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# Common coupled fixed point results for contractive type conditions on $S$-metric spaces

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**Abstract.** In this paper, we prove some common coupled fixed point theorems for contractive type conditions on $S$-metric spaces. We give some consequences of the main results. We also give an example to validate the result. The results of findings in this paper generalize, extend and unify several previous results in the existing literature.

## 1. Introduction

Banach contraction principle in metric spaces is one of the most important results in the theory of fixed points and non-linear analysis. From 1922, when Stefan Banach ([4]) formulated the concept of contraction and proved the famous theorem, scientist and mathematicians around the world are publishing new results that are related either to establish a generalization of metric space or to get a improvement of contractive conditions.

The famous Banach contraction principle, which states that every self mapping $\mathcal{H}$ defined on a complete metric space $(X, \rho)$ satisfying

$$\rho(\mathcal{H}(x), \mathcal{H}(y)) \leq s\,\rho(x, y), \tag{1}$$

for all $x, y \in X$, where $s \in (0, 1)$, has a unique fixed point and for every $x_0 \in X$ a sequence $\{\mathcal{H}^n x_0\}_{n\geq 1}$ is convergent to the fixed point. Inequality (1) also implies the continuity of $\mathcal{H}$.

Fixed point problem for contractive mappings in metric spaces with a partial order have been studied by many authors. Guo and Lakshmikantham [9] introduced the notion of coupled fixed point. In 2006, Bhashkar and Lakshmikantham [5] reconsidered the concept of a coupled fixed point of the mapping $F \colon X \times X \to X$ and established some coupled fixed point theorems in partially ordered complete metric spaces. Bhashkar and Lakshmikantham [5] also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. As, an application, they studied the existence and uniqueness of the solution for a periodic boundary value problem associated with first order differential equation. Several other authors such as Ciric and

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Lakshmikantham [6], Sabetghadam *et al.* [18] and Olaleru *et al.* [16] have proved some coupled fixed point theorems in metric spaces (see, also, [14], [15]).

In literature, there are many generalizations of the metric spaces and generalized metric spaces are exist. One of such generalization is an $S$-metric space given by Sedghi *et al.* [19] in 2012.

Sedghi *et al.* [19] introduced a new notion called $S$-metric space and studied its some properties and they also stated that $S$-metric space is a generalization of $G$-metric space. But Dung et al. [7] in 2014 showed by an example that an $S$-metric space is not a generalization of $G$-metric space and conversely. Consequently, the class of $S$-metric spaces and the class of $G$-metric spaces are different.

On the other hand, Jungck and Rhoades [12] introduced the concept of weak compatibility in the year 1998.

Recently, Sedghi *et al.* [21] proved some existence results of the unique common fixed point for a pair of weakly compatible self-mappings satisfying some $\Phi$-type contractive conditions in the framework of an $S$-metric spaces and give example to validate the results. The results presented in this paper extend and improve several results in the literature.

## 2. Preliminaries

In this section, we need some definitions, lemmas and auxiliary results of an $S$-metric spaces to prove the main results (see, [19]).

**Definition 2.1.** *([19]) Let $X$ be a nonempty set and let $\mathcal{S}\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x,\ y,\ z,\ a \in X$:*
*(S1) $0 < S(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z$;*
*(S2) $S(x,y,z) = 0$ if and only if $x = y = z$;*
*(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.*
*Then the function $S$ is called an $S$-metric on $X$ and the pair $(X,S)$ is called an $S$-metric space.*

**Example 2.2.** *([19])*
*(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $X$, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an $S$-metric on $X$.*
*(2)Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $X$, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an $S$-metric on $X$.*

**Example 2.3.** *([20]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an $S$-metric on $X$. This $S$-metric on $X$ is called the usual $S$-metric on $X$.*

**Example 2.4.** *([13]) Let $X$ be a non-empty set and $d$ be an ordinary metric on $X$. Then $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}$ is an $S$-metric on $X$.*

**Example 2.5.** *([22]) Let $X$ be a non-empty set and $d_1$, $d_2$ be two ordinary metrics on $X$. Then $S(x,y,z) = d_1(x,z) + d_2(y,z)$ for all $x,y,z \in X$ is an $S$-metric on $X$.*

**Example 2.6.** *([19]) Let $X = \mathbb{R}^2$ and $d$ an ordinary metric on $X$. Put $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}^2$, that is, $S$ is the perimeter of the triangle given $x,y,z$. Then $S$ is an $S$-metric on $X$.*

**Definition 2.7.** *Let $(X,S)$ be an $S$-metric space. For $r > 0$ and $x \in X$ we define the open ball $\mathcal{B}_S(x,r)$ and closed ball $\mathcal{B}_S[x,r]$ with center $x$ and radius $r$ as follows, respectively:*

$$\mathcal{B}_S(x,r) = \{y \in X : S(y,y,x) < r\},$$

$$\mathcal{B}_S[x,r] = \{y \in X : S(y,y,x) \leq r\}.$$

**Example 2.8.** *([20]) Let $X = \mathbb{R}$. Denote by $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Then*

$$\begin{aligned} \mathcal{B}_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2), \end{aligned}$$

*and*

$$\begin{aligned}\mathcal{B}_S[2,4] &= \{y \in \mathbb{R} : S(y,y,2) \leq 4\} = \{y \in \mathbb{R} : |y-2| \leq 2\} \\ &= \{y \in \mathbb{R} : 0 \leq y \leq 4\} = [0,4].\end{aligned}$$

**Definition 2.9.** *([19], [20]) Let $(X,S)$ be an S-metric space and $A \subset X$.*

(1) *The subset A is said to be an open subset of X, if for every $x \in A$ there exists $r > 0$ such that $\mathcal{B}_S(x,r) \subset A$.*

(2) *A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

(3) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$ we have $S(x_n,x_n,x_m) < \varepsilon$.*

(4) *The S-metric space $(X,S)$ is called complete if every Cauchy sequence in X is convergent in X.*

(5) *Let $\tau$ be the set of all $A \subset X$ with the property that for each $x \in A$ and there exists $r > 0$ such that $\mathcal{B}_S(x,r) \subset A$. Then $\tau$ is a topology on X (induced by the S-metric space).*

(6) *A nonempty subset A of X is S-closed if closure of A coincides with A.*

**Definition 2.10.** *([19]) Let $(X,S)$ be an S-metric space. A mapping $\mathcal{A}\colon X \to X$ is said to be a contraction if there exists a constant $0 \leq t < 1$ such that*

$$S(\mathcal{A}x,\mathcal{A}y,\mathcal{A}z) \leq t\, S(x,y,z), \tag{2}$$

*for all $x,y,z \in X$.*

**Remark 2.11.** *If the S-metric space $(X,S)$ is complete then the mapping defined as above has a unique fixed point (see [19], Theorem 3.1).*

**Definition 2.12.** *([19]) Let $(X,S)$ and $(Y,S')$ be two S-metric spaces. A function $A\colon X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every sequence $\{x_n\}$ in X with $S(x_n,x_n,x_0) \to 0$, $S'(A(x_n),A(x_n),A(x_0)) \to 0$ as $n \to \infty$. We say that A is continuous on X if A is continuous at every point $x_0 \in X$.*

**Definition 2.13.** *Let X be a non-empty set and let $A,B\colon X \to X$ be two self mappings of X. Then a point $z \in X$ is called a*

($i$) *fixed point of operator A if $A(z) = z$;*

($ii$) *common fixed point of A and B if $A(z) = B(z) = z$.*

**Definition 2.14.** *([1]) Let A and B be single valued self-mappings on a set X. If $u = Av = Bv$ for some $v \in X$, then v is called a coincidence point point of A and B, and u is called a point of coincidence of A and B.*

**Definition 2.15.** *([11]) Let A and B be single valued self-mappings on a set X. Mappings A and B are said to be commuting if $ABv = BAv$ for all $v \in X$.*

**Example 2.16.** *Let $X = [0,\frac{3}{4}]$ and define $A,B\colon X \to X$ defined by $A(x) = \frac{x^3}{4}$ and $B(x) = x^4$ for all $x,y \in X$. Then the mappings A and B have two coincidence points 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.*

**Definition 2.17.** *([3]) An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping $F\colon X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.*

**Example 2.18.** *Let $X = [0,+\infty)$ and $F\colon X \times X \to X$ defined by $F(x,y) = \frac{x+y}{3}$ for all $x,y \in X$. One can easily see that F has a unique coupled fixed point $(0,0)$.*

**Example 2.19.** *Let $X = [0,+\infty)$ and $F\colon X \times X \to X$ be defined by $F(x,y) = \frac{x+y}{2}$ for all $x,y \in X$. Then we see that F has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.*

**Definition 2.20.** *([6]) Let X be a nonempty set. Then we say that the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ are commutative if $A(F(x,y)) = F(Ax,Ay)$.*

**Definition 2.21.** *([6]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F: X \times X \to X$ and $A: X \to X$ if $F(x, y) = Ax$ and $F(y, x) = Ay$.*

**Definition 2.22.** *([2]) The mappings $F: X \times X \to X$ and $A: X \to X$ are called weakly compatible if $A(F(x, y)) = F(Ax, Ay)$ and $A(F(y, x)) = F(Ay, Ax)$ for all $x, y \in X$, whenever $A(x) = F(x, y)$ and $A(y) = F(y, x)$.*

**Lemma 2.23.** *([19], Lemma 2.5) Let $(X, S)$ be an S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

**Lemma 2.24.** *([19], Lemma 2.12) Let $(X, S)$ be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $S(x_n, x_n, y_n) \to S(x, x, y)$ as $n \to \infty$.*

**Lemma 2.25.** *([8], Lemma 8) Let $(X, S)$ be an S-metric space and A is a nonempty subset of X. Then A is said to be S-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

**Lemma 2.26.** *([19]) Let $(X, S)$ be an S-metric space. If $r > 0$ and $x \in X$, then the ball $\mathcal{B}_S(x, r)$ is an open subset of X.*

**Lemma 2.27.** *([20]) The limit of a sequence $\{x_n\}$ in an S-metric space $(X, S)$ is unique.*

**Lemma 2.28.** *([19]) Let $(X, S)$ be an S-metric space. Then any convergent sequence $\{x_n\}$ in X is Cauchy.*

In the following lemma we see the relationship between a metric and $S$-metric.

**Lemma 2.29.** *([10]) Let $(X, d)$ be a metric space. Then the following properties are satisfied:*
*(1) $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.*
*(2) $x_n \to x$ in $(X, d)$ if and only if $x_n \to x$ in $(X, S_d)$.*
*(3) $\{x_n\}$ is Cauchy in $(X, d)$ if and only if $\{x_n\}$ is Cauchy in $(X, S_d)$.*
*(4) $(X, d)$ is complete if and only if $(X, S_d)$ is complete.*

We call the function $S_d$ defined in Lemma 2.29 (1) as the $S$-metric generated by the metric $d$. It can be found an example of an $S$-metric which is not generated by any metric in [10, 17].

**Example 2.30.** *([10]) Let $X = \mathbb{R}$ and the function $S: X^3 \to [0, \infty)$ be defined as*

$$S(x, y, z) = |x - z| + |x + z - 2y|,$$

*for all $x, y, z \in \mathbb{R}$. Then the function S is an S-metric on X and $(X, S)$ is an S-metric space. Now, we prove that there does not exists any metric d such that $S = S_d$. On the contrary, suppose that there exists a metric d such that*

$$S(x, y, z) = d(x, z) + d(y, z),$$

*for all $x, y, z \in \mathbb{R}$. Hence, we obtain*

$$S(x, x, z) = 2d(x, z) = 2|x - z|,$$

*and*

$$d(x, z) = |x - z|.$$

*Similarly, we get*

$$S(y, y, z) = 2d(y, z) = 2|y - z|,$$

*and*

$$d(y, z) = |y - z|,$$

*for all $x, y, z \in \mathbb{R}$. Hence, we have*

$$|x - z| + |x + z - 2y| = |x - z| + |y - z|,$$

*which is a contradiction. Therefore, $S \neq S_d$ and $(\mathbb{R}, S)$ is a complete S-metric space.*

Recently, Aydi [3] obtained the following results in partial metric space.

**Theorem 2.31.** *Let $(X, p)$ be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies one of the following contractive conditions $(C_1)$, $(C_2)$, $(C_3)$:*

$(C_1)$ *for all $x, y, u, v \in X$ and nonnegative constants $k, l$ with $k + l < 1$,*

$$p(F(x, y), F(u, v)) \leq kp(x, u) + lp(y, v), \tag{3}$$

$(C_2)$ *for all $x, y, u, v \in X$ and nonnegative constants $k, l$ with $k + l < 1$,*

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), x) + lp(F(u, v), u), \tag{4}$$

$(C_3)$ *for all $x, y, u, v \in X$ and nonnegative constants $k, l$ with $k + 2l < 1$,*

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), u) + lp(F(u, v), x). \tag{5}$$

*Then $F$ has a unique coupled fixed point.*

Motivated by Aydi [3] and some others, the purpose of this paper is to establish some unique common coupled fixed point theorems for contractive type conditions in the setting of $S$-metric spaces and give some corollaries of the main results. We also illustrate an example to support the result. Our results generalize, extend and enrich several results from the existing literature.

## 3. Main Results

In this section, we prove some unique common coupled fixed point theorems for contractive type mappings in the setting of $S$-metric spaces.

**Theorem 3.1.** *Let $(X, S)$ be a complete $S$-metric space. Let $F\colon X \times X \to X$ and $A\colon X \to X$ be two functions such that*

$$\begin{aligned} S(F(x, y), F(u, v), F(z, w)) \quad \leq \quad & r_1\, S(Ax, Au, Az) + r_2\, S(Ay, Av, Aw) \\ & + r_3\, S(F(x, y), F(x, y), Ax) \\ & + r_4\, S(F(u, v), F(u, v), Au) \\ & + r_5\, S(F(z, w), F(z, w), Az), \end{aligned} \tag{6}$$

*for all $x, y, u, v, z, w \in X$, where $r_1, r_2, r_3, r_4, r_5 > 0$ are nonnegative reals such that $r_1 + r_2 + r_3 + r_4 + r_5 < 1$. Assume that $F$ and $A$ satisfy the following conditions:*

$(i)$ *$F(X \times X) \subseteq A(X)$,*

$(ii)$ *$A(X)$ is complete, and*

$(iii)$ *$A$ is continuous and commute with $F$.*

*Then $F$ and $A$ have a coupled coincidence point in $X$. Moreover, if $F$ and $A$ are weakly compatible, then $F$ and $A$ have a unique common coupled fixed point.*

*Proof.* Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq A(X)$, for , we can choose $x_1, y_1 \in X$ such that $Ax_1 = F(x_0, y_0)$ and $Ay_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq A(X)$, we can choose $x_2, y_2 \in X$ such that $Ax_2 = F(x_1, y_1)$ and $Ay_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$Ax_{n+1} = F(x_n, y_n)$ and $Ay_{n+1} = F(y_n, x_n)$. For $n \in \mathbb{N}$, by equation (6), and using Lemma 2.23, we have

$$\begin{aligned} S(Ax_n, Ax_n, Ax_{n+1}) &= S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\le r_1\, S(Ax_{n-1}, Ax_{n-1}, Ax_n) + r_2\, S(Ay_{n-1}, Ay_{n-1}, Ay_n) \\ &\quad + r_3\, S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), Ax_{n-1}) \\ &\quad + r_4\, S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), Ax_{n-1}) \\ &\quad + r_5\, S(F(x_n, y_n), F(x_n, y_n), Ax_n) \\ &= r_1\, S(Ax_{n-1}, Ax_{n-1}, Ax_n) + r_2\, S(Ay_{n-1}, Ay_{n-1}, Ay_n) \\ &\quad + r_3\, S(Ax_n, Ax_n, Ax_{n-1}) + r_4\, S(Ax_n, Ax_n, Ax_{n-1}) \\ &\quad + r_5\, S(Ax_{n+1}, Ax_{n+1}, Ax_n) \\ &= (r_1 + r_3 + r_4)\, S(Ax_{n-1}, Ax_{n-1}, Ax_n) \\ &\quad + r_2\, S(Ay_{n-1}, Ay_{n-1}, Ay_n) \\ &\quad + r_5\, S(Ax_n, Ax_n, Ax_{n+1}). \end{aligned} \tag{7}$$

Similarly by equation (6), we have

$$\begin{aligned} S(Ay_n, Ay_n, Ay_{n+1}) &= S(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\le (r_1 + r_3 + r_4)\, S(Ay_{n-1}, Ay_{n-1}, Ay_n) \\ &\quad + r_2\, S(Ax_{n-1}, Ax_{n-1}, Ax_n) \\ &\quad + r_5\, S(Ay_n, Ay_n, Ay_{n+1}). \end{aligned} \tag{8}$$

Set

$$\mathcal{K}_n = S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1}). \tag{9}$$

Hence, we have

$$\begin{aligned} \mathcal{K}_n &= S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1}) \\ &\le (r_1 + r_3 + r_4)\, S(Ax_{n-1}, Ax_{n-1}, Ax_n) \\ &\quad + r_2\, S(Ay_{n-1}, Ay_{n-1}, Ay_n) \\ &\quad + r_5\, S(Ax_n, Ax_n, Ax_{n+1}) \\ &\quad + (r_1 + r_3 + r_4)\, S(Ay_{n-1}, Ay_{n-1}, Ay_n) \\ &\quad + r_2\, S(Ax_{n-1}, Ax_{n-1}, Ax_n) \\ &\quad + r_5\, S(Ay_n, Ay_n, Ay_{n+1}) \\ &= (r_1 + r_2 + r_3 + r_4)\, [S(Ax_{n-1}, Ax_{n-1}, Ax_n) \\ &\quad + S(Ay_{n-1}, Ay_{n-1}, Ay_n)] + r_5\, [S(Ax_n, Ax_n, Ax_{n+1}) \\ &\quad + S(Ay_n, Ay_n, Ay_{n+1})] \\ &= (r_1 + r_2 + r_3 + r_4)\, \mathcal{K}_{n-1} + r_5\, \mathcal{K}_n. \end{aligned} \tag{10}$$

This implies that

$$\mathcal{K}_n \le \Big(\frac{r_1 + r_2 + r_3 + r_4}{1 - r_5}\Big)\mathcal{K}_{n-1} = q\, \mathcal{K}_{n-1}, \tag{11}$$

where $q = \left(\frac{r_1+r_2+r_3+r_4}{1-r_5}\right) < 1$, since $r_1 + r_2 + r_3 + r_4 + r_5 < 1$.

Consequently, for each $n \in \mathbb{N}$, we obtain

$$\mathcal{K}_n \le q\, \mathcal{K}_{n-1} \le q^2\, \mathcal{K}_{n-2} \le \cdots \le q^n\, \mathcal{K}_0. \tag{12}$$

If $\mathcal{K}_0 = 0$, then $S(Ax_0, Ax_0, Ax_1) + S(Ay_0, Ay_0, Ay_1) = 0$. Hence, by condition $(S2)$, we get $Ax_0 = Ax_1 = F(x_0, y_0)$ and $Ay_0 = Ay_1 = F(y_0, x_0)$, means that $(Ax_0, Ay_0)$ is a coupled fixed point of $F$ and $A$. Now, we assume that $\mathcal{K}_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using $(S3)$, we have

$$\begin{aligned}
&S(Ax_n, Ax_n, Ax_m) + S(Ay_n, Ay_n, Ay_m) \\
&\quad \leq \; 2S(Ax_n, Ax_n, Ax_{n+1}) + S(Ax_m, Ax_m, Ax_{n+1}) \\
&\quad\quad +2S(Ay_n, Ay_n, Ay_{n+1}) + S(Ay_m, Ay_m, Ay_{n+1}) \\
&\quad = \; 2(S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1})) \\
&\quad\quad +S(Ax_m, Ax_m, Ax_{n+1}) + S(Ay_m, Ay_m, Ay_{n+1}) \\
&\quad \leq \; \dots \\
&\quad \leq \; 2\Big(\mathcal{K}_n + \mathcal{K}_{n+1} + \dots + \mathcal{K}_{m-1} + \mathcal{K}_m\Big) \\
&\quad \leq \; 2(q^n + q^{n+1} + \dots + q^{m-1} + q^m)\mathcal{K}_0 \\
&\quad \leq \; 2q^n(1 + q + q^2 + \dots)\mathcal{K}_0 \\
&\quad \leq \; \Big(\frac{2q^n}{1-q}\Big)\mathcal{K}_0 \\
&\quad \to \; 0 \text{ as } n \to \infty,
\end{aligned}$$

since $0 < q < 1$. Thus, $\{Ax_n\}$ and $\{Ay_n\}$ are $S$-Cauchy sequence in $A(X)$. Since $A(X)$ is complete, we get $\{Ax_n\}$ and $\{Ay_n\}$ are $S$-convergent to some $x \in X$ and $y \in X$ respectively. Since $A$ is continuous, we have $\{AAx_n\}$ is $S$-convergent to $Ax$ and $\{AAy_n\}$ is $S$-convergent to $Ay$. Also, since $A$ and $F$ are commute, we have

$$AAx_{n+1} = A(F(x_n, y_n)) = F(Ax_n, Ay_n),$$

and

$$AAy_{n+1} = A(F(y_n, x_n)) = F(Ay_n, Ax_n).$$

Thus,

$$\begin{aligned}
S(AAx_{n+1}, AAx_{n+1}, F(x, y)) &= S(F(Ax_n, Ay_n), F(Ax_n, Ay_n), F(x, y)) \\
&\leq r_1\, S(AAx_n, AAx_n, Ax) + r_2\, S(AAy_n, AAy_n, Ay) \\
&\quad +r_3\, S(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n) \\
&\quad +r_4\, S(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n) \\
&\quad +r_5\, S(F(x, y), F(x, y), Ax) \\
&= r_1\, S(AAx_n, AAx_n, Ax) + r_2\, S(AAy_n, AAy_n, Ay) \\
&\quad +r_3\, S(AAx_{n+1}, AAx_{n+1}, AAx_n) \\
&\quad +r_4\, S(AAx_{n+1}, AAx_{n+1}, AAx_n) \\
&\quad +r_5\, S(F(x, y), F(x, y), Ax).
\end{aligned}$$

Passing to the limit as $n \to \infty$, using Lemma 2.23, Lemma 2.24 and the condition $(S2)$, we get that

$$\begin{aligned}
S(Ax, Ax, F(x, y)) &\leq r_5\, S(Ax, Ax, F(x, y)) \\
&\leq (r_1 + r_2 + r_3 + r_4 + r_5)S(Ax, Ax, F(x, y)) \\
&< S(Ax, Ax, F(x, y)),
\end{aligned}$$

which is a contradiction, since $r_1+r_2+r_3+r_4+r_5 < 1$. Hence, we get $S(Ax, Ax, F(x, y)) = 0$, that is, $Ax = F(x, y)$. Similarly, we may show that $Ay = F(x, y)$. Thus $(x, y)$ is a coupled coincidence point of the mappings $F$ and $A$. Since the pair $(F, A)$ is weakly compatible, so by weak compatibility of $F$ and $A$, we have

$$A(F(x, y)) = F(Ax, Ay) \text{ and } A(F(y, x)) = F(Ay, Ax).$$

Hence $(Ax, Ay)$ is a common coupled fixed point of $F$ and $A$.

Now, we show the uniqueness of the common coupled fixed point of $F$ and $A$. Assume that $(Ax_1, Ay_1)$ is another common coupled fixed point of $F$ and $A$ with $Ax \neq Ax_1$ and $Ay \neq Ay_1$, that is, $(Ax, Ay) \neq (Ax_1, Ay_1)$. Then by using equation (6), using Lemma 2.23 and the condition $(S2)$, we have

$$\begin{aligned} S(Ax, Ax, Ax_1) &= S(F(x,y), F(x,y), F(x_1,y_1)) \\ &\leq r_1\, S(Ax, Ax, Ax_1) + r_2\, S(Ay, Ay, Ay_1) \\ &\quad + r_3\, S(F(x,y), F(x,y), Ax) \\ &\quad + r_4\, S(F(x,y), F(x,y), Ax) \\ &\quad + r_5\, S(F(x_1,y_1), F(x_1,y_1), Ax_1) \\ &= r_1\, S(Ax, Ax, Ax_1) + r_2\, S(Ay, Ay, Ay_1) \\ &\quad + r_3\, S(Ax, Ax, Ax) + r_4\, S(Ax, Ax, Ax) \\ &\quad + r_5\, S(Ax_1, Ax_1, Ax_1) \\ &= r_1\, S(Ax, Ax, Ax_1) + r_2\, S(Ay, Ay, Ay_1). \end{aligned} \tag{13}$$

Similarly, we obtain

$$S(Ay, Ay, Ay_1) \leq r_1\, S(Ay, Ay, Ay_1) + r_2\, S(Ax, Ax, Ax_1). \tag{14}$$

Hence from equations (13) and (14), we get

$$\begin{aligned} S(Ax, Ax, Ax_1) + S(Ay, Ay, Ay_1) &\leq r_1\, S(Ax, Ax, Ax_1) + r_2\, S(Ay, Ay, Ay_1) \\ &\quad + r_1\, S(Ay, Ay, Ay_1) + r_2\, S(Ax, Ax, Ax_1) \\ &= (r_1 + r_2)\, [S(Ax, Ax, Ax_1) + S(Ay, Ay, Ay_1)] \\ &\leq (r_1 + r_2 + r_3 + r_4 + r_5)\, [S(Ax, Ax, Ax_1) \\ &\quad + S(Ay, Ay, Ay_1)] \\ &< S(Ax, Ax, Ax_1) + S(Ay, Ay, Ay_1), \end{aligned}$$

which is a contradiction, since $r_1 + r_2 + r_3 + r_4 + r_5 < 1$. Hence, we get $S(Ax, Ax, Ax_1) + S(Ay, Ay, Ay_1) = 0$, and so, $Ax = Ax_1$ and $Ay = Ay_1$. Thus, $F$ and $A$ have a unique common coupled fixed point. This completes the proof. □

**Remark 3.2.** *(1) Theorem 3.1 is an extension and a generalization of the results of Aydi [3] from partial metric space to the setting of S-metric space.*

*(2) Theorem 3.1 also extends the results of Sabetghadam et al. [18] from cone metric space to the setting of S-metric space.*

**Theorem 3.3.** *Let $(X, S)$ be a complete S-metric space. Let $F\colon X \times X \to X$ and $A\colon X \to X$ be two functions such that*

$$S(F(x,y), F(u,v), F(z,w)) \leq \lambda\, \mathcal{L}(x,y,u,v,z,w), \tag{15}$$

*for all $x, y, u, v, z, w \in X$, where $\lambda \in [0, 1)$ is a constant and*

$$\begin{aligned} \mathcal{L}(x,y,u,v,z,w) = \max\Big\{ & S(Ax, Au, Az), S(Ay, Av, Aw), \\ & S(F(x,y), F(x,y), Ax), \\ & S(F(u,v), F(u,v), Au), \\ & \frac{S(F(z,w), F(z,w), Az)}{1 + S(F(z,w), F(z,w), Az)} \Big\}. \end{aligned} \tag{16}$$

*Assume that $F$ and $A$ satisfy the following conditions:*

*$(i)$ $F(X \times X) \subseteq A(X)$,*

*$(ii)$ $A(X)$ is complete, and*

(*iii*) *A is continuous and commute with F.*

*Then F and A have a coupled coincidence point in X. Moreover, if F and A are weakly compatible, then F and A have a unique common coupled fixed point.*

*Proof.* Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq A(X)$, for , we can choose $x_1, y_1 \in X$ such that $Ax_1 = F(x_0, y_0)$ and $Ay_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq A(X)$, we can choose $x_2, y_2 \in X$ such that $Ax_2 = F(x_1, y_1)$ and $Ay_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Ax_{n+1} = F(x_n, y_n)$ and $Ay_{n+1} = F(y_n, x_n)$. Let $G_n = S(Ax_n, Ax_n, Ax_{n+1})$ and $H_n = S(Ay_n, Ay_n, Ay_{n+1})$. For $n \in \mathbb{N}$, by equations (15), (16) and using Lemma 2.23, we have

$$\begin{aligned} S(Ax_n, Ax_n, Ax_{n+1}) &= S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \lambda\, \mathcal{L}(x_{n-1}, y_{n-1}, x_{n-1}, y_{n-1}, x_n, y_n), \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\mathcal{L}(x_{n-1}, y_{n-1}, x_{n-1}, y_{n-1}, x_n, y_n) \\ &= \max\Big\{S(Ax_{n-1}, Ax_{n-1}, Ax_n), S(Ay_{n-1}, Ay_{n-1}, Ay_n), \\ &\qquad S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), Ax_{n-1}), \\ &\qquad S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), Ax_{n-1}), \\ &\qquad \frac{S(F(x_n, y_n), F(x_n, y_n), Ax_n)}{1 + S(F(x_n, y_n), F(x_n, y_n), Ax_n)}\Big\} \\ &= \max\Big\{S(Ax_{n-1}, Ax_{n-1}, Ax_n), S(Ay_{n-1}, Ay_{n-1}, Ay_n), \\ &\qquad S(Ax_n, Ax_n, Ax_{n-1}), S(Ax_n, Ax_n, Ax_{n-1}), \\ &\qquad \frac{S(Ax_{n+1}, Ax_{n+1}, Ax_n)}{1 + S(Ax_{n+1}, Ax_{n+1}, Ax_n)}\Big\} \\ &= \max\Big\{S(Ax_{n-1}, Ax_{n-1}, Ax_n), S(Ay_{n-1}, Ay_{n-1}, Ay_n), \\ &\qquad S(Ax_{n-1}, Ax_{n-1}, Ax_n), S(Ax_{n-1}, Ax_{n-1}, Ax_n), \\ &\qquad \frac{S(Ax_n, Ax_n, Ax_{n+1})}{1 + S(Ax_n, Ax_n, Ax_{n+1})}\Big\} \\ &= \max\Big\{S(Ax_{n-1}, Ax_{n-1}, Ax_n), S(Ay_{n-1}, Ay_{n-1}, Ay_n), \\ &\qquad \frac{S(Ax_n, Ax_n, Ax_{n+1})}{1 + S(Ax_n, Ax_n, Ax_{n+1})}\Big\} \\ &= \max\Big\{G_{n-1}, H_{n-1}, \frac{G_n}{1 + G_n}\Big\} \\ &= \max\Big\{G_{n-1}, H_{n-1}\Big\}. \end{aligned} \tag{18}$$

Similarly, we have

$$\begin{aligned} S(Ay_n, Ay_n, Ay_{n+1}) &= S(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq \lambda\, \mathcal{L}(y_{n-1}, x_{n-1}, y_{n-1}, x_{n-1}, y_n, x_n), \end{aligned} \tag{19}$$

where

$$\mathcal{L}(y_{n-1}, x_{n-1}, y_{n-1}, x_{n-1}, y_n, x_n)$$

$$\begin{aligned}
&= \max\Big\{S(Ay_{n-1}, Ay_{n-1}, Ay_n), S(Ax_{n-1}, Ax_{n-1}, Ax_n), \\
&\qquad \frac{S(Ay_n, Ay_n, Ay_{n+1})}{1 + S(Ay_n, Ay_n, Ay_{n+1})}\Big\} \\
&= \max\Big\{H_{n-1}, G_{n-1}, \frac{H_n}{1 + H_n}\Big\} \\
&= \max\Big\{H_{n-1}, G_{n-1}\Big\}. \qquad (20)
\end{aligned}$$

Set

$$\begin{aligned}
W_n &= S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1}) \\
&= G_n + H_n. \qquad (21)
\end{aligned}$$

Now consider the following **possible cases**.

**Case 1**. If $\max\Big\{G_{n-1}, H_{n-1}\Big\} = G_{n-1}$, then from equations (17)-(21), we obtain

$$\begin{aligned}
W_n &\leq S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1}) = G_n + H_n \\
&\leq \lambda\,[G_{n-1} + G_{n-1}] = 2\lambda\, G_{n-1}. \qquad (22)
\end{aligned}$$

**Case 2**. If $\max\Big\{G_{n-1}, H_{n-1}\Big\} = H_{n-1}$, then from equations (17)-(21), we obtain

$$\begin{aligned}
W_n &\leq S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1}) = G_n + H_n \\
&\leq \lambda\,[H_{n-1} + H_{n-1}] = 2\lambda\, H_{n-1}. \qquad (23)
\end{aligned}$$

Hence from equations (22) and (23), we obtain

$$2W_n \leq 2\lambda\,[G_{n-1} + H_{n-1}] = 2\lambda\, W_{n-1},$$

or

$$W_n \leq \lambda\, W_{n-1}.$$

Consequently, for all $n \in \mathbb{N}$, we obtain that

$$W_n \leq \lambda\, W_{n-1} \leq \lambda^2\, W_{n-2} \leq \cdots \leq \lambda^n\, W_0. \qquad (24)$$

If $W_0 = 0$, then $S(Ax_0, Ax_0, Ax_1) + S(Ay_0, Ay_0, Ay_1) = 0$. Hence, from condition $(S2)$, we get $Ax_0 = Ax_1 = F(x_0, y_0)$ and $Ay_0 = Ay_1 = F(y_0, x_0)$, means that $(Ax_0, Ay_0)$ is a coupled fixed point of $F$ and $A$. Now, we assume that $W_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using $(S3)$, we have

$$\begin{aligned}
S(Ax_n, Ax_n, Ax_m) + S(Ay_n, Ay_n, Ay_m) &\leq 2S(Ax_n, Ax_n, Ax_{n+1}) + S(Ax_m, Ax_m, Ax_{n+1}) \\
&\quad + 2S(Ay_n, Ay_n, Ay_{n+1}) + S(Ay_m, Ay_m, Ay_{n+1}) \\
&= 2(S(Ax_n, Ax_n, Ax_{n+1}) + S(Ay_n, Ay_n, Ay_{n+1})) \\
&\quad + S(Ax_m, Ax_m, Ax_{n+1}) + S(Ay_m, Ay_m, Ay_{n+1}) \\
&\leq \dots \\
&\leq 2\Big(W_n + W_{n+1} + \cdots + W_{m-1} + W_m\Big) \\
&\leq 2(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1} + \lambda^m)W_0 \\
&\leq 2\lambda^n(1 + \lambda + \lambda^2 + \dots)W_0 \\
&\leq \Big(\frac{2\lambda^n}{1-\lambda}\Big)W_0 \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

since $0 \leq \lambda < 1$. Thus, $\{Ax_n\}$ and $\{Ay_n\}$ are $S$-Cauchy sequence in $A(X)$. Since $A(X)$ is complete, we get $\{Ax_n\}$ and $\{Ay_n\}$ are $S$-convergent to some $p \in X$ and $r \in X$ respectively. Since $A$ is continuous, we have $\{AAx_n\}$ is $S$-convergent to $Ap$ and $\{AAy_n\}$ is $S$-convergent to $Ar$. Also, since $A$ and $F$ are commute, we have

$$AAx_{n+1} = A(F(x_n, y_n)) = F(Ax_n, Ay_n),$$

and

$$AAy_{n+1} = A(F(y_n, x_n)) = F(Ay_n, Ax_n).$$

Thus,

$$\begin{aligned} S(AAx_{n+1}, AAx_{n+1}, F(p, r)) &= S(F(Ax_n, Ay_n), F(Ax_n, Ay_n), F(p, r)) \\ &\leq \lambda\, \mathcal{L}(x_n, y_n, x_n, y_n, p, r), \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathcal{L}(x_n, y_n, x_n, y_n, p, r) &= \max\Big\{S(AAx_n, AAx_n, Ap), S(AAy_n, AAy_n, Ar), \\ &\qquad S(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n), \\ &\qquad S(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n), \\ &\qquad \frac{S(F(p, r), F(p, r), Ap)}{1 + S(F(p, r), F(p, r), Ap)}\Big\} \\ &= \max\Big\{S(AAx_n, AAx_n, Ap), S(AAy_n, AAy_n, Ar), \\ &\qquad S(AAx_{n+1}, AAx_{n+1}, AAx_n), \\ &\qquad S(AAx_{n+1}, AAx_{n+1}, AAx_n), \\ &\qquad \frac{S(F(p, r), F(p, r), Ap)}{1 + S(F(p, r), F(p, r), Ap)}\Big\}. \end{aligned} \tag{26}$$

Passing to the limit as $n \to \infty$ in equation (26), using Lemma 2.23, Lemma 2.24 and the condition ($S2$), we get that

$$\begin{aligned} \mathcal{L}(x_n, y_n, x_n, y_n, p, r) &= \max\Big\{S(Ap, Ap, Ap), S(Ar, Ar, Ar), S(Ap, Ap, Ap), \\ &\qquad S(Ap, Ap, Ap), \frac{S(Ap, Ap, F(p, r))}{1 + S(Ap, Ap, F(p, r))}\Big\} \\ &= \max\Big\{0, 0, 0, 0, \frac{S(Ap, Ap, F(p, r))}{1 + S(Ap, Ap, F(p, r))}\Big\} \\ &= \frac{S(Ap, Ap, F(p, r))}{1 + S(Ap, Ap, F(p, r))}. \end{aligned} \tag{27}$$

Passing to the limit as $n \to \infty$ in equation (25) and using equation (27), we get

$$\begin{aligned} S(Ap, Ap, F(p, r)) &\leq \lambda\, \frac{S(Ap, Ap, F(p, r))}{1 + S(Ap, Ap, F(p, r))} \\ &\leq \lambda\, S(Ap, Ap, F(p, r)), \end{aligned} \tag{28}$$

which is a contradiction, since $0 \leq \lambda < 1$. Hence, we have $S(Ap, Ap, F(p, r)) = 0$, that is, $F(p, r) = Ap$. Similarly, we can show that $F(r, p) = Ar$. Thus $(Ap, Ar)$ is a coupled coincidence point of the mappings $F$ and $A$. Since the pair $(F, A)$ is weakly compatible, so by weak compatibility of $F$ and $A$, we have

$$A(F(p, r)) = F(Ap, Ar) \text{ and } A(F(r, p)) = F(Ar, Ap). \tag{29}$$

Hence $(Ap, Ar)$ is a common coupled fixed point of $F$ and $A$.

Now, we show the uniqueness of the common coupled fixed point of $F$ and $A$. Assume that $(Ap_1, Ar_1)$ is another common coupled fixed point of $F$ and $A$ with $Ap \neq Ap_1$ and $Ar \neq Ar_1$, that is, $(Ap, Ar) \neq (Ap_1, Ar_1)$. Then by using equations (15), (16), using Lemma 2.23 and the condition $(S2)$, we have

$$\begin{aligned} S(Ap, Ap, Ap_1) &= S(F(p,r), F(p,r), F(p_1, r_1)) \\ &\leq \lambda\, \mathcal{L}(p, r, p, r, p_1, r_1), \end{aligned} \tag{30}$$

where

$$\begin{aligned} \mathcal{L}(p, r, p, r, p_1, r_1) &= \max\Big\{S(Ap, Ap, Ap_1), S(Ar, Ar, Ar_1), S(F(p,r), F(p,r), Ap), \\ &\qquad S(F(p,r), F(p,r), Ap), \frac{S(F(p_1,r_1), F(p_1,r_1), Ap_1)}{1 + S(F(p_1,r_1), F(p_1,r_1), Ap_1)}\Big\} \\ &= \max\Big\{S(Ap, Ap, Ap_1), S(Ar, Ar, Ar_1), S(Ap, Ap, Ap), \\ &\qquad S(Ap, Ap, Ap), \frac{S(Ap_1, Ap_1, Ap_1)}{1 + S(Ap_1, Ap_1, Ap_1)}\Big\} \\ &= \max\Big\{S(Ap, Ap, Ap_1), S(Ar, Ar, Ar_1), 0, 0, 0\Big\} \\ &= \max\Big\{S(Ap, Ap, Ap_1), S(Ar, Ar, Ar_1), 0\Big\}. \end{aligned} \tag{31}$$

Similarly, we obtain

$$\begin{aligned} S(Ar, Ar, Ar_1) &= S(F(r,p), F(r,p), F(r_1, p_1)) \\ &\leq \lambda\, \mathcal{L}(r, p, r, p, r_1, p_1), \end{aligned} \tag{32}$$

where

$$\mathcal{L}(r, p, r, p, r_1, p_1) = \max\Big\{S(Ap, Ap, Ap_1), S(Ar, Ar, Ar_1), 0\Big\}. \tag{33}$$

From equations (30) and (32), we obtain

$$\begin{aligned} & S(Ap, Ap, Ap_1) + S(Ar, Ar, Ar_1) \\ &\qquad \leq \lambda\, [\mathcal{L}(p, r, p, r, p_1, r_1) + \mathcal{L}(r, p, r, p, r_1, p_1)]. \end{aligned} \tag{34}$$

Set

$$Q = S(Ap, Ap, Ap_1) + S(Ar, Ar, Ar_1), \tag{35}$$

and

$$M = S(Ap, Ap, Ap_1),\ N = S(Ar, Ar, Ar_1). \tag{36}$$

Now, we consider the following **possible cases**.

**Case $1^0$**. If $\max\{M, N, 0\} = M$, then from equations (34)-(36), we obtain

$$\begin{aligned} Q &= S(Ap, Ap, Ap_1) + S(Ar, Ar, Ar_1) = M + N \\ &\leq \lambda\, [M + M] = 2\lambda\, M. \end{aligned} \tag{37}$$

**Case $2^0$**. If $\max\{M, N, 0\} = N$, then from equations (34)-(36), we obtain

$$\begin{aligned} Q &= S(Ap, Ap, Ap_1) + S(Ar, Ar, Ar_1) = M + N \\ &\leq \lambda\, [N + N] = 2\lambda\, N. \end{aligned} \tag{38}$$

Hence from equations (37) and (38), we obtain that $2Q \leq 2\lambda\, [M + N] = 2\lambda\, Q$ or $Q \leq \lambda\, Q$, which is a contradiction, since $0 \leq \lambda < 1$. Hence, we conclude that $Q = 0$, that is, $S(Ap, Ap, Ap_1) + S(Ar, Ar, Ar_1) = 0$, and so,

$Ap = Ap_1$ and $Ar = Ar_1$.

**Case 3⁰**. If $\max\left\{M, N, 0\right\} = 0$, then from equations (34)-(36), we obtain $Q \leq \lambda\,[0+0] = 0$. Hence, we conclude that $Q = 0$, that is, $S(Ap, Ap, Ap_1) + S(Ar, Ar, Ar_1) = 0$, and so, $Ap = Ap_1$ and $Ar = Ar_1$.

Thus in both the above cases, we obtain $Ap = Ap_1$ and $Ar = Ar_1$. Consequently, $F$ and $A$ have a unique common coupled fixed point. This completes the proof. □

Now, we give an example in support of the result.

**Example 3.4.** *Let $X = [0,1]$ and the function $S\colon X^3 \to [0,\infty)$ be defined as*

$$S(x,y,z) = |y-z| + |y+z-2x|,$$

*for all $x, y, z \in X$. Then the function $S$ is an S-metric on $X$ and $(X, S)$ is an S-metric space. Define a map $F\colon X \times X \to X$ by $F(x,y) = \frac{x}{128} + \frac{y}{256}$ for $x, y \in X$. Also, define $A\colon X \to X$ by $A(x) = \frac{x}{4}$. We have*

$$\begin{aligned}
S(F(x,y),F(u,v),F(z,w)) &= |F(u,v)+F(z,w)-2F(x,y)| \\
&\quad +|F(u,v)-F(z,w)| \\
&= \left|\frac{u}{128}+\frac{v}{256}+\frac{z}{128}+\frac{w}{256}-\frac{2x}{128}-\frac{2y}{256}\right| \\
&\quad +\left|\frac{u}{128}+\frac{v}{256}-\frac{z}{128}-\frac{w}{256}\right| \\
&= \frac{1}{128}|u+z-2x| + \frac{1}{256}|v+w-2y| \\
&\quad +\frac{1}{128}|u-z| + \frac{1}{256}|v-w| \\
&= \frac{1}{128}\Big(|u+z-2x| + |u-z|\Big) \\
&\quad +\frac{1}{256}\Big(|v+w-2y| + |v-w|\Big) \\
&= \frac{1}{32}\Big(\left|\frac{u}{4}+\frac{z}{4}-\frac{2x}{4}\right| + \left|\frac{u}{4}-\frac{z}{4}\right|\Big) \\
&\quad +\frac{1}{64}\Big(\left|\frac{v}{4}+\frac{w}{4}-\frac{2y}{4}\right| + \left|\frac{v}{4}-\frac{w}{4}\right|\Big) \\
&= \frac{1}{32}S(Ax,Au,Az) + \frac{1}{64}S(Ay,Av,Aw) \\
&\leq \frac{1}{32}\Big(S(Ax,Au,Az) + S(Ay,Av,Aw)\Big) \\
&\leq \frac{1}{32}\Big(S(Ax,Au,Az) + S(Ay,Av,Aw) \\
&\qquad\qquad +S(F(x,y),F(x,y),Ax) \\
&\qquad\qquad +S(F(u,v),F(u,v),Au) \\
&\qquad\qquad +S(F(z,w),F(z,w),Az)\Big),
\end{aligned}$$

*holds for all $x, y, z, u, v, w \in X$. It is easy to see that $F$ and $A$ satisfy all the conditions of Theorem 3.1 for $r_1 = r_2 = r_3 = r_4 = r_5 = \frac{1}{32}$ with $r_1 + r_2 + r_3 + r_4 + r_5 = \frac{5}{32} < 1$. Thus $F$ and $A$ have a unique common coupled fixed point. Here $F(0,0) = 0$ and $A(0) = 0$.*

## 4. Conclusion

In this paper, we prove some unique common coupled fixed point theorems in the setting of $S$-metric spaces for a pair of weakly compatible mappings. Our results of findings extend and generalize several previously published findings from the existing literature.

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## References

[1] M. Abbas and B. E. Rhoades, Common fixed point results for non-commuting mappings without continuity generalized metric spaces, Appl. Math. Computation 215 (2009), 262–269.
[2] M. Abbas, M. Ali Khan and S. Radenović, Common coupled fixed point theorems in cone metric spaces for $w$-compatible mappings, Appl. Math. Comput. 217 (2010), 195–202.
[3] H. Aydi, Some coupled fixed point results on partial metric spaces, International J. Math. Math. Sci. 2011, Article ID 647091, 11 pages.
[4] S. Banach, Sur les operation dans les ensembles abstraits et leur application aux equation integrals, Fund. Math. 3 (1922), 133–181.
[5] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: TMA, 65(7) (2006), 1379–1393.
[6] L. Ciric and V. Lakshmikantham, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: TMA, 70(12) (2009), 4341–4349.
[7] N. V. Dung, N. T. Hieu and S. Radojević, Fixed point theorems for $g$-monotone maps on partially ordered $S$-metric spaces, Filomat 28(9) (2014), 1885–1898.
[8] A. Gupta, Cyclic contraction on $S$-metric space, Int. J. Anal. Appl. 3(2) (2013), 119–130.
[9] D. Guo and V. Lakshmikantham, Coupled fixed point of nonlinear operator with application, Nonlinear Anal. TMA., 11 (1987), 623–632.
[10] N. T. Hieu, N. T. Ly and N. V. Dung, A generalization of Ciric quasi-contractions for maps on $S$-metric spaces, Thai J. Math. 13(2) (2015), 369–380.
[11] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986), 771–779.
[12] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29 1998), 227–238.
[13] J. K. Kim, S. Sedghi, A. Gholidahneh and M. M. Rezaee, Fixed point theorems in $S$-metric spaces, East Asian Math. J. 32(5) (2016), 677–684.
[14] N. V. Luong and N. X. Thuan, Coupled fixed points theorems for mixed monotone mappings and an application to integral equations, Comput. Math. Appl. 62 (2011), 4238–4248.
[15] H. K. Nashine, J. K. Kim, A. K. Sharma and G. S. Saluja, Some coupled fixed point without mixed monotone mappings, Nonlinear Funct. Anal. Appl. 21(2) (2016), 235–247.
[16] J. O. Olaleru, G. A. Okeke and H. Akewe, Coupled fixed point theorems for generalized $\varphi$-mappings satisfying contractive condition of integral type on cone metric spaces, Int. J. Math. Model. Comput. 2(2) (2012), 87–98.
[17] N. Y. Özgür and N. Tas, Some new contractive mappings on $S$-metric spaces and their relationships with the mapping (**S25**), Math. Sci. 11(7) (2017), 7–16.
[18] F. Sabetghadam, H. P. Mashiha and A. H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory Appl. (2009), Article ID 125426, 8 pages.
[19] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in $S$-metric spaces, Mat. Vesnik 64(3) (2012), 258–266.
[20] S. Sedghi and N. V. Dung, Fixed point theorems on $S$-metric spaces, Mat. Vesnik 66(1) (2014), 113–124.
[21] S. Sedghi, M. M. Rezaee, T. Dosenovic and S. Radenovic, Common fixed point theorems for contractive mappings satisfying $\Phi$-maps in $S$-metric spaces, Acta Univ. Sapientiae Math. 8(2) (2016), 298–311.
[22] S. Sedghi, N. Shobkolaei, M. Shahraki and T. Dosenovic, Common fixed point of four maps in $S$-metric space, Math. Sci. 12 (2018), 137–143.