Functional Analysis,
Approximation and
Computation
15 (1) (2023), 55–63

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Besicovitch ρ-almost periodic type functions in $\mathbb{R}^n$

Marko Kostić[a]

[a] *Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia*

Abstract. In this paper, we analyze several new classes of multi-dimensional Besicovitch ρ-almost periodic type functions. We clarify the basic features of the introduced classes of functions and propose some open problems.

1. Introduction and preliminaries

The notion of an almost periodic function was introduced by the Danish mathematician H. Bohr around 1924–1926 and later generalized by many others (see the research monographs [2], [3], [7], [8], [9], [10], [14] and [15] for more details concerning almost periodic functions and their applications). Suppose that $(X, \|\cdot\|)$ is a complex Banach space and $F : \mathbb{R}^n \to X$ is a continuous function ($n \in \mathbb{N}$). Then we say that the function $F(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$ with

$$\left\|F(\mathbf{t}+\tau) - F(\mathbf{t})\right\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n;$$

here, $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$ and τ is usually called an ϵ-almost period of $F(\cdot)$. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic, and we know that a continuous function $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges uniformly to $F(\cdot)$.

We refer the reader to [13] for the notion of a Stepanov-p-almost periodic function $F : \mathbb{R}^n \to X$ and the notion of an equi-Weyl-p-almost periodic $F : \mathbb{R}^n \to X$ $(1 \leq p < +\infty)$. We know that any Stepanov-p-almost periodic function $F : \mathbb{R}^n \to X$ is equi-Weyl-p-almost periodic as well as that any equi-Weyl-p-almost periodic function $F : \mathbb{R}^n \to X$ is Besicovitch-p-almost periodic. The notion of Besicovitch-p-almost periodicity is traditionally introduced as follows; if $F \in L^p_{loc}(\mathbb{R}^n : X)$, then we first define

$$\|F\|_{\mathcal{M}^p} := \limsup_{t \to +\infty} \left[\frac{1}{(2t)^n} \int_{[-t,t]^n} \|F(\mathbf{s})\|^p \, d\mathbf{s}\right]^{1/p}.$$

2020 *Mathematics Subject Classification.* Primary 42A75; Secondary 43A60.
Keywords. Besicovitch ρ-almost periodic functions; Doss ρ-almost periodic functions; Partial differential equations.
Received: 10 October 2022; Accepted: 29 January 2023
Communicated by Dragan S. Djordjević
This research was partially supported by grant no. 51-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia.
Email address: marco.s@verat.net (Marko Kostić)

Then $\|\cdot\|_{\mathcal{M}^p}$ is a seminorm on the space $\mathcal{M}^p(\mathbb{R}^n : X)$ consisting of those $L^p_{loc}(\mathbb{R}^n : X)$-functions $F(\cdot)$ for which $\|F\|_{\mathcal{M}^p} < \infty$. Set $K_p(\mathbb{R}^n : X) := \{f \in \mathcal{M}^p(\mathbb{R}^n : X) \,;\, \|F\|_{\mathcal{M}^p} = 0\}$ and

$$M_p(\mathbb{R}^n : X) := \mathcal{M}^p(\mathbb{R}^n : X)/K_p(\mathbb{R}^n : X).$$

The seminorm $\|\cdot\|_{\mathcal{M}^p}$ on $\mathcal{M}^p(\mathbb{R}^n : X)$ induces the norm $\|\cdot\|_{M^p}$ on $M^p(\mathbb{R}^n : X)$ under which $M^p(\mathbb{R}^n : X)$ is complete; therefore, $(M^p(\mathbb{R}^n : X), \|\cdot\|_{M^p})$ is a Banach space. It is said that a function $F \in L^p_{loc}(\mathbb{R}^n : X)$ is Besicovitch-p-almost periodic if and only if there exists a sequence of trigonometric polynomials (almost periodic functions, equivalently) which converges to $F(\cdot)$ in the space $(M^p(\mathbb{R}^n : X), \|\cdot\|_{M^p})$. The vector space consisting of all Besicovitch-p-almost periodic functions is denoted by $B^p(\mathbb{R}^n : X)$. We know that $B^p(\mathbb{R}^n : X)$ is a closed subspace of $M^p(\mathbb{R}^n : X)$ and therefore a Banach space itself.

The main aim of this paper is to introduce and analyze several new classes of multi-dimensional Besicovitch ρ-almost periodic type functions in a Bohr-like manner. The basic structural properties of the introduced classes of Besicovitch ρ-almost periodic type functions are presented as well as some illustrative examples and an application to the partial differential equations. We also propose many useful remarks and two open problems about the function spaces under our consideration.

We use the same notation as in our recent research article [13]. We assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $n \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ and $\mathcal{B}$ is a non-empty collection of non-empty subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By I we denote the identity operator on Y. Set, finally, $\mathbb{N}_n := \{1, ..., n\}$ and $\Delta_n := \{(t, t, ..., t) \in \mathbb{R}^n : t \in \mathbb{R}\}$.

Lebesgue spaces with variable exponents $L^{p(x)}$. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a non-empty Lebesgue measurable subset and $M(\Omega : X)$ is the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. By $\mathcal{P}(\Omega)$ we denote the vector space of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. If $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, then we define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, \quad t \geq 0, \quad 1 \leq p(x) < \infty, \\ 0, \quad 0 \leq t \leq 1, \quad p(x) = \infty, \\ \infty, \quad t > 1, \quad p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_\Omega \varphi_{p(x)}(\|f(x)\|)\, dx.$$

The Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent is defined by

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}.$$

We know that

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\}.$$

For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ by

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{ \lambda > 0 : \rho(u/\lambda) \leq 1 \Big\}.$$

Equipped with the Luxemburg norm, $L^{p(x)}(\Omega : X)$ becomes a Banach space, coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geq 1$ is a constant function.

For further information concerning the theory of Lebesgue spaces with variable exponents, we refer the reader to the research monograph [4] by L. Diening et al.

2. Multi-dimensional Besicovitch ρ-almost periodic type functions

In this paper, we will always assume that $\rho_1 \subseteq Y \times Y$ and $\rho_2 \subseteq Y \times Y$ are binary relations, Λ is a general non-empty subset of $\mathbb{R}^n$ as well as that $p \in \mathcal{P}(\Lambda)$, $\phi : [0, \infty) \to [0, \infty)$ is measurable, $\mathrm{F} : (0, \infty) \to (0, \infty)$, $\Lambda'' := \{\tau \in \mathbb{R}^n : \tau + \Lambda \subseteq \Lambda\}$, $\Lambda_t := \{\mathbf{t} \in \Lambda : |\mathbf{t}| \leq t\}$ $(t > 0)$ and the following condition holds $(1 \leq i \leq 3;$ $1 \leq j \leq 2)$:

$\phi_i : [0, \infty) \to [0, \infty)$ is measurable, $\mathrm{F}_j : (0, \infty) \to (0, \infty)$ and $p \in \mathcal{P}(\Lambda)$.

Now we are ready to introduce the following notion (if considered henceforth, we assume that the discrete set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ satisfies that $\tau_i \neq \tau_j$ for all $i,\ j \in \mathbb{Z}$ with $i \neq j$):

Definition 2.1. Let $F : \Lambda \times X \to Y$.

(i) Suppose that the set $A \subseteq \Lambda' \subseteq \mathbb{R}^n$ has no finite accumulation point and $\delta > 1$. Then we say that the set A is Λ'_δ-satisfactorily uniform if and only if there exists a finite real number $l > 0$ such that, for every $\mathbf{t}_0,\ \mathbf{t}_1 \in \Lambda'$, we have

$$\delta \cdot |A|_{B(\mathbf{t}_0, l)} > |A|_{B(\mathbf{t}_1, l)}; \tag{2.1}$$

here and hereafter, $|A|_D$ denotes the number of elements of set $A \cap D$, where $D \subseteq \mathbb{R}^n$.

(ii) We say that the function $F(\cdot; \cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists a Λ'_δ-satisfactorily uniform set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ such that, for every $\tau \in A$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho_1(F(\cdot; x))$ such that

$$\limsup_{t \to +\infty} \mathrm{F}_1(t) \sup_{x \in B} \Bigl[\phi_1\bigl(\|F(\cdot + \tau; x) - y_{\cdot;x}\|_Y\bigr)\Bigr]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon, \tag{2.2}$$

as well as, for every $l > 0$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t + l\Omega$, there exists an element $z_{\cdot;x} \in \rho_2(F(\cdot; x))$ such that

$$\limsup_{t \to +\infty} \mathrm{F}_2(t) \sup_{x \in B} \Biggl[\phi_2\Biggl(\limsup_{k \to +\infty} \frac{1}{(2k+1)} \times \sum_{i=-k}^{k} \Bigl[\phi_3\Bigl(l^{-n}\bigl\|F(\cdot + \tau_i; x) - z_{\cdot;x}\bigr\|_Y\Bigr)\Bigr]_{L^{p(\cdot)}(y + l\Omega : Y)}\Biggr)\Biggr]_{L^{p(y)}(\Lambda_t)} < \epsilon. \tag{2.3}$$

(iii) We say that the function $F(\cdot; \cdot)$ is Besicovitch-$(p, \phi_2, \phi_3, \mathrm{F}_2, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type 1 if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists a Λ'_δ-satisfactorily uniform set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t + \Omega$, there exists an element $z_{\cdot;x} \in \rho_2(F(\cdot; x))$ such that (2.3) holds with $l = 1$.

(iv) Suppose $p(\cdot) \equiv p \in [1, \infty)$. Then we say that the function $F(\cdot; \cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic of type 2 if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists a Λ'_δ-satisfactorily uniform set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ such that, for every $\tau \in A$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho_1(F(\cdot; x))$ such that (2.2) holds, as well as that, for every $l > 0$, $t > 0$, $x \in B$ and $\mathbf{t} \in \Lambda_t + l\Omega$, there exists an element $z_{\mathbf{t};x} \in \rho_2(F(\mathbf{t}; x))$ such that

$$\limsup_{t \to +\infty} \mathrm{F}_2(t) \sup_{x \in B} \Biggl[\int_{\Lambda_t} \phi_2\Biggl(\limsup_{k \to +\infty} \frac{1}{(2k+1)} \times \sum_{i=-k}^{k} \Biggl[l^{-n} \int_{y + l\Omega} \Bigl[\phi_3\Bigl(\bigl\|F(\mathbf{t} + \tau_i; x) - z_{\mathbf{t};x}\bigr\|_Y\Bigr)\Bigr]^p d\mathbf{t}\Biggr]\Biggr) dy\Biggr]^{1/p} < \epsilon. \tag{2.4}$$

(v) Suppose $p(\cdot) \equiv p \in [1, \infty)$. Then we say that the function $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type 3 if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists a Λ'_δ-satisfactorily uniform set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ such that, for every $t > 0$, $x \in B$ and $\mathbf{t} \in \Lambda_t + \Omega$, there exists an element $z_{\mathbf{t};x} \in \rho_2(F(\mathbf{t}; x))$ such that (2.4) holds with $l = 1$.

It is clear that (2.1) implies

$$\delta \cdot \inf_{\mathbf{t}\in\mathbb{R}^n} |A|_{B(\mathbf{t},l)} \geq \sup_{\mathbf{t}\in\mathbb{R}^n} |A|_{B(\mathbf{t},l)}.$$

The usual notion of a satisfactorily uniform set $A \subseteq \mathbb{R}^n$ is obtained by plugging $\Lambda' = \mathbb{R}$ and $\delta = 2$ in Definition 2.1(i); we say that the discrete set $A \subseteq \mathbb{R}^n$ is δ-satisfactorily uniform if and only if A is Λ'_δ-satisfactorily uniform with $\Lambda' = \mathbb{R}^n$. Concerning this notion, we would like to observe, probably for the first time in the existing literature, that the use of number $\delta = 2$ is completely meaningless in the structural characterizations of the space of one-dimensional Besicovitch-p-almost periodic functions $f : \mathbb{R} \to \mathbb{C}$ given in [2]. Let us explain this fact in more detail: the statement of [2, Corollary 2], which continues to hold for the vector-valued functions, is essentially used in the proofs of [2, Theorem 7°, $(IC)_B(A)$, p. 95; Theorem 1°, $(IC)_{B^r}(A)$, p. 100]. All other technical lemmas and results from [2] continue to hold with δ-satisfactorily uniform sets and here we will only note that, in the concrete situation of [2, Remark, pp. 93–94], we have the following estimate with the use of δ-satisfactorily uniform sets:

$$\frac{\mu(b)}{b} \leq p \leq \frac{\nu(b)}{b} \leq \frac{\delta\mu(b)}{b} \leq \delta p.$$

This simply implies that for each number $\delta' > \delta$ we have the existence of a real number $b' > b$, where b has the same meaning as in the above-mentioned remark, such that the estimate

$$\frac{n(t,T)}{2T} \leq \delta' p, \quad T > b'$$

holds; see the equation [2, (1), p. 94]. Therefore, we have the following result:

Theorem 2.2. *Suppose that $1 \leq p < +\infty$ and the function $f : \mathbb{R} \to X$ is locally p-integrable. Then $f(\cdot)$ is Besicovitch-p-almost periodic if and only if for each (some) $\delta > 1$ there exists a δ-satisfactorily uniform set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}$ such that*

$$\limsup_{t\to+\infty}\left(\frac{1}{2t}\int_{-t}^{t}\left\|f(s+\tau_i)-f(s)\right\|^p ds\right)^{1/p} < \epsilon$$

and, for every $l > 0$,

$$\limsup_{t\to+\infty}\left(\frac{1}{2t}\int_{-t}^{t}\left[\limsup_{k\to+\infty}\frac{1}{2k+1}\sum_{i=-k}^{k} l^{-1}\int_{x}^{x+l}\left\|f(s+\tau_i)-f(s)\right\|^p ds\right]dx\right)^{1/p} < \epsilon.$$

The notion of a Bohr (Besicovitch) almost periodic set in the Euclidean space $\mathbb{R}^n$ can be also introduced and analyzed; see the research article [5] by S. Favorov for more details about this non-trivial problematic. Concerning the notion of a Λ'_δ-satisfactorily uniform set, we will provide the following illustrative examples:

Example. (i) If A is a Λ'_δ-satisfactorily uniform set, then $A - A$ need not be a Λ'_δ-satisfactorily uniform set. In actual fact, the set $A - A$ can have a finite accumulation point as the following special case with $\Lambda' = \mathbb{R}$ and $\delta = 2$ shows. Let $A := \mathbb{Z} \cup \{k + k^{-1} : k \in \mathbb{Z} \setminus \{0\}\}$. Then A is a satisfactorily uniform set but 0 is an accumulation point of the set $A - A$, as easily approved.

(ii) The set $\mathbb{Z}^n$ is δ-satisfactorily uniform for each number $\delta > 1$. Given a number $\delta_0 > 1$, it could be interesting to construct a set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ which is δ-satisfactorily uniform for $\delta > \delta_0$ but not δ_0-satisfactorily uniform.

(iii) In the multi-dimensional setting, it is not completely clear how one can order a discrete set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$; this fact has a series of obvious unpleasant consequences in applications to the partial differential equations. We propose the following unsatisfactory approach: Let A_0 consist of those elements of A with the minimal norm. Then A_0 can be written as the union of disjoint sets $A_{-s_2}, ..., A_{s_1}$, where A_j is determined in the following way $(-s_2 \leq j \leq s_1)$: Let $x_{-s_2} < ... < x_{-2} < x_{-1} < 0 \leq x_0 < x_1 < ... < x_{s_1}$, and $\{x_{-s_2}, ..., x_{s_1}\}$ is the first projection of set A_0. We first order the elements of A_0 whose first projection is x_0 by the minimal second coordinate, the minimal third coordinate, etc.; we continue the enumeration with the sets $A_1, A_2, ..., A_{s_1}$. After that we proceed to the elements of sets $A_{-1}, A_{-2}, ..., A_{-s_2}$ and give them the negative values of indices i by the maximal second coordinate, the maximal third coordinate, etc.; and so on and so forth. After ordering the elements with the minimal norm, we proceed to the elements with the second minimal norm and order them as above (we continue the enumeration). It is worth noting that, in the one-dimensional setting, we obtain the enumeration $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}$ with $\tau_i < \tau_j$ for $i < j$.

Now we would like to raise the following important issue:

Problem 2.3. Can we prove a multi-dimensional analogue of Theorem 2.2?

For every space of Besicovitch almost periodic type functions introduced in Definition 2.1, we can also introduce the corresponding space of Besicovitch uniformly recurrent type functions:

Definition 2.4. Let $F : \Lambda \times X \to Y$.

(i) We say that the function $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2)$-uniformly recurrent if and only if, for every $B \in \mathcal{B}$, there exists an unbounded set $\{\tau_i : i \in \mathbb{N}\} \subseteq \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho_1(F(\cdot; x))$ such that

$$\lim_{i\to+\infty} \limsup_{t\to+\infty} \mathrm{F}_1(t) \sup_{x\in B} \Big[\phi_1\big(\|F(\cdot + \tau_i; x) - y_{\cdot;x}\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} = 0, \tag{2.5}$$

as well as, for every $l > 0$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t + l\Omega$, there exists an element $z_{\cdot;x} \in \rho_2(F(\cdot; x))$ such that

$$\lim_{i\to+\infty} \limsup_{t\to+\infty} \mathrm{F}_2(t) \sup_{x\in B} \Bigg[\phi_2\Bigg(\limsup_{k\to+\infty} \frac{1}{(2k+1)} \times \sum_{i=-k}^{k} \Big[\phi_3\Big(l^{-n}\big\|F(\cdot + \tau_i; x) - z_{\cdot;x}\big\|_Y\Big)\Big]_{L^{p(\cdot)}(y+l\Omega:Y)}\Bigg)\Bigg]_{L^{p(y)}(\Lambda_t)} = 0. \tag{2.6}$$

(ii) We say that the function $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_2, \phi_3, \mathrm{F}_2, \mathcal{B}, \Lambda', \rho_2)$-uniformly recurrent of type 1 if and only if, for every $B \in \mathcal{B}$, there exists an unbounded set $\{\tau_i : i \in \mathbb{N}\} \subseteq \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t + \Omega$, there exists an element $z_{\cdot;x} \in \rho_2(F(\cdot; x))$ such that (2.6) holds with $l = 1$.

(iii) Suppose $p(\cdot) \equiv p \in [1, \infty)$. Then we say that the function $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2)$-uniformly recurrent of type 2 if and only if, for every $B \in \mathcal{B}$, there exists an unbounded set $\{\tau_i : i \in \mathbb{N}\} \subseteq \Lambda'$ such that, for every $\tau \in A$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho_1(F(\cdot; x))$ such that (2.5) holds, as well as that, for every $l > 0$, $t > 0$, $x \in B$ and $\mathbf{t} \in \Lambda_t + l\Omega$, there exists an element $z_{\mathbf{t};x} \in \rho_2(F(\mathbf{t}; x))$ such that

$$\lim_{i\to+\infty} \limsup_{t\to+\infty} \mathrm{F}_2(t) \sup_{x\in B} \Bigg[\int_{\Lambda_t} \phi_2\Bigg(\limsup_{k\to+\infty} \frac{1}{(2k+1)} \times \sum_{i=-k}^{k} \bigg[\int_{y+l\Omega} \Big[\phi_3\Big(l^{-n}\big\|F(\mathbf{t} + \tau_i; x) - y_{\mathbf{t};x}\big\|_Y\Big)\Big]^p d\mathbf{t}\bigg]\Bigg) dy\Bigg]^{1/p} = 0. \tag{2.7}$$

(iv) Suppose $p(\cdot) \equiv p \in [1,\infty)$. Then we say that the function $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2)$-uniformly recurrent of type 3 if and only if, for every $B \in \mathcal{B}$, there exists an unbounded set $\{\tau_i : i \in \mathbb{N}\} \subseteq \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\mathbf{t} \in \Lambda_t + \Omega$, there exists an element $z_{\mathbf{t};x} \in \rho_2(F(\mathbf{t};x))$ such that (2.7) holds with $l = 1$.

Remark 2.5. We accept the usual terminology agreements from our previous research studies: If $F : \Lambda \to Y$, then we omit the term "$\mathcal{B}$" from the notation. Further on, if $\rho_i = c_i\mathrm{I}$ for some $c_i \in \mathbb{C}$, where $i = 1, 2$, then we also write "c_i" in place of "ρ"; we omit "c_i" from the notation if $c_i = 1$. We also omit the term "Λ'" from the notation if $\Lambda' = \Lambda$. If $\phi_i(x) \equiv x$ for $i = 1, 2, 3$, then we omit it from the notation. The usual notion is obtained by plugging $c_1 = c_2 = 1$, $\Lambda' = \mathbb{R}^n$, $\phi_i(x) \equiv x$ for $i = 1, 2, 3$ and $\mathrm{F}_j(\mathbf{t}) \equiv \mathbf{t}^{-(n/p)}$ for $j = 1, 2$, in Definition 2.1(iv); we define the notion of Besicovitch-p-uniform recurrence in a similar fashion. For some important counterexamples in the theory of one-dimensional Besicovitch-p-almost periodic functions, we refer the reader to [1, Example 6.24-Example 6.27].

It is clear that the notion introduced here provides a very general approach to the notion of Besicovitch almost periodicity (uniform recurrence) as well as that we work with general subsets Λ of $\mathbb{R}^n$ here. Concerning this issue, we would like to mention the following: on [12, p. 16], we have proved that, under certain reasonable conditions, any Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-almost periodic function $F : \Lambda \times X \to Y$ can be extended to a Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho_1)$-almost periodic function $\tilde{F} : \mathbb{R}^n \times X \to Y$, defined by $\tilde{F}(\mathbf{t}) := 0$, $t \notin \Lambda$, $\tilde{F}(\mathbf{t}) := F(\mathbf{t})$, $t \in \Lambda$, with $\rho_1 := \rho \cup \{(0,0)\}$ (the corresponding analysis from [12] contains small typographycal errors that will be corrected in our forthcoming monograph [11]).

Before we go any further, we would like to emphasize that a similar analysis cannot be carried out for Besicovitch almost periodic type functions. Because of that, we would like to introduce the following notion:

Definition 2.6. Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, and let $\mathcal{C}_\Lambda$ denote any class of functions $F : \Lambda \to Y$ introduced in Definition 2.1 or Definition 2.4. Then we say that a set Λ is admissible with respect to the class $\mathcal{C}_\Lambda$ if and only if for any complex Banach space Y and for any function $F : \Lambda \to Y$ there exists a function $\tilde{F} \in \mathcal{C}_{\mathbb{R}^n}$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$.

In connection with Definition 2.6, we would like to emphasize that we do not yet know whether the set $[0,\infty) \subseteq \mathbb{R}$ is admissible with respect to the class of Besicovitch-p-almost periodic functions, i.e., whether a Besicovitch-p-almost periodic function $f : [0,\infty) \to Y$ can be extended to a Besicovitch-p-almost periodic function $\tilde{f} : \mathbb{R} \to Y$ defined on the whole real line ($1 \leq p < \infty$). Furthermore, the following question is meaningful:

Problem 2.7. Let $(v_1, v_2, ..., v_n)$ is a basis of $\mathbb{R}^n$ and let

$$\Lambda = \left\{\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n : \alpha_i \geq 0 \text{ for all } i \in \mathbb{N}_n\right\}$$

be a convex polyhedral in $\mathbb{R}^n$. Further on, let a function $F \in e - (\cdot^{-n/p}) - B^p(\Lambda : Y)$ be given; cf. [13, Definition 2.1] and the paragraph following it for the notion. Is it true that there exists a function $\tilde{F} \in e - (\cdot^{-n/p}) - B^p(\mathbb{R}^n : Y)$ such that $\tilde{F} = F$ on Λ?

In support of our investigation of the general case $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, we would like to present the following example, as well:

Example. (cf. also [10, Example 6.1.15]) Suppose that $L > 0$ is a fixed real number, $p \in [1,\infty)$ as well as the functions $t \mapsto f(t)$, $t \in \mathbb{R}$ and $t \mapsto g(t)$, $t \in \mathbb{R}$ are Besicovitch-p-almost periodic. Set $\Lambda := \{(x, y) \in \mathbb{R}^2 : |x - y| \geq L\}$ and $\Lambda' := \{(\tau, \tau) : \tau \in \mathbb{R}\}$. From the classical theory of PDEs, we know that the function

$$u(x, y) := \frac{f(x) + g(y)}{x - y}, \quad (x, y) \in \Lambda,$$

is a solution of the partial differential equation

$$u_{xy} - \frac{u_x}{x - y} + \frac{u_y}{x - y} = 0,$$

if we impose certain regularity conditions. Further on, if $\epsilon > 0$ is given, then we can find a satisfactorily uniform set $V_\epsilon = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}$ such that, for every $i \in \mathbb{Z}$, we can find a finite real number $t_0 > 0$ such that, for every real number $t \geq t_0$, we have

$$\int_{-t}^{t} \bigl|f(x+\tau_i) - f(x)\bigr|^p \, dx + \int_{-t}^{t} \bigl|g(y+\tau_i) - g(y)\bigr|^p \, dy < \epsilon t^p, \quad t \geq t_0, \tag{2.8}$$

and, for every real number $l > 0$,

$$\begin{aligned} &\left[\int_{-t}^{t} \limsup_{k\to+\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} l^{-1} \int_x^{x+l} \bigl|f(s+\tau_i) - f(s)\bigr|^p \, ds\right] dx \\ &+ \int_{-t}^{t} \left[\limsup_{k\to+\infty} \frac{1}{2k+1} \sum_{i=-k}^{k} l^{-1} \int_x^{x+l} \bigl|g(s+\tau_i) - g(s)\bigr|^p \, ds\right] dx < \epsilon t^p, \quad t \geq t_0. \end{aligned} \tag{2.9}$$

It is clear that

$$\begin{aligned} \|u(x+\tau, y+\tau) - u(x,y)\| &\leq \frac{\|f(x+\tau) - f(x)\| + \|g(y+\tau) - g(y)\|}{|x-y|} \\ &\leq \frac{\|f(x+\tau) - f(x)\| + \|g(y+\tau) - g(y)\|}{L}, \quad (x,y) \in \Lambda, \ \tau \in V_\epsilon. \end{aligned}$$

Set $W_\epsilon := \{(\tau_i, \tau_i) : i \in \mathbb{Z}\}$. Then W_ϵ is Λ'-satisfactorily uniform with $\Lambda' := \Delta_2$ and the function $u(x,y)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic of type 2 ($\phi_i(x) \equiv x$, $\mathrm{F}_j(t) \equiv t^{-(2/p)}$, $\rho_1 = \rho_2 = \mathrm{I}$, $\delta = 2$), as a very simple computation involving the Fubini theorem and (2.8)-(2.9) shows. Observe, finally, that this example also indicates that the situation in which Λ' is not a subset of Λ is meaningful.

It is clear that we have the following:

(i) If $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic, then $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type 1.

(ii) If $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic of type 2, then $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type 3.

(iii) If $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic (of type 2), then $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2)$-uniformly recurrent (of type 2). Furthermore, if $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type 1 (of type 3), then $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2)$-uniformly recurrent of type 1 (of type 3).

Now we will prove the following simple result (it is clear that the converse statement can be proved in the case that $p = 1$, when the corresponding classes coincide):

Proposition 2.8. *Suppose $p(\cdot) \equiv p \in [1,\infty)$ and $F : \Lambda \times X \to Y$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic of type* 2 *(Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type* 3*) (Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-uniformly recurrent of type* 2 *(Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-uniformly recurrent of type* 3*)). Let the function $\phi_2(\cdot)$ be monotonically increasing and satisfy that $\phi_2(x) \geq [\phi_2(x^{1/p})]^p$ for all $x \geq 0$. Then the function $F(\cdot;\cdot)$ is Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic (Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2, \delta)$-almost periodic of type* 1*) (Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2)$-uniformly recurrent (Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_2)$-uniformly recurrent of type* 1*)).*

Proof. We will only prove that the Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodicity of type 2 implies the Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodicity for the function $F(\cdot;\cdot)$. Keeping in mind the corresponding definitions, we only need to prove that, for every sequence (b_i) of positive real numbers, we have:

$$\left[\phi_2\left(\limsup_{k\to+\infty}\frac{1}{2k+1}\sum_{j=1}^{(2k+1)} b_i^{1/p}\right)\right]^p \leq \phi_2\left(\limsup_{k\to+\infty}\frac{1}{2k+1}\sum_{j=1}^{2k+1} b_i\right).$$

This simply follows from the fact that the function $\phi_2(\cdot)$ is monotonically increasing, the inequality

$$\frac{1}{(2k+1)}\sum_{j=1}^{2k+1} b_i^{1/p} \leq \left[\frac{1}{(2k+1)}\sum_{j=1}^{2k+1} b_i\right]^{1/p}$$

between the exponential means, the equality

$$\limsup_{k\to+\infty}\left[\frac{1}{2k+1}\sum_{j=1}^{(2k+1)} b_i\right]^{1/p} = \left[\limsup_{k\to+\infty}\frac{1}{2k+1}\sum_{j=1}^{2k+1} b_i\right]^{1/p},$$

and our assumption that $\phi_2(x) \geq [\phi_2(x^{1/p})]^p$ for all $x \geq 0$. □

Now we will reconsider [6, Proposition 2.2] for multi-dimensional Besicovitch almost periodic type functions; the best we can do is the following (see also Example 2):

Proposition 2.9. *Suppose that* $p(\cdot) \equiv p \in [1,\infty)$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda+\Lambda' \subseteq \Lambda$, $\Lambda'+(\Lambda'-\Lambda') \subseteq \Lambda'$, ρ_1 *is a binary relation on* Y *satisfying* $R(F) \subseteq D(\rho_1)$ *and* $\rho_1(y)$ *is a singleton for any* $y \in R(F)$. *If for each* $\tau \in \Lambda'$ *we have* $\tau + \Lambda = \Lambda$, $F(\cdot;\cdot)$ *is Besicovitch-*$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$*-almost periodic (of type* 2*) and there exists a finite real constant* $c > 0$ *such that*

$$\phi_1(x+y) \leq c\big[\phi_1(x)+\phi_1(y)\big], \quad x,\ y \geq 0, \tag{2.10}$$

then $\Lambda + (\Lambda' - \Lambda') \subseteq \Lambda$ *and* $F(\cdot;\cdot)$ *is Besicovitch-*$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda'-\Lambda', \mathrm{I}, \rho_2, \delta)$*-almost periodic (of type* 2*). The same statement holds for the corresponding classes of Besicovitch uniformly recurrent functions.*

Proof. We will consider the Besicovitch-$(p, \phi_i, \mathrm{F}_j, \mathcal{B}, \Lambda', \rho_1, \rho_2, \delta)$-almost periodic functions, only. The inclusion $\Lambda + (\Lambda' - \Lambda') \subseteq \Lambda$ can be proved as in [6]; suppose that a number $\epsilon > 0$ and a set $B \in \mathcal{B}$ are given. Then we know that there exists a Λ'_δ-satisfactorily uniform set $A = \{\tau_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}^n$ such that, for every $\tau \in A$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho_1(F(\cdot;x))$ such that (2.3) holds as well as, for every $l > 0$, $t > 0$, $x \in B$ and $\cdot \in \Lambda_t + l\Omega$, there exists an element $z_{\cdot;x} \in \rho_2(F(\cdot;x))$ such that (2.3) holds. Let $j \in \mathbb{Z}$ be fixed; then a very simple argumentation shows that the set $A_j := \{\tau_i - \tau_j : i \in \mathbb{Z}\}$ is $(\Lambda' - \Lambda')$-satisfactorily uniform since we have assumed that $\Lambda' + (\Lambda' - \Lambda') \subseteq \Lambda'$, and we only need to show that, for every fixed index $i \in \mathbb{Z}$, we have:

$$\limsup_{t\to+\infty} \mathrm{F}_1(t)\sup_{x\in B}\Big[\phi_1\Big(\big\|F\big(\cdot+\tau_i;x\big) - F\big(\cdot+\tau_j;x\big)\big\|_Y\Big)\Big]_{L^p(\Lambda_t)} < \text{Const.}\cdot\epsilon. \tag{2.11}$$

Toward this end, observe that our assumptions imply that there exists a sufficiently large real number $t_0(\epsilon) > 0$ such that, for every $t \geq t_0(\epsilon)$ and $x \in B$, we have

$$\begin{aligned}&\mathrm{F}_1(t)\Big[\phi_1\big(\|F(\cdot+\tau_i;x) - \rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)} < \epsilon\\ &\quad\text{and } \mathrm{F}_1(t)\Big[\phi_1\big(\|F(\cdot+\tau_j;x) - \rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)} < \epsilon.\end{aligned} \tag{2.12}$$

Keeping in mind (2.10), the above yields

$$
\begin{aligned}
&\mathrm{F}_1(t)\Big[\phi_1\big(\|F(\cdot+\tau_i;x)-F(\cdot+\tau_j;x)\|_Y\big)\Big]_{L^p(\Lambda_t)}\\
&\quad\leq c\mathrm{F}_1(t)\Big[\phi_1\big(\|F(\cdot+\tau_i;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)}\\
&\quad+c\mathrm{F}_1(t)\Big[\phi_1\big(\|F(\cdot+\tau_j;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)}\\
&\quad\leq 2c\mathrm{F}_1(t)\epsilon.
\end{aligned}
$$

This implies (2.12) and completes the proof. □

It is important to state the following corollary of Proposition 2.9:

Corollary 2.10. *Suppose that $\Lambda=\Lambda'=\mathbb{R}^n$, $\delta\in[1,2]$, $p(\cdot)\equiv p\in[1,\infty)$, $\phi_i(x)\equiv x$ for $i=1,2,3$, $\mathrm{F}_j(\mathbf{t})\equiv\mathbf{t}^{-n/p}$ for $j=1,2$, ρ_1 is a binary relation on Y satisfying $R(F)\subseteq D(\rho_1)$ and $\rho_1(y)$ is a singleton for any $y\in R(F)$. If for each $\tau\in\Lambda'$ we have $\tau+\Lambda=\Lambda$, $F(\cdot;\cdot)$ is Besicovitch-$(p,\phi_i,\mathrm{F}_j,\Lambda',\rho_1,\rho_2,\delta)$-almost periodic of type 2, and $\rho_2=\mathrm{I}$, then the function $F(\cdot)$ has the mean value*

$$
\mathcal{M}_\lambda(F):=\lim_{T\to+\infty}\frac{1}{(2T)^n}\int_{K_T}e^{-i\langle\lambda,\mathbf{t}\rangle}F(\mathbf{t})\,d\mathbf{t}=\lim_{T\to+\infty}\frac{1}{T^n}\int_{L_T}e^{-i\langle\lambda,\mathbf{t}\rangle}F(\mathbf{t})\,d\mathbf{t} \tag{2.13}
$$

for any $\lambda\in\mathbb{R}^n$, and the set of all points $\lambda\in\mathbb{R}^n$ for which $\mathcal{M}_\lambda(F)\neq 0$ is at most countable; here, for every $T>0$, $K_T:=\{(t_1,t_2,...,t_n)\in\mathbb{R}^n:|t_i|\leq T,\ i\in\mathbb{N}_n\}$ and $L_T:=\{(t_1,t_2,...,t_n)\in[0,\infty)^n:t_i\leq T,\ i\in\mathbb{N}_n\}$.

We close the paper with the observation that it would be very tempting to provide some new applications of the notion introduced in Definition 2.1 and Definition 2.4 to the (abstract) partial differential equations.

References

[1] J. Andres, A. M. Bersani, R. F. Grande, Hierarchy of almost-periodic function spaces, Rend. Mat. Appl. (7) **26** (2006), 121-188.

[2] A. S. Besicovitch, Almost Periodic Functions, Dover Publ., New York, 1954.

[3] T. Diagana, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, Springer-Verlag, New York, 2013.

[4] L. Diening, P. Harjulehto, P. Hästüso, M. Ruzicka, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, Springer, Heidelberg, 2011.

[5] S. Favorov, Bohr and Besicovitch almost periodic sets and quasicrystals, Proc. Amer. Math. Soc. **140** (2012), 1761–1767.

[6] M. Fečkan, M. T. Khalladi, M. Kostić, A. Rahmani, Multi-dimensional ρ-almost periodic type functions and applications, Appl. Anal., (2022), doi:10.1080/00036811.2022.2103678.

[7] A. M. Fink, Almost Periodic Differential Equations, Springer-Verlag, Berlin, 1974.

[8] G. M. N'Guérékata, Almost Automorphic and Almost Periodic Functions in Abstract Spaces, Kluwer Acad. Publ, Dordrecht, 2001.

[9] M. Kostić, Almost Periodic and Almost Automorphic Type Solutions to Integro-Differential Equations, W. de Gruyter, Berlin, 2019.

[10] M. Kostić, Selected Topics in Almost Periodicity, W. de Gruyter, Berlin, 2022.

[11] M. Kostić, Metrical ρ-Almost Periodicity and Applications, Book Manuscript, 2023.

[12] M. Kostić, W.-S. Du, V. E. Fedorov, Doss ρ-almost periodic type functions in $\mathbb{R}^n$, Mathematics **9** (2021), 2825. https://doi.org/10.3390/math9212825.

[13] M. Kostić, Multi-dimensional Besicovitch almost periodic type functions and applications, Comm. Pure Appl. Anal. **21** (2022), 4215–4250.

[14] M. Levitan, Almost Periodic Functions, G.I.T.T.L., Moscow, 1953 (in Russian).

[15] A. A. Pankov, Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations, Kluwer Acad. Publ., Dordrecht, 1990.