p(Ax, Ay)]}\Big), \end{aligned}$$

(ii) *the pair (A, S) is weakly compatible;*
(iii) *the pair (A, S) satisfying $(E.A)$-property;*
(iv) *$A(X) \subseteq S(X)$.*
Also, assume that $A(\mathcal{X})$ is closed subset of $\mathcal{X}$. Then A and S have a unique common fixed point $z \in \mathcal{X}$ with $p(z, z) = 0$.

If we take $B = A$ and $\alpha(t_1, t_2, t_3, t_4, t_5) = \alpha(t_1, t_2, t_3, t_4)$ in Theorem 3.1, then we have the following result.

Corollary 3.3. *Let $(\mathcal{X}, p)$ be a partial metric space and let $A, S, T \colon \mathcal{X} \to \mathcal{X}$ be three self-mappings of $\mathcal{X}$. If there exists some $\alpha \in A_\phi$ such that for all $x, y \in \mathcal{X}$ satisfying the following conditions:*
(i)

$$p(Tx, Sy) \leq \alpha\Big(p(Ax, Ay), p(Ax, Tx), p(Sy, Ay), p(Tx, Ay)\Big),$$

(ii) *the pairs (A, S) and (A, T) are weakly compatible;*
(iii) *the pairs (A, S) and (A, T) satisfying common $(E.A)$-property;*
(iv) *$A(X) \subseteq S(X)$ and $A(X) \subseteq T(X)$.*
Also, assume that $A(\mathcal{X})$ is closed subset of $\mathcal{X}$. Then A, S and T have a unique common fixed point $z \in \mathcal{X}$ with $p(z, z) = 0$.

If we take $B = A$ and $\alpha(t_1, t_2, t_3, t_4, t_5) = \alpha(t_1, t_2, t_3)$ in Theorem 3.1, then we have the following result.

Corollary 3.4. *Let $(\mathcal{X}, p)$ be a partial metric space and let $A, S, T\colon \mathcal{X} \to \mathcal{X}$ be three self-mappings of $\mathcal{X}$. If there exists some $\alpha \in A_\phi$ such that for all $x, y \in \mathcal{X}$ satisfying the following conditions:*
(i)

$$p(Tx, Sy) \leq \alpha\Big(p(Ax, Ay), p(Ax, Tx), p(Sy, Ay)\Big),$$

(ii) *the pairs (A, S) and (A, T) are weakly compatible;*
(iii) *the pairs (A, S) and (A, T) satisfying common $(E.A)$-property;*
(iv) *$A(X) \subseteq S(X)$ and $A(X) \subseteq T(X)$.*
Also, assume that $A(\mathcal{X})$ is closed subset of $\mathcal{X}$. Then A, S and T have a unique common fixed point $z \in \mathcal{X}$ with $p(z, z) = 0$.

Remark 3.5. *By using similar fashion we can find some more results from Theorem 3.1.*

If we take $A = I$ (where I is an identity mapping) and $\alpha(t_1, t_2, t_3, t_4, t_5) = q\, t_1$, where $q \in [0, 1)$ in Corollary 3.2, then we obtain the following result.

Corollary 3.6. *([25]) Let $(\mathcal{X}, p)$ be a complete partial metric space and let $S\colon \mathcal{X} \to \mathcal{X}$ be a self-mapping of $\mathcal{X}$ satisfying the condition:*

$$p(Sx, Sy) \leq q\, p(x, y),$$

for all $x, y \in \mathcal{X}$, where $q \in [0, 1)$ is a constant. Then S has a unique fixed point $z \in \mathcal{X}$ with $p(z, z) = 0$.

Remark 3.7. *Corollary 3.6 extends the well-known Banach fixed point theorem [9] from complete metric space to the setting of complete partial metric space.*

Corollary 3.8. *Let $(\mathcal{X}, p)$ be a complete partial metric space and let $S\colon \mathcal{X} \to \mathcal{X}$ be a self-mapping of $\mathcal{X}$ satisfying the condition:*

$$p(S^n x, S^n y) \leq r\, p(x, y),$$

for all $x, y \in \mathcal{X}$, where n is some positive integer and $r \in [0, 1)$ is a constant. Then S has a unique fixed point $z \in \mathcal{X}$ with $p(z, z) = 0$.

Proof. By Corollary 3.6, there exists $u \in \mathcal{X}$ such that $S^n u = u$. Then

$$\begin{aligned} p(Su, u) &= p(SS^n u, S^n u) \\ &= p(S^n Su, S^n u) \\ &\leq r\, p(Su, u), \end{aligned}$$

which is a contradiction, since $0 \leq r < 1$ and so $p(Su, u) = 0$, that is, $Su = u$. This shows that S has a unique fixed point in $\mathcal{X}$. This completes the proof. □

Remark 3.9. *Corollary 3.6 is a special case of Corollary 3.8 for $n = 1$.*

Now, we give some examples to demonstrate the validity of Theorem 3.1 and Corollary 3.6.

Example 3.10. *Let $\mathcal{X} = [3, 15]$. We define the function $p\colon \mathrm{X}^2 \to [0, +\infty)$ by $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathcal{X}$. Then $(\mathcal{X}, p)$ is a partial metric space.*
Define four self-maps $A, B, S, T\colon \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$ by

$$A(x) = \begin{cases} 3, & \text{if } x \in \{3\} \cup (5, 15], \\ 5, & \text{if } x \in (3, 5], \end{cases}$$

$$B(x) = \begin{cases} 3, & \textit{if } x \in \{3\} \cup (5, 15], \\ 4, & \textit{if } x \in (3, 5], \end{cases}$$

$$S(x) = \begin{cases} 3, & \textit{if } x = 3, \\ 10, & \textit{if } x \in (3, 5], \\ \frac{x+1}{2}, & \textit{if } x \in (5, 15], \end{cases}$$

$$T(x) = \begin{cases} 3, & \textit{if } x = 3, \\ 5 - x, & \textit{if } x \in (3, 5], \\ \frac{x+1}{2}, & \textit{if } x \in (5, 15], \end{cases}$$

(1) Now, we take the sequences $\{x_n\} = \{5 + \frac{1}{n}\}$ *and* $\{y_n\} = \{3\}$*. Now, we observe that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = 3 \in \mathcal{X}.$$

Thus the pairs (A, T) *and* (B, S) *satisfy common* $(E.A)$*-property.*

Again, we observe that the pairs of mappings (A, T) *and* (B, S) *commute at* 3 *which is the coincidence point. Also,*

$$A(X) = \{3, 5\} \subseteq [3, 8] \cup \{10\} = S(X) \ \textit{ and } \ B(X) = \{3, 4\} \subseteq [3, 8] \cup (8, 10] = T(X).$$

Now, we can verify the contractive condition (3) of Theorem 3.1 for the case $x, y \in [3, 5]$*, by a simple calculation we see that*

$$p(Tx, Sy) = 10, \ p(Ax, By) = 5, \ p(Ax, Tx) = 10, \ p(Sy, By) = 10, p(By, Tx) = 10,$$

$$\frac{p(Sy, By)[1 + p(By, Tx)]}{[1 + p(Ax, By)]} = \frac{10[1 + 10]}{[1 + 5]} = \frac{55}{3}.$$

Now using the inequality (3), which yields

$$10 \leq \alpha\Big(5, 10, 10, 10, \frac{55}{3}\Big),$$

where $\alpha(x, y, z, t, w) = \max\{x, y, z, t, w\}$ *and* $\phi(t) = \frac{2t}{3}$*. Thus, we see that*

$$10 \leq \phi\Big(\frac{55}{3}\Big) = \frac{110}{9} \ \textit{ or } \ 90 \leq 110,$$

which is true. Similarly, we can verify for other cases. Thus all the conditions of Theorem 3.1 are satisfied and 3 *is the unique common fixed point of the mappings A, B, S and T.*

(2) Now using inequality of Corollary 3.6, if we take $x = 3$ *and* $y = 7$*, then we see that* $S(3) = 3$ *and* $S(7) = 4$*. Now, we have*

$$p(Sx, Sy) = \max\{3, 4\} = 4 \quad \textit{and} \quad p(x, y) = \max\{3, 7\} = 7.$$

Consequently, we have

$$p(Sx, Sy) = 4 \leq q\, p(x, y) = 7\, q,$$

or,

$$4 \leq 7q,$$

or,

$$q \geq \frac{4}{7}.$$

If we take $0 \leq q < 1$, then all the conditions of Corollary 3.6 are satisfied and 3 is the unique fixed point of S. Hence we conclude that

$$p(Sx, Sy) \leq q\, p(x, y).$$

Example 3.11. *Let $\mathcal{X} = \{1, 2, 3, 4\}$ and $p\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be defined by*

$$p(x, y) = \begin{cases} |x - y| + \max\{x, y\}, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1, \end{cases}$$

for all $x, y \in \mathcal{X}$. Then $(\mathcal{X}, p)$ is a complete partial metric space.

Define the mapping $S\colon \mathcal{X} \to \mathcal{X}$ by

$$S(1) = 1,\ S(2) = 1,\ S(3) = 2,\ S(4) = 2.$$

Now, we have

$$p(S(1), S(2)) = p(1, 1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4} p(1, 2),$$

$$p(S(1), S(3)) = p(1, 2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4} p(1, 3),$$

$$p(S(1), S(4)) = p(1, 2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4} p(1, 4),$$

$$p(S(2), S(3)) = p(1, 2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4} p(2, 3),$$

$$p(S(2), S(4)) = p(1, 2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4} p(2, 4),$$

$$p(S(3), S(4)) = p(2, 2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4} p(3, 4).$$

Thus, S satisfies all the conditions of Corollary 3.6 with $q = \frac{3}{4} < 1$. Now by applying Corollary 3.6, S has a unique fixed point. Indeed 1 is the required unique fixed point in this case.

4. Conclusion

In this paper, we prove some common fixed point theorems in the setting partial metric spaces using common $(E.A)$-property and an implicit relation. We give some consequences of the main result as corollaries. We also give some examples to demonstrate the validity of the results. The results presented in this paper extend, generalize and enrich several results from the existing literature regarding partial metric spaces.

5. Acknowledgement

The author would like to thanks the anonymous referee and the Editor for their careful reading and kind comments to improve this paper.

References

[1] M. Aamri and D. El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., (2002), 181-188.
[2] M. Abbas, and B. E. Rhoades, *Common fixed point results for noncommuting mappings without continuity in generalized metric spaces*, Appl. Math. Comput., 215(1) (2009), 262-269.
[3] M. Abbas, T. Nazir and S. Ramaguera, *Fixed point results for generalized cyclic contraction mappings in partial metric spaces*, Rev. R. Acad. Cienc. Exactas. Fis. Nat. Ser. A Mat., RACSAM, 106(1) (2012), 287-297.
[4] T. Abdeljawad, E. Karapinar and K. Tas, *Existence and uniqueness of a common fixed point on partial metric spaces*, Appl. Math. Lett., 24 (2011), 1900-1904.
[5] O. Acar, V. Berinde and I. Altun, *Fixed point theorems for Ciric-type strong almost contractions on partial metric spaces*, Fixed Point Theory Appl., 12 (2012), 247-259.
[6] J. Ali and M. Imdad, *An implicit function implies several contraction conditions*, Sara. J. of Math., 4(17) (2008), 269-285.
[7] A. Aliouche and V. Popa, *General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications*, Novi Sad J. Math., 39(1) (2009), 89-109.
[8] H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology and Its Appl., 159 (2012), No. 14, 3234-3242.
[9] S. Banach, *Sur les operation dans les ensembles abstraits et leur application aux equation integrals*, Fund. Math., 3 (1922), 133-181.
[10] V. Berinde, *Approximating fixed points of implicit almost contractions*, Hacettepe J. Math. Stat., 41(1) (2012), 93-102.
[11] V. Berinde and F. Vetro, *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory Appl., (2012), 2012:105.
[12] S. Chauhan, S. Dalal, W. Sintunavarat and J. Vujakovic, *Common property* (*E.A*) *and existence of fixed points in Menger spaces*, J. Inequ. Appl., 2014, 2014:56.
[13] F. Gu and Y. Yin, *Common fixed point for three pairs of self maps satisfying common* (*E.A*)*-property in generalized metric spaces*, Abst. Appl. Anal., Volume 2013, specil issue (2013), 11 pages, Article ID 808092.
[14] M. Imdad, S. Kumar and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Radovi Math., 1 (2002), 135-143.
[15] G. Jungck, *Commuting maps and fixed points*, Am. Math. Monthly, 83 (1976), 261-263.
[16] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., 9 (1986), 771-779.
[17] G. Jungck, *Common fixed points for noncontinuous, nonself maps on nonnumetric spaces*, Far East J. Math. Sci., 4(2) (1996), 195-215.
[18] E. Karapinar, *Generalization of Caristi-Kirk's theorem on partial metric spaces*, Fixed Point Theorem Appl., 2011(4) (2011).
[19] E. Karapinar and U. Yüksel, *Some common fixed point theorems in partial metric space*, J. Appl. Math., 2011, Article ID: 263621, 2011.
[20] E. Karapinar, I. M. Erhan and A. Y. Ulus, *Fixed point theorem for cyclic maps on partial metric spaces*, Appl. Math. Inf. Sci., 6 (2012), 239-244.
[21] E. Karapinar, W. Shatanawi and K. Tas, *Fixed point theorems on partial metric spaces involving rational expressions*, Miskolc Math. Notes, 14 (2013), 135-142.
[22] H. P. A. Künzi, *Nonsymmetric distances and their associated topologies about the origins of basic ideas in the area of asymptotic topology*, Handbook of the History Gen. Topology (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., 3 (2001), 853-868.
[23] W. Liu, J. Wu and Z. Li, *Common fixed point of single-valued and multi-valued maps*, Int. J. Math. Sci., 19 (2005), 3045-3055.
[24] S. G. Matthews, *Partial metric topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.
[25] S. G. Matthews, *Partial metric topology*, Proceedings of the 8th summer conference on topology and its applications, Annals of the New York Academy of Sciences, 728 (1994), 183-197.
[26] H. K. Nashine, Z. Kadelburg, S. Radenovic and J. K. Kim, *Fixed point theorems under Hardy-Rogers contractive conditions on* 0*-complete ordered partial metric spaces*, Fixed Point Theory Appl., 2012 (2012), 1-15.
[27] M. Neog, Mohammed M. M. Jaradat and P. Debnath, *Common fixed point results of set valued maps for A_φ-contraction and generalized ϕ-type weak contraction*, Symmetry, 11 (2019), 894, doi:10.3390/sym11070894.
[28] S. Oltra and O. Oltra, *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste, 36(1-2) (2004), 17-26.
[29] R. P. Pant, *Common fixed point of contractive maps*, J. Math. Anal. Appl., 226 (1998), 251-258.
[30] H. K. Pathak, R. R. Lopez and R. K. Verma, *A common fixed point theorem using implicit relation and property* (*E.A*) *in metric spaces*, Filomat, 21(2) (2007), 211-234.
[31] H. K. Pathak, R. R. Lopez and R. K. Verma, *A common fixed point theorem of integral type using implicit relation*, Nonlinear Funct. Anal. Appl., 15 (2009), 1-12.
[32] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet. Stiin. Ser. Mat. Univ. Bacau, 7 (1997), 127-133.
[33] V. Popa, *On some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstr. Math., 32(1) (1999), 157-163.
[34] V. Popa, *A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation*, Filomat, 19 (2005), 45-51.
[35] V. Popa and A. -M. Patriciu, *A general Fixed point theorem for pairs of weakly compatible mappings in G-metric spaces*, J. Nonlinear Sci. Appl., 5 (2012), 151-160.
[36] A. Roldan, J. Martinez-Moreno, C. Roldan and E. Karapinar, *Multidimensional fixed point theorems in partially ordered complete partial metric spaces under* $(\psi - \varphi)$*-contractivity conditions*, Abstract and Applied Analysis, (2013), Article ID:634371.
[37] I. A. Rus, A. Petrusel and G. Petrusel, Fixed Point Theory, Cluj Univ. Press, 2008.
[38] G. S. Saluja, *Some fixed point results on S-metric spaces satisfying implicit relation*, J. Adv. Math. Stud., 12(3) (2019), 256-267.
[39] G. S. Saluja, *Fixed point results under implicit relation in S-metric spaces*, The Mathematics Student, 89, Nos. 3-4, (2020), 111-126.
[40] G. S. Saluja, *Fixed point theorems on cone S-metric spaces using implicit relation*, Cubo, A Mathematical Journal, 22(2) (2020), 273-288.
[41] G. S. Saluja, *Fixed point results for $\mathcal{F}_{(\mathcal{S},\mathcal{T})}$-contraction in S-metric spaces using implicit relation with applications*, International J. Math. Combin., 1 (2022), 17-30.

[42] G. S. Saluja, *On common fixed point and coincidence point theorems in weak partial metric spaces using an implicit relation*, Funct. Anal. Approxi. Comput., 14(1) (2022), 39-46.
[43] S. Sessa, *On a weak commutative condition in fixed point consideration*, Publ. Inst. Math. (Beograd), 32 (1982), 146-153.
[44] R. Tiwari and S. Gupta, *Some common fixed point theorems in metric spaces satisfying an implicit relation involving quadratic terms*, Funct. Anal. Approxi. Comput., 8(2) (2016), 45-51.
[45] R. Tiwari and S. Thakur, *Common fixed point theorems for pair of mappings satisfying common* $(E.A)$*-property in complete metric spaces with application*, Electronic J. Math. Anal. Appl., 9(1) (2021), 334-342.
[46] Oscar Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol., 6(2) (2005), 229-240.