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New results of the pseudospectra of linear operators in a Banach space

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Abstract. In this work, we investigate the S-pseudospectra of closed linear operators defined by non-strict inequality in Banach space. We begin the analysis by studying some of this basic properties. After that, we characterize the S-pseudospectra of closed linear operator by means the S-spectra of all perturbed operators with perturbations that have norms strictly less than ε, where $\varepsilon > 0$, in Banach spaces.

1. Introduction

Let X, Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ (resp. $\mathcal{C}(X,Y)$) the set of all bounded (resp. closed) linear operators from X into Y. For $A \in \mathcal{C}(X,Y)$, we will denote by $\mathcal{D}(A)$ the domain, $N(A)$ the null space and $R(A)$ the range of A. The nullity $\alpha(A)$ of A is defined as the dimension of $N(A)$ and the deficiency $\beta(A)$ of A is defined as the codimension of $R(A)$ in Y. If the range $R(A)$ of A is closed and both $\alpha(A)$ and $\beta(A)$ are finite, then A is called a Fredholm operator denoted by $\Phi(X,Y)$. The number $i(A) = \alpha(A) - \beta(A)$ is called the index of A. It is clear that if $A \in \Phi(X,Y)$, then $i(A)$ is finite.
If $X = Y$, then we write $\mathcal{L}(X,Y) = \mathcal{L}(X)$, $\mathcal{C}(X,Y) = \mathcal{C}(X)$ and $\Phi(X,Y) = \Phi(X)$.

Let F be a subspace of X and E be a subspace of X' (the dual space of X), then we have

$$F^{\perp} = \{x' \in X' : x'(x) = 0, \text{ for all } x \in F\}$$

and

$$E^{\top} = \{x \in X : x'(x) = 0, \text{ for all } x' \in E\}.$$

The resolvent set of closed linear operator A acting on X is define by:

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ is invertible and } (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

and the spectrum set of A is define by: $\sigma(A) = \mathbb{C} \backslash \rho(A)$.

In the last years, the spectral theory, which attracted the attention of many researchers, has witnessed an explosive development. It has numerous applications in many branches of sciences for example mathematics and physics. The goal of this theory can be described as trying to give useful information about

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linear operators. In the order to determine and localize $\sigma(\cdot)$, many mathematician propose to investigate the concept of the pseudospectra. This concept was introduced and studied by several authors. We can cite J. M. Varah (1967), H. J. Landau (1975), L. N. Trefethen (1992) and E. B. Davies (1996). There are many ways to define the pseudospectra of a closed linear operator in a Banach space (see, for example [1–4, 6, 13–15]). Among them we are interested of this

$$\Sigma_\varepsilon(A) = \sigma(A) \bigcup \left\{\lambda \in \mathbb{C} : \|(\lambda - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\},$$

where $A \in \mathcal{C}(X)$ and $\varepsilon > 0$. By convention $\|(\lambda - A)^{-1}\| = +\infty$ if, and only if, $\lambda \in \sigma(A)$(see [13]). In [10], E. Shargorodsky has proved that we have the following relationship

$$\bigcup_{\|D\| \leq \varepsilon} \sigma(A + D) \subsetneq \Sigma_\varepsilon(A).$$

This is equivalent to say that the pseudospectrum of closed linear operator is not equal to the union of the spectra of all perturbed operators with perturbations that have norms non-strictly less than ε. However, in [3], F. C. Chetelin and A. Harrabi have proved that if the resolvent norm of the closed linear operator A acting in Banach space cannot be constant on an open set of $\rho(A)$, then we can obtain

$$\Sigma_\varepsilon(A) = \overline{\bigcup_{\|D\| \leq \varepsilon} \sigma(A + D)}.$$

This paper is devoted to examine some properties of the S-pseudospectrum of closed linear operator in Banach spaces and achieve a characterization of this notion.

The contents of the paper are as follows. Section 2 contains preliminary properties that we will need to prove the main results of the other sections. In Section 3, we begin giving the definition and some proprieties of S-pseudospectrum of linear operators in the Banach space. After that, we establish a characterizaton of the S-pseudospectrum of linear operators by means of perturbation of its spectrum in a Banach space (see Theorem 3.14).

2. Preliminary results

The goal of this section consists in establishing some preliminary results which will be needed in the sequel.

Theorem 2.1. [9, Theorem 2.7] *Let X be a normed vector space and x a nonzero element of X. Then, there exist $x' \in X'$ such that $\|x'\| = 1$ and $x'(x) = \|x\|$.* ◊

Theorem 2.2. [8, Theorem 7.3.1] *Let X be a Banach space and let $A \in \mathcal{L}(X)$. If $\|A\| < 1$, then $(I - A)^{-1}$ exists as a bounded linear operator on X and*

$$(I - A)^{-1} = \sum_{n=0}^{+\infty} A^n. \qquad \diamond$$

Definition 2.3. *Let X, Y be two Banach spaces. Let $S \in \mathcal{L}(X, Y)$, for $A \in \mathcal{C}(X, Y)$ such that $A \neq S$ and $S \neq 0$, we define the S-resolvent set of A by:*

$$\rho_S(A) = \{\lambda \in \mathbb{C} : \lambda S - A \text{ has a bounded inverse}\},$$

and the S-spectrum set of A by: $\sigma_S(A) = \mathbb{C} \backslash \rho_S(A)$. ◊

The following result is developed by A. Jeribi in [7, Chapter 3].

Lemma 2.4. *Let X, Y be two Banach spaces. Let $A \in \mathcal{C}(X, Y)$ and $S \in \mathcal{L}(X, Y)$ such that $S \neq A$ and $S \neq 0$. Then, we have*

(i) $(\lambda S - A)^{-1} - (\mu S - A)^{-1} = (\mu - \lambda)(\lambda S - A)^{-1} S (\mu S - A)^{-1}$, *for all* $\lambda, \mu \in \rho_S(A)$.

(ii) *The S-resolvent set* $\rho_S(A)$ *is open.*

(iii) *The function* $\lambda \longmapsto (\lambda S - A)^{-1}$ *is holomorphic at any point of* $\rho_S(A)$. ◇

Lemma 2.5. *Let X, Y be two Banach spaces. Let $A \in \mathcal{C}(X, Y)$ and S is an invertible bounded operator from X to Y such that $S \neq A$. Then, we have*

$$\sigma_S(A) = \sigma(S^{-1}A) \bigcap \sigma(AS^{-1}). \quad \diamond$$

Proof. The proof is similarly to proof of [7, Remark 3.3.1]. □

3. Main results

We start our investigation with the following definition of the S-pseudospectra of a closed linear operators on Banach spaces.

Definition 3.1. *Let X, Y be Banach spaces, $\varepsilon > 0$ and let $A \in \mathcal{C}(X, Y)$. Let S be a non-null bounded operator acting from X into Y such that $S \neq A$. We define the S-pseudospectra and S-pseudoresolvent set of the operator A, respectively, by:*

$$\Sigma_{S,\varepsilon}(A) = \sigma_S(A) \bigcup \left\{ \lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon} \right\},$$

and

$$\rho_{S,\varepsilon}(A) = \rho_S(A) \bigcap \left\{ \lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon} \right\},$$

by convention $\|(\lambda S - A)^{-1}\| = +\infty$ *if, and only if,* $\lambda \in \sigma_S(A)$. ◇

Now, let us study some basic properties of the S-pseudospectra.

Proposition 3.2. *Let X, Y be Banach spaces, $\varepsilon > 0$, $A \in \mathcal{C}(X, Y)$ and let S be a non-null bounded operator acting from X into Y such that $S \neq A$.*

(i) $\Sigma_{S,\varepsilon}(A)$ *is not empty.*

(ii) $\Sigma_{S,\varepsilon}(A)$ *is closed.*

(iii) *If* $0 < \varepsilon_1 < \varepsilon_2$, *then* $\sigma_S(A) \subset \Sigma_{S,\varepsilon_1}(A) \subset \Sigma_{S,\varepsilon_2}(A)$.

(iv) $\displaystyle\bigcap_{\varepsilon > 0} \Sigma_{S,\varepsilon}(A) = \sigma_S(A)$. ◇

Proof. (i) We argue by contradiction. Let us assume that $\Sigma_{S,\varepsilon}(A) = \emptyset$. Then, we have $\|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon}$, for all $\lambda \in \mathbb{C}$. Let $\varphi : \mathbb{C} \longrightarrow \mathcal{L}(X)$ be defined by: $\varphi(\lambda) = (\lambda S - A)^{-1}$. By using the fact that $\|\varphi(\lambda)\| < \frac{1}{\varepsilon}$, Lemma 2.4 (iii) and Liouville's theorem, we infer that φ is a constant function. Therefore, for all $\lambda, \mu \in \mathbb{C}$ such that $\lambda \neq \mu$, we get that

$$(\lambda S - A)^{-1} - (\mu S - A)^{-1} = 0. \tag{1}$$

Let $\lambda, \mu \in \mathbb{C}$ such that $\lambda \neq \mu$, then it follows from (1) and Lemma 2.4 (i) that

$$(\mu - \lambda)(\lambda S - A)^{-1} S (\mu S - A)^{-1} = 0.$$

Using the fact that $\lambda \neq \mu$ and $S \neq 0$, we deduce that $(\lambda S - A)^{-1}$ is null, for all $\lambda \in \mathbb{C}$, which is a contradiction.

(ii) Since $\lambda \longmapsto \|(\lambda S - A)^{-1}\|$ is continuous, then $\left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\}$ is closed. Moreover, from Lemma 2.4 (ii), we get that $\sigma_S(A)$ is closed. Hence, we conclude that $\Sigma_{S,\varepsilon}(A) = \sigma_S(A) \bigcup \left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\}$ is closed.

(iii) For $\varepsilon > 0$, we have

$$\sigma_S(A) \subset \Sigma_{S,\varepsilon}(A). \tag{2}$$

Let $0 < \varepsilon_1 < \varepsilon_2$, we show that

$$\Sigma_{S,\varepsilon_1}(A) \subset \Sigma_{S,\varepsilon_2}(A).$$

Let $\lambda \in \Sigma_{S,\varepsilon_1}(A) \backslash \sigma_S(A)$. Then, we have $\lambda \in \rho_S(A)$ and $\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon_1}$. The fact that $\frac{1}{\varepsilon_2} < \frac{1}{\varepsilon_1}$ implies that $\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon_2}$. Hence, we infer that

$$\Sigma_{S,\varepsilon_1}(A) \backslash \sigma_S(A) \subset \Sigma_{S,\varepsilon_2}(A).$$

Finally, the use of (2) gives the wanted inclusion and achieves the proof of (iii).

(iv) It is clear that

$$\bigcap_{\varepsilon>0} \Sigma_{S,\varepsilon}(A) = \sigma_S(A) \bigcup \left(\bigcap_{\varepsilon>0} \left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\}\right).$$

Then, it is sufficient to show that

$$\bigcap_{\varepsilon>0} \left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\} \subset \sigma_S(A).$$

Let us assume that $\lambda \in \bigcap_{\varepsilon>0} \left\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}\right\}$. Then, for all $\varepsilon > 0$, we have $\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}$. As $\varepsilon \to 0^+$, we obtain $\|(\lambda S - A)^{-1}\| = +\infty$. This implies that $\lambda \in \sigma_S(A)$. Hence, we infer that $\sigma_S(A) = \bigcap_{\varepsilon>0} \Sigma_{S,\varepsilon}(A)$. $\square$

The following lemma give a relationship between the S-pseudospectra of A and the pseudospectra of $S^{-1}A$, for all bounded and invertible linear operator S.

Lemma 3.3. *Let X, Y be Banach spaces, $\varepsilon > 0$, $A \in \mathcal{C}(X, Y)$ and let $S \in \mathcal{L}(X, Y)$ such that $S \neq A$, $0 \in \rho(S)$ and S^{-1} commute with A. Then,*

(i) $\Sigma_{\varepsilon\|S\|^{-1}}(S^{-1}A) \subset \Sigma_{S,\varepsilon}(A)$.

(ii) $\Sigma_{S,\varepsilon}(A) \subset \Sigma_{\varepsilon\|S^{-1}\|}(S^{-1}A)$. $\diamond$

Proof. (i) Let $\lambda \notin \Sigma_{S,\varepsilon}(A)$. Then, we have $\lambda \in \rho_S(A)$ and $\|(\lambda S - A)^{-1}\| < \frac{1}{\varepsilon}$. By referring to Lemma 2.5, we have $\lambda \in \rho(S^{-1}A)$. Therefore, $(\lambda - S^{-1}A)^{-1}$ can be expressed in the form

$$(\lambda - S^{-1}A)^{-1} = (S^{-1}(\lambda\, S - A))^{-1} = S(\lambda\, S - A)^{-1}. \tag{3}$$

This implies that

$$\begin{aligned} \|(\lambda - S^{-1}A)^{-1}\| &\leq \|S\|\, \|(\lambda S - A)^{-1}\| \\ &< \frac{1}{\varepsilon\|S\|^{-1}}. \end{aligned}$$

Hence, we conclude that $\lambda \notin \Sigma_{\varepsilon\|S\|^{-1}}(S^{-1}A)$. Thus, $\Sigma_{\varepsilon\|S\|^{-1}}(S^{-1}A) \subset \Sigma_{S,\varepsilon}(A)$.

(ii) Let $\lambda \notin \Sigma_{\varepsilon\|S^{-1}\|}(S^{-1}A)$. Then, $\lambda \in \rho(S^{-1}A)$ and $\|(\lambda - S^{-1}A)^{-1}\| < \dfrac{1}{\varepsilon\|S^{-1}\|}$. Since $S^{-1}A = AS^{-1}$, then by Lemma 2.5, we obtain $\lambda \in \rho_S(A)$. Hence, we deduce from (3) that

$$\begin{aligned}\|(\lambda S - A)^{-1}\| &\leq \|S^{-1}\| \, \|(\lambda - S^{-1}A)^{-1}\| \\ &< \frac{1}{\varepsilon}.\end{aligned}$$

As a result, $\Sigma_{S,\varepsilon}(A) \subset \Sigma_{\varepsilon\|S^{-1}\|}(S^{-1}A)$ as desired. □

Proposition 3.4. *Let X, Y be Banach spaces, $\varepsilon > 0$, $A \in \mathcal{C}(X, Y)$ and let $S \in \mathcal{L}(X, Y)$ such that $S \neq A$ and $S \neq 0$. For $D \in \mathcal{L}(X, Y)$ satisfying $\|D\| \leq \varepsilon$ and $\delta < \varepsilon - \|D\|$, we have*

$$\Sigma_{S,\varepsilon-\delta-\|D\|}(A + D) \subset \Sigma_{S,\varepsilon}(A). \qquad \diamond$$

Proof. Let us assume that $\lambda \notin \Sigma_{S,\varepsilon}(A)$. Then, $\lambda \in \rho_S(A)$ and $\|(\lambda S - A)^{-1}\| < \dfrac{1}{\varepsilon}$. Therefore, we can write

$$\lambda S - A - D = (\lambda S - A)(I - (\lambda S - A)^{-1}D) \tag{4}$$

Using the fact that $\|(\lambda S - A)^{-1}D\| < 1$, together with Theorem 2.2, we get $(I - (\lambda S - A)^{-1}D)^{-1} \in \mathcal{L}(X)$ and

$$\begin{aligned}\|(I - (\lambda S - A)^{-1}D)^{-1}\| &\leq \frac{1}{1 - \|(\lambda S - A)^{-1}D\|} \\ &\leq \frac{\varepsilon}{\varepsilon - \|D\|}.\end{aligned} \tag{5}$$

Hence, it follows from (4) that $\lambda \in \rho_S(A + D)$ and

$$\|(\lambda S - A - D)^{-1}\| \leq \|(\lambda S - A)^{-1}\| \, \|(I - (\lambda S - A)^{-1}D)^{-1}\|.$$

The use of (5) makes us conclude that

$$\begin{aligned}\|(\lambda S - A - D)^{-1}\| &\leq \frac{1}{\varepsilon - \|D\|} \\ &< \frac{1}{\varepsilon - \delta - \|D\|}.\end{aligned}$$

This shows that $\Sigma_{S,\varepsilon-\delta-\|D\|}(A + D) \subset \Sigma_{S,\varepsilon}(A)$. □

Proposition 3.5. *Let X, Y be Banach spaces, $\varepsilon > 0$, $A \in \mathcal{C}(X, Y)$ and let $S \in \mathcal{L}(X, Y)$ such that $S \neq A$, $0 \in \rho(S)$ and S^{-1} commute with A. For any $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}\backslash\{0\}$, we have*

$$\Sigma_{S,\varepsilon\|S^{-1}\|^{-1}}(\alpha S + \beta A) \subset \alpha + \beta \, \Sigma_{\varepsilon\,|\beta|^{-1}}(S^{-1}A) \subset \Sigma_{S,\varepsilon\|S\|}(\alpha S + \beta A). \qquad \diamond$$

Proof. In view of Lemma 3.3, we have

$$\Sigma_{S,\varepsilon\|S^{-1}\|^{-1}}(\alpha S + \beta A) \subset \Sigma_\varepsilon(S^{-1}(\alpha S + \beta A)) \subset \Sigma_{S,\varepsilon\|S\|}(\alpha S + \beta A). \tag{6}$$

Then, it is sufficient to show that

$$\Sigma_\varepsilon(S^{-1}(\alpha S + \beta A)) = \alpha + \beta \, \Sigma_{\varepsilon\,|\beta|^{-1}}(S^{-1}A). \tag{7}$$

For $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$, we have

$$\Sigma_\varepsilon(S^{-1}(\alpha S + \beta A)) = \Sigma_\varepsilon(\alpha + \beta S^{-1}A). \tag{8}$$

Now, we have to prove that

$$\Sigma_\varepsilon(\alpha + \beta A) = \alpha + \beta \, \Sigma_{\varepsilon\,|\beta|^{-1}}(A). \tag{9}$$

For $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$, we can write

$$\beta^{-1}(\lambda - \alpha) - A = \beta^{-1}(\lambda - \alpha - \beta A). \tag{10}$$

Let us assume that $\lambda \notin \Sigma_\varepsilon(\alpha + \beta A)$. Then, we have $\lambda \in \rho(\alpha + \beta A)$ and $\|(\lambda - \alpha - \beta A)^{-1}\| < \frac{1}{\varepsilon}$. Hence, it follows from (10) that $\beta^{-1}(\lambda - \alpha) \in \rho(A)$ and $\|(\beta^{-1}(\lambda - \alpha) - A)^{-1}\| < \frac{1}{\varepsilon\,|\beta|}$. This is equivalent to saying that

$$\beta^{-1}(\lambda - \alpha) \notin \Sigma_{\varepsilon\ |\beta|^{-1}}(A).$$

Hence, we conclude that $\lambda \notin \alpha + \beta\ \Sigma_{\varepsilon/|\beta|^{-1}}(A)$. Conversely, a same reasoning as before leads to the result. Finally, it follows from (8) and (9) that (7) holds. □

Now, our goal is to give a characterization of S-pseudospectra of closed linear operators.

Theorem 3.6. *Let X, Y be Banach spaces, $\varepsilon > 0$ and let $A \in \mathcal{C}(X, Y)$. Let S be a non-null bounded operator acting from X into Y such that $S \neq A$. Then, the following assertions are equivalent:*

(*i*) $\Sigma_{S,\varepsilon}(A)\backslash\sigma_S(A)$.

(*ii*) $\left\{\lambda \in \mathbb{C} : \exists (x_n) \subset \mathcal{D}(A),\ \|x_n\| = 1 \text{ and } \lim\limits_{n\to+\infty} \|(\lambda S - A)x_n\| \leq \varepsilon\right\}\backslash\sigma_S(A)$. ◊

Proof. (*i*) ⇒ (*ii*): Let us assume that $\lambda \in \Sigma_{S,\varepsilon}(A)\backslash\sigma_S(A)$. Then, we get $\lambda \in \rho_S(A)$ and $\|(\lambda S - A)^{-1}\| \geq \frac{1}{\varepsilon}$. The fact that

$$\|(\lambda S - A)^{-1}\| = \sup_{\|y\|=1} \|(\lambda S - A)^{-1}y\|,$$

implies that for every $n \in \mathbb{N}\backslash\{0\}$, there exists (y_n) such that $\|y_n\| = 1$ and

$$\|(\lambda S - A)^{-1}\| - \frac{1}{n} \leq \|(\lambda S - A)^{-1}y_n\| \leq \|(\lambda S - A)^{-1}\|.$$

Hence, we infer that

$$\begin{aligned} \lim_{n\to+\infty} \|(\lambda S - A)^{-1}y_n\| &= \|(\lambda S - A)^{-1}\| \\ &\geq \frac{1}{\varepsilon}. \end{aligned} \tag{11}$$

Putting $x_n = \|(\lambda S - A)^{-1}y_n\|^{-1}(\lambda S - A)^{-1}y_n$. Then, $x_n \in \mathcal{D}(A)$, $\|x_n\| = 1$ and

$$\|(\lambda S - A)x_n\| = \|(\lambda S - A)^{-1}y_n\|^{-1}. \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned} \lim_{n\to+\infty} \|(\lambda S - A)x_n\| &= \lim_{n\to+\infty} \|(\lambda S - A)^{-1}y_n\|^{-1} \\ &= \left(\lim_{n\to+\infty} \|(\lambda S - A)^{-1}y_n\|\right)^{-1} \\ &\leq \varepsilon. \end{aligned}$$

(*ii*) ⇒ (*i*): Let us assume that $\lambda \notin \sigma_S(A)$ and there exists $(x_n) \subset \mathcal{D}(A)$, $\|x_n\| = 1$ and $\lim\limits_{n\to+\infty} \|(\lambda S - A)x_n\| \leq \varepsilon$. Putting $y_n = \|(\lambda S - A)x_n\|^{-1}(\lambda S - A)x_n$. Then, we have $\|y_n\| = 1$ and

$$\|(\lambda S - A)^{-1}y_n\| = \|(\lambda S - A)x_n\|^{-1}. \tag{13}$$

The fact that $|(\lambda S - A)^{-1}y_n\| \leq \|(\lambda S - A)^{-1}\|$, for $\|y_n\| = 1$, implies from (13) that

$$\begin{aligned}\lim_{n\to+\infty} \|(\lambda S - A)^{-1}\| &\geq \lim_{n\to+\infty} \|(\lambda S - A)x_n\|^{-1}\\ &\geq \frac{1}{\varepsilon}.\end{aligned}$$

This completes the proof. □

As a direct consequence of Theorem 3.6, we have the following:

Corollary 3.7. *Let X be a Banach space, $\varepsilon > 0$ and let $A \in \mathcal{C}(X)$. Let S be a non-null bounded operator such that $S \neq A$. Then, the following sets are equivalents*

(*i*) $\sigma_S(A) \bigcup \Big\{\lambda \in \mathbb{C} : \|(\lambda S - A)^{-1}\| \geq \dfrac{1}{\varepsilon}\Big\}$.

(*ii*)$\sigma_S(A) \bigcup \Big\{\lambda \in \mathbb{C} : \exists (x_n) \subset \mathcal{D}(A),\ \|x_n\| = 1$ *and* $\lim\limits_{n\to+\infty} \|(\lambda S - A)x_n\| \leq \varepsilon\Big\}$. ◊

Theorem 3.8. *Let X, Y be Banach spaces, $\varepsilon > 0$ and let $A \in \mathcal{C}(X, Y)$. Let $S \in \mathcal{L}(X, Y)$ such that $S \neq 0$ and $S \neq A + D$, for all $D \in \mathcal{L}(X, Y)$ with $\|D\| \leq \varepsilon$, then*

$$\bigcup_{\|D\|\leq\varepsilon} \sigma_S(A + D) \subset \Sigma_{S,\varepsilon}(A). \qquad ◊$$

Proof. Let $\lambda \in \bigcup\limits_{\|D\|\leq\varepsilon} \sigma_S(A + D)$. Then, there exists $D \in \mathcal{L}(X, Y)$ such that $\|D\| \leq \varepsilon$ and $\lambda \in \sigma_S(A + D)$. We derive a contradiction from the assumption that $\lambda \in \rho_S(A)$ and $\|(\lambda S - A)^{-1}\| < \dfrac{1}{\varepsilon}$. For $\lambda \in \rho_S(A)$, we can write

$$\lambda S - A - D = (\lambda S - A)\Big(I - (\lambda S - A)^{-1}D\Big). \tag{14}$$

By using Theorem 2.2, we infer that $I - (\lambda S - A)^{-1}D$ is invertible, and by applying (14), we conclude that $\lambda S - A - D$ is invertible. This is equivalent to saying that $\lambda \in \rho_S(A + D)$. □

Remark 3.9. *We should notice that if X, Y are Banach spaces, $\varepsilon > 0$, $A \in \mathcal{C}(X, Y)$ and $S \in \mathcal{L}(X, Y)$ such that $S \neq 0$ and $S \neq A + D$, for all $D \in \mathcal{L}(X, Y)$ with $\|D\| \leq \varepsilon$, then we do not have*

$$\bigcup_{\|D\|\leq\varepsilon} \sigma_S(A + D) = \Sigma_{S,\varepsilon}(A).$$

In fact, it suffices to consider the following examples: ◊

Example 3.10. *Let* $l^1(\mathbb{N}) = \left\{(x_j) : x_j \in \mathbb{N} \text{ and } \sum\limits_{j=1}^{+\infty} |x_j| < \infty\right\}$ *with the standard norm defined by* $\|x\| = \sum\limits_{j=1}^{+\infty} |x_j|$, $\varepsilon_1 \in]0, 1]$ *and (ε_n) be a sequence of positive numbers monotonically decreasing to 0 . Consider the linear operator $K : l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N})$ defined by*

$$Kx = \Big((1 + 2\varepsilon_1)x_1 - \sum_{j=3}^{+\infty} x_j, -\varepsilon_2 x_2, \ldots, -\varepsilon_n x_n, \ldots\Big),$$

and the linear operator $S : l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N})$ defined by:

$$Sx = \Big(x_1, \frac{x_2}{\varepsilon_1}, x_3, \ldots, x_n, \ldots\Big), \text{ where } x = (x_1, \ldots, x_n, \ldots) \in l^1(\mathbb{N}).$$

Then,
(i) $S \in \mathcal{L}(l^1(\mathbb{N}))$ *and* K *is a compact operator on* $l^1(\mathbb{N})$.
(ii) $2\varepsilon_1 \in \Sigma_{S,\varepsilon_1}(K)$.
(iii) $2\varepsilon_1 \notin \bigcup\limits_{\|D\|\leq\varepsilon_1} \sigma_S(K+D)$. ◊

Proof. (i) Let $x \in l^1(\mathbb{N}) = \mathcal{D}(S) = \mathcal{D}(K)$. Then, we have

$$\begin{aligned} \|Sx\| &= \sum_{j=1}^{+\infty} |x_j| - |x_2| + \left|\frac{x_2}{\varepsilon_1}\right| \\ &= \|x\| + \left(\frac{1}{\varepsilon_1} - 1\right)|x_2| \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|Kx\| &= \left|(1+2\varepsilon_1)x_1 - \sum_{j=3}^{+\infty} x_j\right| + \sum_{j=2}^{+\infty} \varepsilon_j |x_j| \\ &\leq (1+2\varepsilon_1)\,|x_1| + \left|\sum_{j=3}^{+\infty} x_j\right| + \sum_{j=2}^{+\infty} \varepsilon_j|x_j| \\ &\leq (1+2\varepsilon_1)\,|x_1| + (1+2\varepsilon_1)\left|\sum_{j=3}^{+\infty} x_j\right| + \sum_{j=2}^{+\infty} \varepsilon_j|x_j| \\ &\leq (1+2\varepsilon_1)\|x\| + \sum_{j=2}^{+\infty} \varepsilon_j|x_j|. \end{aligned} \tag{16}$$

Using the fact that $0 < \varepsilon_1 \leq 1$, we get $\dfrac{1}{\varepsilon_1} - 1 \geq 0$. Hence, it follows from (15) that

$$\|Sx\| \leq \|x\| + \left(\frac{1}{\varepsilon_1} - 1\right)\|x\| = \frac{1}{\varepsilon_1}\|x\|.$$

Therefore, we infer that $S \in \mathcal{L}(l^1(\mathbb{N}))$.
Now, we show that K is a compact operator on $l^1(\mathbb{N})$. Using the fact that (ε_n) is a sequence monotonically decreasing, we infer that $0 < \varepsilon_1 \leq 1$. This implies that $\varepsilon_j \leq \varepsilon_1$ for all $j \geq 1$. By referring to (16), we obtain

$$\begin{aligned} \|Kx\| &\leq (1+2\varepsilon_1)\|x\| + \sum_{j=2}^{+\infty} \varepsilon_1|x_j| \\ &\leq (1+2\varepsilon_1)\|x\| + \varepsilon_1\|x\| \\ &\leq (1+3\varepsilon_1)\|x\|. \end{aligned}$$

This implies that $K \in \mathcal{L}(l^1(\mathbb{N}))$.
Let the finite rank operator K_n defined by

$$K_n x = \left((1+2\varepsilon_1)x_1 - \sum_{j=3}^{+\infty} x_j, -\varepsilon_2 x_2, \ldots, -\varepsilon_n x_n, 0, \ldots\right), \text{ for } x \in l^1(\mathbb{N}).$$

Since

$$\begin{aligned}\lim_{n\to\infty}\|K_nx-Kx\| &= \lim_{n\to\infty}\sum_{j=n+1}^{+\infty}\varepsilon_j|x_j|\\ &\le \varepsilon_1\lim_{n\to\infty}\sum_{j=n+1}^{+\infty}|x_j|\\ &\le 0,\end{aligned}$$

then we infer that K is a compact operator.
(ii) We have

$$(K-2\varepsilon_1S)x=\Big(x_1-\sum_{j=2}^{+\infty}x_j,-(2+\varepsilon_2)x_2,\dots,-(2\varepsilon_1+\varepsilon_n)x_n,\dots\Big)$$

and

$$\Big\|(K-2\varepsilon S)x\Big\|=\Big|x_1-\sum_{j=3}^{+\infty}x_j\Big|+2|x_2|+2\varepsilon_1\sum_{j=3}^{+\infty}|x_j|+\sum_{j=2}^{+\infty}\varepsilon_j|x_j|.$$

Let

$$\psi(x)=\Big|x_1-\sum_{j=3}^{+\infty}x_j\Big|+2|x_2|+2\varepsilon_1\sum_{j=3}^{+\infty}|x_j|.$$

Assume $\|x\|=1$, so $\psi(x)=\Big|x_1-\sum_{j=3}^{+\infty}x_j\Big|+2|x_2|+2\varepsilon_1\Big(1-|x_1|-|x_2|\Big)$.
If $|x_1|<\frac{1}{2}$, then we have

$$\begin{aligned}\psi(x) &\ge 2|x_2|+2\varepsilon_1\Big(1-|x_1|-|x_2|\Big)\\ &\ge 2\varepsilon_1|x_2|+2\varepsilon_1\Big(1-|x_1|-|x_2|\Big),\ \ (\text{as } \varepsilon_1\le 1)\\ &\ge 2\varepsilon_1\Big(1-|x_1|\Big)\\ &\ge \varepsilon_1.\end{aligned}$$

If $|x_1|\ge\frac{1}{2}$, then we have

$$\begin{aligned}\psi(x) &\ge |x_1|-\Big|\sum_{j=3}^{+\infty}x_j\Big|+2|x_2|+2\varepsilon_1\Big(1-|x_1|-|x_2|\Big)\\ &\ge |x_1|-\Big(1-|x_1|-|x_2|\Big)+2|x_2|+2\varepsilon_1\Big(1-|x_1|-|x_2|\Big)\\ &\ge |x_1|+(2\varepsilon_1-1)\Big(1-|x_1|-\Big)+(-2\varepsilon_1+3)|x_2|\\ &\ge (2\varepsilon_1-1)+2(1-\varepsilon_1)|x_1|\ \ (\text{as } 0<\varepsilon_1\le 1)\\ &\ge \varepsilon_1.\end{aligned}$$

Consequently,

$$\psi(x)\ge\varepsilon_1,\ \|x\|=1,$$

where equality is reached if, and only if, $x_2 = 0$, $|x_1| = \dfrac{1}{2}$ and $\displaystyle\sum_{j=3}^{+\infty} x_j = x_1$. Hence,

$$\begin{aligned} \left\|(K - 2\varepsilon_1 S)x\right\| &= \left|x_1 - \sum_{j=3}^{+\infty} x_j\right| + |(-\varepsilon_2 - 2)x_2| + \sum_{j=3}^{+\infty} \left|(-\varepsilon_j - 2\varepsilon_1)x_j\right| \\ &= \left|x_1 - \sum_{j=3}^{+\infty} x_j\right| + 2\,|x_2| + 2\varepsilon_1 \sum_{j=3}^{+\infty} \left|x_j\right| + \sum_{j=3}^{+\infty} \varepsilon_j \left|x_j\right| + \varepsilon_2\,|x_2| \\ &= \psi(x) + \sum_{j=2}^{+\infty} \varepsilon_j |x_j| > \varepsilon_1,\ \|x\| = 1 \end{aligned} \tag{17}$$

and $\left\|(K - 2\varepsilon_1 S)x^{(k)}\right\| \to \varepsilon_1$ as $k \to +\infty$, where $x^{(k)} = \Big(\dfrac{1}{2}, \underbrace{0, \ldots, 0}_{k \text{ zeros}}, \dfrac{1}{2}, 0, \ldots\Big)$, $k \in \mathbb{N}$. Thus, we obtain

$$\inf_{\|x\|=1} \left\|(K - 2\varepsilon_1 S)x\right\| = \varepsilon_1. \tag{18}$$

Now, we can see that the operator $K - 2\varepsilon_1 S$ is invertible and $(K - 2\varepsilon_1 S)^{-1} : l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N})$ is defined by

$$(K - 2\varepsilon_1 S)^{-1} y = \Big(y_1 - \sum_{j=3}^{+\infty} (2\varepsilon_1 + \varepsilon_j)^{-1} y_j, -(2 + \varepsilon_2)^{-1} y_2, -(2\varepsilon_1 + \varepsilon_3)^{-1} y_3, \ldots, -(2\varepsilon_1 + \varepsilon_n)^{-1} y_n, \ldots\Big).$$

It follows from (18) that $\left\|(K - 2\varepsilon_1 S)^{-1}\right\| = \dfrac{1}{\varepsilon_1}$. This means that

$$2\varepsilon_1 \in \Sigma_{S,\varepsilon_1}(K).$$

(*iii*) Let $D \in \mathcal{L}(l^1(\mathbb{N}))$ such that $\|D\| \le \varepsilon_1$. It is clear that S is invertible, with its inverse $S^{-1} : l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N})$ is defined by:

$$x \mapsto \Big(x_1, \varepsilon_1 x_2, x_3, \ldots, x_n, \ldots\Big) \text{ where } x = (x_1, \ldots, x_n, \ldots) \in l^1(\mathbb{N}).$$

Moreover, we have

$$\begin{aligned} \|S^{-1}x\| &= |x_1| + \varepsilon_1 |x_2| + \sum_{j=3}^{+\infty} |x_j|,\ x \in\in l^1(\mathbb{N}) \\ &\le \sum_{j=3}^{+\infty} |x_j| = \|x\|,\ \text{(as } \varepsilon_1 \le 1). \end{aligned}$$

Consequently, $\|S^{-1}\| \le 1$. Since S is invertible and $\varepsilon_1 \neq 0$, then we can write

$$D - 2\varepsilon_1 S = 2\varepsilon_1 S\Big(\frac{1}{2\varepsilon_1} S^{-1} D - I\Big), \tag{19}$$

and

$$\begin{aligned} \left\|\frac{1}{2\varepsilon_1} S^{-1} D\right\| &\le \frac{1}{2}\, \varepsilon_1^{-1} \|S^{-1}\| \|D\| \\ &\le \frac{1}{2}. \end{aligned}$$

By referring to Theorem 2.2, we infer that $(2\varepsilon_1)^{-1}S^{-1}D - I$ is invertible operator. Hence, it follows from (19) that $D - 2\varepsilon_1 S$ is invertible operator. The use of [6, Theorems 2.2.17 and 2.2.44] makes us conclude that $K + D - 2\varepsilon_1 S \in \Phi(l^1(\mathbb{N}))$ and

$$i(K + D - 2\varepsilon_1 S) = 0. \tag{20}$$

Suppose that $N(K + D - 2\varepsilon_1 S) \neq 0$. Then, there exists $x \in l^1(\mathbb{N})$ such that $\|x\| = 1$ and $(K - 2\varepsilon_1 S)x = -Dx$. By referring to (17), we have

$$\|D\| \quad \geq \quad \|Dx\| = \left\|(K - 2\varepsilon_1 S)x\right\| > \varepsilon_1, \|x\| = 1.$$

This contradiction implies that $N(K - 2\varepsilon_1 S + D) = 0$. Hence, by using (20), we infer that

$$\beta(K - 2\varepsilon_1 S + D) = \alpha(K - 2\varepsilon_1 S + D) = 0.$$

This leads to $K + D - 2\varepsilon_1 S$ is invertible. Thus, we deduce that $2\varepsilon_1 \in \rho_S(K + D)$. This is equivalent to say that

$$2\varepsilon_1 \notin \bigcup_{\|D\|<\varepsilon_1} \sigma_S(K + D).$$

□

Remark 3.11. (*i*) *The above Example is a generalization of* [10, Theorem 1.1].

(*ii*) *In* [3], *F. Chaitin-Chetelin and A. Harrabi have proved that*

$$\Sigma_\varepsilon(A) = \overline{\bigcup_{\|D\|\leq\varepsilon} \sigma(A + D)},$$

if the resolvent norm of the closed linear operator A acting in Banach space cannot be constant on an open set of $\rho(A)$. Now, the goal is to extend this result for the S-pseudospectra. ◇

The aim of the following lemma consists in studying the S-resolvent of a closed linear operator in Banach space can have constant norm.

Lemma 3.12. *Let X, Y be Banach spaces, $A \in \mathcal{C}(X, Y)$, $S \in \mathcal{L}(X, Y)$ such that $S \neq A$, $0 \in \rho(S)$ and $\lambda \in \rho_S(A)$. Suppose there $(\lambda S - A)^{-1}$ attains its norm, that is, there is a vector $x \in Y$ of norm one such that*

$$\|(\lambda S - A)^{-1}\| = \|(\lambda S - A)^{-1}x\|. \tag{21}$$

Then, any neighborhood of λ contains a point μ such that

$$\|(\mu S - A)^{-1}\| > \|(\lambda S - A)^{-1}\|.$$

◇

Proof. Since $\lambda \in \rho_S(A)$, then we have $(\lambda S - A)^{-1} \neq 0$. This implies from (21) that

$$(\lambda S - A)^{-1}x \neq 0, \ \ \|x\| = 1.$$

It follows from Theorem 2.1 that there exists $x' \in X'$ such that $\|x'\| = 1$ and

$$\begin{aligned} x'((\lambda S - A)^{-1}x) &= \|(\lambda S - A)^{-1}x\| \\ &= \|(\lambda S - A)^{-1}\|, \ \text{(by (21))}. \end{aligned} \tag{22}$$

Consider the function $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ defined by: $\psi(\lambda) = x'((\lambda S - A)^{-1}x)$. We divide this part of the proof into two steps.

Step 1. We show that ψ is a holomorphic function.
Let $\lambda, \mu \in \mathbb{C}$, then

$$\psi(\lambda) - \psi(\mu) = x'\left[\left((\lambda S - A)^{-1} - (\mu S - A)^{-1}\right)x\right]. \tag{23}$$

Hence, it follows from (23) and Lemma 2.4 (i) that

$$\begin{aligned}\lim_{\lambda\to\mu}\frac{\psi(\lambda)-\psi(\mu)}{\lambda-\mu} &= \lim_{\lambda\to\mu}-(\lambda-\mu)\frac{x'\left[\left((\lambda S-A)^{-1}S(\mu S-A)^{-1}\right)x\right]}{\lambda-\mu}\\ &= -x'\left[\left((\mu S-A)^{-1}S(\mu S-A)^{-1}\right)x\right].\end{aligned}$$

Step 2. We show that ψ is a not constant function.
Assume that ψ is constant function. Let $\lambda,\mu\in\mathbb{C}$ such that $\lambda\neq\mu$ and $\psi(\lambda)=\psi(\mu)$. Then, by using (23) and Lemma 2.4 (i), we infer that

$$x'\left[(\lambda S-A)^{-1}S(\mu S-A)^{-1}x\right]=0.$$

Therefore, $x'\in X^{\perp}=\{0\}$. This means that x' is null operator. This contradicts the fact that $\|x'\|=1$. Hence, we conclude that ψ is a not constant function. Thus, ψ not have a maximum locally. This implies that there exists μ belongs to the

neighborhood of λ such that $|x'((\mu S-A)^{-1}x)|>|x'((\lambda S-A)^{-1}x)|$. Hence, by refereing to (22), we obtain

$$x'((\mu S-A)^{-1}x)>\|(\lambda S-A)^{-1}\|.$$

Moreover, we have

$$\begin{aligned}|x'((\mu S-A)^{-1}x)| &\le \|x'\|\|(\mu S-A)^{-1}\|\|x\|\\ &\le \|(\mu S-A)^{-1}\|.\end{aligned}$$

This proves that $\|(\mu S-A)^{-1}\|>\|(\lambda S-A)^{-1}\|$. □

Remark 3.13. *It follows from Lemma 3.12 that a closed operator whose S-resolvent norm is constant on no open set $\rho_S(A)$. In all that follows we will make the following assumption*

$$(\mathcal{H}):\left\{\begin{array}{l}\textit{The S-resolvent norm of a closed operator acting}\\ \textit{in Banach space cannot be constant on an open set.}\end{array}\right.$$

◊

The following theorem gives a characterization of the S-pseudospectrum of closed operators on Banach spaces satisfying the hypothesis $(\mathcal{H})$ by means of its S-spectrum.

Theorem 3.14. *Let X, Y be Banach spaces, $\varepsilon>0$ and let $A\in\mathcal{C}(X,Y)$. Let $S\in\mathcal{L}(X,Y)$ such that $S\neq A+D$ for all $D\in\mathcal{L}(X,Y)$ with $\|D\|\le\varepsilon$ and $0\in\rho(S)$. We assume that A satisfies the hypothesis $(\mathcal{H})$, then the following sets are equivalents:*

(i) $\overline{\bigcup_{\|D\|\le\varepsilon}\sigma_S(A+D)}$.

(ii) $\sigma_S(A)\bigcup\left\{\lambda\in\mathbb{C}:\|(\lambda S-A)^{-1}\|\ge\frac{1}{\varepsilon}\right\}$.

(iii) $\sigma_S(A)\bigcup\left\{\lambda\in\mathbb{C}:\exists\, x\in\mathcal{D}(A),\|x\|=1 \text{ and } \|(\lambda S-A)x\|\le\varepsilon\right\}$. ◊

Proof. (i) ⇒ (ii): From Theorem 3.8, we have $\bigcup_{\|D\|\le\varepsilon}\sigma_S(A+D)\subset\Sigma_{S,\varepsilon}(A)$, and by using Proposition 3.2 (ii), we infer that

$$\overline{\bigcup_{\|D\|\le\varepsilon}\sigma_S(A+D)}\subset\Sigma_{S,\varepsilon}(A).$$

$(ii) \Rightarrow (iii)$: Let us assume that $\lambda \notin \sigma_S(A)$ and $\|(\lambda S - A)^{-1}\| \geq \dfrac{1}{\varepsilon}$. Then, by using the hypothesis $(\mathcal{H})$, there exists μ belongs to the neighborhood of λ such that

$$\begin{aligned} \|(\mu S - A)^{-1}\| &> \|(\lambda S - A)^{-1}\|, \\ &> \frac{1}{\varepsilon}. \end{aligned} \tag{24}$$

This implies that there exists $y \in Y$ such that $\|y\| = 1$ and $\|(\mu S - A)^{-1} y\| > \dfrac{1}{\varepsilon}$. Putting $x = \|(\mu S - A)^{-1} y\|^{-1} (\mu S - A)^{-1} y$. Therefore, $x \in \mathcal{D}(A)$, $\|x\| = 1$ and

$$\|(\mu S - A)x\| = \|(\mu S - A)^{-1} y\|^{-1}. \tag{25}$$

Since $\|(\mu S - A)^{-1} y\| \leq \|(\mu S - A)^{-1}\|$, then by using (24) and (25), we obtain $\|(\mu S - A)x\| < \varepsilon$. Hence, we deduce that $\|(\lambda S - A)x\| \leq \varepsilon$. This enables us to conclude that

$$\Sigma_{S,\varepsilon}(A) \backslash \sigma_S(A) \subset \Big\{\lambda \in \mathbb{C} : \exists\, x \in \mathcal{D}(A),\ \|x\| = 1 \text{ and } \|(\lambda S - A)x\| \leq \varepsilon\Big\}.$$

This shows that

$$\Sigma_{S,\varepsilon}(A) \subset \sigma_S(A) \bigcup \Big\{\lambda \in \mathbb{C} : \exists\, x \in \mathcal{D}(A), \|x\| = 1 \text{ and } \|(\lambda S - A)x\| \leq \varepsilon\Big\}.$$

$(iii) \Rightarrow (i)$: Suppose that $\lambda \in \mathbb{C}$ such that there exists $x_0 \in \mathcal{D}(A)$, $\|x_0\| = 1$ and $\|(\lambda S - A)x_0\| \leq \varepsilon$. It follows from Theorem 2.1 that there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'(x_0) = 1$. We consider the following linear operator

$$D(x) = x'(x)\ (\lambda S - A)x_0, \ \ x \in X.$$

Let us observe that

$$\begin{aligned} \|D(x)\| &\leq \|x'\|\,\|x\|\,\|(\lambda S - A)x_0\| \\ &\leq \varepsilon \|x\|, \end{aligned}$$

then we have $\|D\| \leq \varepsilon$ and D is everywhere defined. Hence, D is a bounded operator. Moreover, we have $(\lambda S - A - D)x_0 = 0$. Therefore, $\lambda S - A - D$ is not

injective. This implies that $\lambda \in \bigcup\limits_{\|D\| \leq \varepsilon} \sigma_S(A + D)$. Thus, we conclude that

$$\lambda \in \overline{\bigcup_{\|D\| \leq \varepsilon} \sigma_S(A + D)}. \qquad \square$$

As a direct consequence of Theorem 3.14, we have the following:

Corollary 3.15. *Let X, Y be Banach spaces, $\varepsilon > 0$ and let $A \in \mathcal{C}(X, Y)$. Let $S \in \mathcal{L}(X, Y)$ such that $S \neq A + D$ for all $D \in \mathcal{L}(X, Y)$ with $\|D\| \leq \varepsilon$ and $0 \in \rho(S)$. We assume that A satisfies the hypothesis $(\mathcal{H})$, then*

$$\Sigma_{S,\varepsilon}(A) = \overline{\bigcup_{\|D\| \leq \varepsilon} \sigma_S(A + D)}. \qquad \diamond$$

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