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Coupled fixed point results based on control function in complex partial metric spaces

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Abstract. The purpose of this article is to establish some coupled fixed point results based on control function in the setting of complex partial metric spaces. Furthermore, we give some consequences of the established result. An example is given to support the result. The results of findings in this article extend and generalize several results from the existing literature. Specially, our results extend the corresponding results of *Aydi* [4].

1. Introduction

In 1922, *Stefan Banach* ([6]) has proved a fixed point theorem for a contraction mapping in a complete metric space. It plays an important role in analysis to find a unique solution of many mathematical problems. It is very popular tool in many branches of mathematics for solving existing problems.

There are many extensions of the famous Banach contraction principle, which states that every self mapping $\mathcal{R}$ defined on a complete metric space $(\mathcal{X}, d)$ satisfying

$$d(\mathcal{R}(s), \mathcal{R}(t)) \leq \delta\, d(s, t), \tag{1}$$

for all $s, t \in \mathcal{X}$, where $\delta \in (0, 1)$, has a unique fixed point and for every $x_0 \in \mathcal{X}$, a sequence $\{\mathcal{R}^n x_0\}_{n\geq 1}$ is convergent to the fixed point. Inequality (1) also implies the continuity of $\mathcal{R}$.

In 2011, *Azam et al.* [5] introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The results proved by *Azam et al.* [5] and *Bhatt et al.* [8] via rational inequality in a complex valued metric space as a contractive condition. Complex valued metric space is very useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics, applied physics, mechanical engineering, thermodynamics and electrical engineering.

Recently many authors have done a wide range of research in complex valued metric spaces, see for example, Abbas *et al.* [1], Sintunavarat and Kumam [26], Rouzkard and Imdad [20], Kutbi *et al.* [17], Ahmed *et al.* [2], Sittikul and Saejung [27], Chandok and Kumar [9], Ansari et al. [3] and many others (see, also, [11], [13], [28]).

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Very recently, *Dhivya and Marudai* [12] introduced new spaces called complex partial metric space and demonstrated the existence of common fixed point results under contractive condition involving rational expression.

On the other hand, *Bhashkar and Lakshmikantham* [7] introduced the concept of coupled fixed points in ordered spaces and applied their results to boundary value problems for the unique solution. *Ciric and Lakshmikantham* [10] introduced the concept of coupled coincidence, common fixed points to nonlinear contractions in ordered metric spaces. More results on coupled fixed points, coupled coincidence points and common coupled fixed points in various spaces, one can see [4, 14–16, 18, 19, 22–25] and many others.

Aydi [4] demonstrated a number of coupled fixed point results for contractive type conditions in partial metric spaces and gave some examples in support of the results. Recently, *Kim et al.* [15] demonstrated some common coupled fixed point theorems for contractive type conditions in the setting of partial metric spaces.

Motivated by the works of *Aydi* [4], *Azam et al.* [5], *Bhashkar and Lakshmikantham* [7], *Dhivya and Marudai* [12] and some others, the goal of this article is to establish some coupled fixed point results based on control function in the framework of complex partial metric spaces. Also we give an example to support the result. The results obtain in this article extend and generalize several previously published results in the existing literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $c_1, c_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$c_1 \precsim c_2$ if and only if $Re(c_1) \leq Re(c_2)$, $Im(c_1) \leq Im(c_2)$. It follows that $c_1 \precsim c_2$ if one of the following conditions is satisfied:

(i) $Re(c_1) = Re(c_2)$, $Im(c_1) < Im(c_2)$;
(ii) $Re(c_1) < Re(c_2)$, $Im(c_1) = Im(c_2)$;
(iii) $Re(c_1) < Re(c_2)$, $Im(c_1) < Im(c_2)$;
(iv) $Re(c_1) = Re(c_2)$, $Im(c_1) = Im(c_2)$.

In particular, we will write $c_1 \lnsim c_2$ if $c_1 \neq c_2$ and one of (i), (ii), and (iii) is satisfied and we will write $c_1 \prec c_2$ if only (iii) is satisfied. Notice that:

($a1$) If $0 \precsim c_1 \lnsim c_2$, then $|c_1| < |c_2|$,
($a2$) If $c_1 \precsim c_2$ and $c_2 \prec c_3$, then $c_1 \prec c_3$,
($a3$) If $t_1, t_2 \in \mathbb{R}$ and $t_1 \leq t_2$, then $t_1 z \precsim t_2 z$ for all $z \in \mathbb{C}$.

In 2017, Dhivya and Marudai [12] define the following.

Definition 2.1. *([12]) Let $\mathcal{X}$ be a nonempty set and $\mathbb{C}$ be the set of all complex numbers. A complex partial metric space on $\mathcal{X}$ is a function $\mathcal{P}_c \colon \mathcal{X}^2 \to \mathbb{C}^+$ such that for all $p, q, r \in \mathcal{X}$:*

($CPM1$) $0 \precsim \mathcal{P}_c(p,p) \precsim \mathcal{P}_c(p,q)$ *(small self-distance),*
($CPM2$) $\mathcal{P}_c(p,q) = \mathcal{P}_c(q,p)$ *(symmetry),*
($CPM3$) $\mathcal{P}_c(p,p) = \mathcal{P}_c(p,q) = \mathcal{P}_c(q,q)$ *if and only if* $p = q$ *(equality),*
($CPM4$) $\mathcal{P}_c(p,q) \precsim \mathcal{P}_c(p,r) + \mathcal{P}_c(r,q) - \mathcal{P}_c(r,r)$ *(triangularity).*

A complex partial metric space is a pair $(\mathcal{X}, \mathcal{P}_c)$ such that $\mathcal{X}$ is a non-empty set and $\mathcal{P}_c$ is complex partial metric on $\mathcal{X}$.

For the complex partial metric $\mathcal{P}_c$ on $\mathcal{X}$, the function $d_{\mathcal{P}_c} \colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}^+$ given by

$$\mathcal{P}_c^t(p,q) = 2\mathcal{P}_c(p,q) - \mathcal{P}_c(p,p) - \mathcal{P}_c(q,q), \tag{2}$$

is a (usual) metric on $\mathcal{X}$.

Also note that each complex partial metric $\mathcal{P}_c$ on $\mathcal{X}$ generates a T_0 topology $\tau_{\mathcal{P}_c}$ on $\mathcal{X}$ with the base family of open $\mathcal{P}_c$-balls $\{\mathcal{B}_{\mathcal{P}_c}(p, \varepsilon) : p \in \mathcal{X}, \varepsilon > 0\}$ where

$$\mathcal{B}_{\mathcal{P}_c}(p,\varepsilon) = \{q \in \mathcal{X} : \mathcal{P}_c(p,q) < \mathcal{P}_c(p,p) + \varepsilon\},$$

for all $p \in \mathcal{X}$ and $0 \prec \varepsilon \in \mathbb{C}^+$.

Similarly, closed $\mathcal{P}_c$-ball is defined as

$$\mathcal{B}_{\mathcal{P}_c}[p,\varepsilon] = \{q \in \mathcal{X} : \mathcal{P}_c(p,q) \le \mathcal{P}_c(p,p) + \varepsilon\},$$

for all $p \in \mathcal{X}$ and $0 \prec \varepsilon \in \mathbb{C}^+$.

A complex valued metric space is a complex partial metric space. But a complex partial metric space need not be a complex valued metric space. The following example illustrates such a complex partial metric space.

Example 2.2. *([12]) Let $\mathcal{X} = [0,\infty)$ endowed with complex partial metric $\mathcal{P}_c$ is defined by $\mathcal{P}_c\colon \mathcal{X}\times\mathcal{X} \to \mathbb{C}^+$ with*

$$\mathcal{P}_c(p,q) = \max\{p,q\} + i\max\{p,q\} \text{ for all } p,q \in \mathcal{X}.$$

It is easy to verify that $(\mathcal{X},\mathcal{P}_c)$ is a complex partial metric space and note that self distance need not be zero, for example, $\mathcal{P}_c(1,1) = 1 + i \ne 0$. Now the metric induced by $\mathcal{P}_c$ is as follows

$$\mathcal{P}_c^t(p,q) = 2\mathcal{P}_c(p,q) - \mathcal{P}_c(p,p) - \mathcal{P}_c(q,q),$$

without loss of generality suppose $p \ge q$, then

$$\mathcal{P}_c^t(p,q) = 2\Big[\max\{p,q\} + i\max\{p,q\}\Big] - (p+ip) - (q+iq).$$

Therefore,

$$\mathcal{P}_c^t(p,q) = |p-q| + i|p-q| = |p-q|(1+i).$$

Definition 2.3. *([12]) Let $(\mathcal{X},\mathcal{P}_c)$ be a complex partial metric space (CPMS). A sequence $\{p_n\}$ in a CPMS $(\mathcal{X},\mathcal{P}_c)$ converges to p_0 if and only if for every $\varepsilon \in \mathbb{C}^+$ with $0 \prec \varepsilon$, there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$, we have $\mathcal{P}_c(p_n,p_0) \prec \varepsilon$ or $p_n \in \mathcal{B}_{\mathcal{P}_c}(p_0,\varepsilon)$ and we denote this by $p_n \to p_0$ or $\lim_{n\to\infty} p_n = p_0$.*

Definition 2.4. *([12]) Let $(\mathcal{X},\mathcal{P}_c)$ be a complex partial metric space (CPMS). A sequence $\{p_n\}$ in a CPMS $(\mathcal{X},\mathcal{P}_c)$ is called a Cauchy sequence if for every $\varepsilon \in \mathbb{C}^+$ with $0 \prec \varepsilon$ and $a \in \mathbb{C}^+$, there exists $N_0 \in \mathbb{N}$ such that for all $n,m \ge N_0$, we have $|\mathcal{P}_c(p_n,p_m) - a| < \varepsilon$.*

Definition 2.5. *([12]) Let $(\mathcal{X},\mathcal{P}_c)$ be a complex partial metric space (CPMS).*

• A CPMS $(\mathcal{X},\mathcal{P}_c)$ is said to be complete if a Cauchy sequence $\{p_n\}$ in $(\mathcal{X}$ converges with respect to $\tau_{\mathcal{P}_c}$ to a point $p_0 \in \mathcal{X}$ such that $\mathcal{P}_c(p_0,p_0) = \lim_{n,m\to+\infty} \mathcal{P}_c(p_n,p_m)$.

• A napping $\mathcal{G}\colon \mathcal{X} \to \mathcal{X}$ is said to be continuous at $p_0 \in \mathcal{X}$ if for every $0 \prec \varepsilon$, there exists $\alpha > 0$ such that $\mathcal{G}\big(\mathcal{B}_{\mathcal{P}_c}(p_0,\alpha)\big) \subset \mathcal{B}_{\mathcal{P}_c}\big(\mathcal{G}(p_0),\varepsilon\big)$.

Definition 2.6. *Let $\mathcal{X}$ be a non-empty set and let $\mathcal{P},\mathcal{Q}\colon \mathcal{X} \to \mathcal{X}$ be two self mappings of $\mathcal{X}$. Then a point $\gamma \in \mathcal{X}$ is called a*

(Γ_1) *fixed point of operator $\mathcal{P}$ if $\mathcal{P}(\gamma) = \gamma$,*

(Γ_2) *common fixed point of $\mathcal{P}$ and $\mathcal{Q}$ if $\mathcal{P}(\gamma) = \mathcal{Q}(\gamma) = \gamma$.*

Definition 2.7. *Let $(\mathcal{X},\mathcal{P}_c)$ be a complex partial metric space (CPMS). Then an element $(p,q) \in \mathcal{X}\times\mathcal{X}$ is said to be a coupled fixed point of the mapping $F\colon \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ if $F(p,q) = p$ and $F(q,p) = q$.*

Example 2.8. *Let $\mathcal{X} = [0,+\infty)$ and $F\colon \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ defined by $F(p,q) = \frac{p+q}{3}$ for all $p,q \in \mathcal{X}$. One can easily see that F has a unique coupled fixed point $(0,0)$.*

Example 2.9. *Let $\mathcal{X} = [0, +\infty)$ and $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be defined by $F(p, q) = \frac{p+q}{2}$ for all $p, q \in \mathcal{X}$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.*

Lemma 2.10. *([12]) Let $(\mathcal{X}, \mathcal{P}_c)$ be a complex partial metric space (CPMS). A sequence $\{p_n\}$ is a Cauchy sequence in the CPMS $(\mathcal{X}, \mathcal{P}_c)$, then $\{p_n\}$ is Cauchy in a metric space $(\mathcal{X}, \mathcal{P}_c^t)$.*

In the present paper, we will denote the control function Ω as follows:

Definition 2.11. *Let Ω be the set of functions $\omega\colon [0, \infty) \to [0, \infty)$ satisfying the following conditions:*
(Ω_1) *ω is continuous;*
(Ω_2) *$\omega(t) < t$ for all $t > 0$.*
Obviously, if $\omega \in \Omega$, then $\omega(0) = 0$ and $\omega(t) \leq t$ for all $t \geq 0$.

3. Main Results

In this section, we shall prove some unique coupled fixed point theorems based on control function in the framework of complex partial metric spaces.

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $p, q, r, s \in \mathcal{X}$:*

$$\mathcal{P}_c(F(p, q), F(r, s) \precsim R_1\, \Delta_1(p, q, r, s) + R_2\, \Delta_2(p, q, r, s), \tag{3}$$

where

$$\Delta_1(p, q, r, s) = \omega\Big(\mathcal{P}_c(F(r, s), r)\frac{1 + \mathcal{P}_c(F(p, q), p)}{1 + \mathcal{P}_c(p, r)}\Big), \tag{4}$$

$$\begin{aligned}\Delta_2(p, q, r, s) = \max\Big\{&\omega(\mathcal{P}_c(p, r)), \omega(\mathcal{P}_c(q, s)), \omega(\mathcal{P}_c(F(p, q), p)),\\ &\omega(\mathcal{P}_c(F(r, s), r)), \omega(\mathcal{P}_c(F(p, q), r)),\\ &\omega(\mathcal{P}_c(F(q, p), s))\Big\},\end{aligned} \tag{5}$$

R_1, R_2 are nonnegative constants with $R_1 + R_2 < 1$ and $\omega \in \Omega$. Then F has a unique coupled fixed point.

Proof. Let $p_0, q_0 \in \mathcal{X}$ be arbitrary points. Set $p_1 = F(p_0, q_0)$ and $q_1 = F(q_0, p_0)$. Repeating this process, we obtain two sequences $\{p_n\}$ and $\{q_n\}$ in $\mathcal{X}$ such that $p_{n+1} = F(p_n, q_n)$ and $q_{n+1} = F(q_n, p_n)$. Let $\xi_n = \mathcal{P}_c(p_n, p_{n+1})$ and $\rho_n = \mathcal{P}_c(q_n, q_{n+1})$. Then, from equations (3)-(5) and using $(CPM1)$, $(CPM2)$, we have

$$\begin{aligned}\mathcal{P}_c(p_n, p_{n+1}) &= \mathcal{P}_c(F(p_{n-1}, q_{n-1}), F(p_n, q_n))\\ &\precsim R_1\, \Delta_1(p_{n-1}, q_{n-1}, p_n, q_n) + R_2\, \Delta_2(p_{n-1}, q_{n-1}, p_n, q_n),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\Delta_1(p_{n-1}, q_{n-1}, p_n, q_n) &= \omega\Big(\mathcal{P}_c(F(p_n, q_n), p_n)\frac{1 + \mathcal{P}_c(F(p_{n-1}, q_{n-1}), p_{n-1})}{1 + \mathcal{P}_c(p_{n-1}, p_n)}\Big)\\ &= \omega\Big(\mathcal{P}_c(p_{n+1}, p_n)\frac{1 + \mathcal{P}_c(p_n, p_{n-1})}{1 + \mathcal{P}_c(p_{n-1}, p_n)}\Big)\\ &= \omega\Big(\mathcal{P}_c(p_n, p_{n+1})\Big) = \omega(\xi_n),\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\Delta_2(p_{n-1}, q_{n-1}, p_n, q_n) &= \max\Big\{\omega(\mathcal{P}_c(p_{n-1}, p_n)), \omega(\mathcal{P}_c(q_{n-1}, q_n)),\\
&\qquad \omega(\mathcal{P}_c(F(p_{n-1}, q_{n-1}), p_{n-1})), \omega(\mathcal{P}_c(F(p_n, q_n), p_n)),\\
&\qquad \omega(\mathcal{P}_c(F(p_{n-1}, q_{n-1}), p_n)), \omega(\mathcal{P}_c(F(q_{n-1}, p_{n-1}), q_n))\Big\}\\
&= \max\Big\{\omega(\mathcal{P}_c(p_{n-1}, p_n)), \omega(\mathcal{P}_c(q_{n-1}, q_n)), \omega(\mathcal{P}_c(p_n, p_{n-1})),\\
&\qquad \omega(\mathcal{P}_c(p_{n+1}, p_n)), \omega(\mathcal{P}_c(p_n, p_n)), \omega(\mathcal{P}_c(q_n, q_n))\Big\}\\
&\precsim \max\Big\{\omega(\mathcal{P}_c(p_{n-1}, p_n)), \omega(\mathcal{P}_c(q_{n-1}, q_n)), \omega(\mathcal{P}_c(p_{n-1}, p_n)),\\
&\qquad \omega(\mathcal{P}_c(p_n, p_{n+1})), \omega(\mathcal{P}_c(p_n, p_{n+1})), \omega(\mathcal{P}_c(q_n, q_{n+1}))\Big\}\\
&= \max\Big\{\omega(\mathcal{P}_c(p_{n-1}, p_n)), \omega(\mathcal{P}_c(q_{n-1}, q_n)),\\
&\qquad \omega(\mathcal{P}_c(p_n, p_{n+1})), \omega(\mathcal{P}_c(q_n, q_{n+1}))\Big\}\\
&= \max\Big\{\omega(\xi_{n-1}), \omega(\rho_{n-1}), \omega(\xi_n), \omega(\rho_n)\Big\}. \qquad (8)
\end{aligned}$$

From equations (6)-(8), we obtain

$$\mathcal{P}_c(p_n, p_{n+1}) \precsim R_1\, \omega(\xi_n) + R_2\, \max\Big\{\omega(\xi_{n-1}), \omega(\rho_{n-1}), \omega(\xi_n), \omega(\rho_n)\Big\},$$

which implies that

$$|\mathcal{P}_c(p_n, p_{n+1})| \le R_1\, |\omega(\xi_n)| + R_2 \Big|\max\Big\{\omega(\xi_{n-1}), \omega(\rho_{n-1}), \omega(\xi_n), \omega(\rho_n)\Big\}\Big|. \qquad (9)$$

Similarly, we obtain

$$\begin{aligned}
\mathcal{P}_c(q_n, q_{n+1}) &= \mathcal{P}_c(F(q_{n-1}, p_{n-1}), F(q_n, p_n))\\
&\precsim R_1\, \Delta_1(q_{n-1}, p_{n-1}, q_n, p_n) + R_2\, \Delta_2(q_{n-1}, p_{n-1}, q_n, p_n), \qquad (10)
\end{aligned}$$

where

$$\Delta_1(q_{n-1}, p_{n-1}, q_n, p_n) = \omega\Big(\mathcal{P}_c(q_n, q_{n+1})\Big) = \omega(\rho_n), \qquad (11)$$

and

$$\Delta_2(q_{n-1}, p_{n-1}, q_n, p_n) = \max\Big\{\omega(\rho_{n-1}), \omega(\xi_{n-1}), \omega(\rho_n), \omega(\xi_n)\Big\}. \qquad (12)$$

From equations (10)-(12), we obtain

$$\mathcal{P}_c(q_n, q_{n+1}) \precsim R_1\, \omega(\rho_n) + R_2\, \max\Big\{\omega(\rho_{n-1}), \omega(\xi_{n-1}), \omega(\rho_n), \omega(\xi_n)\Big\},$$

which implies that

$$|\mathcal{P}_c(q_n, q_{n+1})| \le R_1\, |\omega(\rho_n)| + R_2 \Big|\max\Big\{\omega(\rho_{n-1}), \omega(\xi_{n-1}), \omega(\rho_n), \omega(\xi_n)\Big\}\Big|. \qquad (13)$$

Set

$$\begin{aligned}
\lambda_n &= |\mathcal{P}_c(p_n, p_{n+1})| + |\mathcal{P}_c(q_n, q_{n+1})|\\
&= |\xi_n| + |\rho_n|, \qquad (14)
\end{aligned}$$

$$\Omega_1 = \Big|\max\Big\{\omega(\xi_{n-1}), \omega(\rho_{n-1}), \omega(\xi_n), \omega(\rho_n)\Big\}\Big|, \tag{15}$$

and

$$\Omega_2 = \Big|\max\Big\{\omega(\rho_{n-1}), \omega(\xi_{n-1}), \omega(\rho_n), \omega(\xi_n)\Big\}\Big|. \tag{16}$$

Consider the following **possible cases**.

Case 1. If $\Omega_1 = |\omega(\xi_{n-1})|$ and $\Omega_2 = |\omega(\xi_{n-1})|$, then from equations (9), (13)-(16), we obtain

$$\lambda_n \le R_1\,(|\omega(\xi_n)| + |\omega(\rho_n)|) + R_2\,(|\omega(\xi_{n-1})| + |\omega(\xi_{n-1})|),$$

using the fact that $\omega(t) < t$ for all $t > 0$, then we obtain

$$\begin{aligned}\lambda_n &\le R_1\,(|\xi_n| + |\rho_n|) + R_2\,(|\xi_{n-1}| + |\xi_{n-1}|)\\ &= R_1\,\lambda_n + 2R_2\,|\xi_{n-1}|.\end{aligned}$$

This implies that

$$(1 - R_1)\,\lambda_n \le 2R_2\,|\xi_{n-1}|. \tag{17}$$

Case 2. If $\Omega_1 = |\omega(\rho_{n-1})|$ and $\Omega_2 = |\omega(\rho_{n-1})|$, then from equations (9), (13)-(16), we obtain

$$\lambda_n \le R_1\,(|\omega(\xi_n)| + |\omega(\rho_n)|) + R_2\,(|\omega(\rho_{n-1})| + |\omega(\rho_{n-1})|),$$

using the fact that $\omega(t) < t$ for all $t > 0$, then we obtain

$$\begin{aligned}\lambda_n &\le R_1\,(|\xi_n| + |\rho_n|) + R_2\,(|\rho_{n-1}| + |\rho_{n-1}|)\\ &= R_1\,\lambda_n + 2R_2\,|\rho_{n-1}|.\end{aligned}$$

This implies that

$$(1 - R_1)\,\lambda_n \le 2R_2\,|\rho_{n-1}|. \tag{18}$$

From equations (17) and (18), we obtain

$$2(1 - R_1)\,\lambda_n \le 2R_2\,(|\xi_{n-1}| + |\rho_{n-1}|) = 2R_2\,\lambda_{n-1},$$

or

$$\lambda_n \le \Big(\frac{R_2}{1 - R_1}\Big)\,\lambda_{n-1}. \tag{19}$$

Case 3. If $\Omega_1 = |\omega(\xi_n)|$ and $\Omega_2 = |\omega(\xi_n)|$, then from equations (9), (13)-(16), we obtain

$$\lambda_n \le R_1\,(|\omega(\xi_n)| + |\omega(\rho_n)|) + R_2\,(|\omega(\xi_n)| + |\omega(\xi_n)|),$$

using the fact that $\omega(t) < t$ for all $t > 0$, then we obtain

$$\begin{aligned}\lambda_n &\le R_1\,(|\xi_n| + |\rho_n|) + R_2\,(|\xi_n| + |\xi_n|)\\ &= R_1\,\lambda_n + 2R_2\,|\xi_n|.\end{aligned}$$

This implies that

$$(1 - R_1)\,\lambda_n \le 2R_2\,|\xi_n|. \tag{20}$$

Case 4. If $\Omega_1 = |\omega(\rho_n)|$ and $\Omega_2 = |\omega(\rho_n)|$, then from equations (9), (13)-(16), we obtain

$$\lambda_n \leq R_1\,(|\omega(\xi_n)| + |\omega(\rho_n)|) + R_2\,(|\omega(\rho_n)| + |\omega(\rho_n)|),$$

using the fact that $\omega(t) < t$ for all $t > 0$, then we obtain

$$\begin{aligned}\lambda_n &\leq R_1\,(|\xi_n| + |\rho_n|) + R_2\,(|\rho_n| + |\rho_n|)\\ &= R_1\,\lambda_n + 2R_2\,|\rho_n|.\end{aligned}$$

This implies that

$$(1 - R_1)\,\lambda_n \leq 2R_2\,|\rho_n|. \tag{21}$$

From equations (20) and (21), we obtain

$$2(1 - R_1)\,\lambda_n \leq 2R_2\,(|\xi_n| + |\rho_n|) = 2R_2\,\lambda_n,$$

or

$$\lambda_n \leq (R_1 + R_2)\,\lambda_n < \lambda_n,$$

which is a contradiction, since $R_1 + R_2 < 1$. Hence the inequality (3) is satisfied.

Thus from equation (19), we have

$$\lambda_n \leq \Big(\frac{R_2}{1 - R_1}\Big)\,\lambda_{n-1} = \tau\,\lambda_{n-1}, \tag{22}$$

where $\tau = \Big(\frac{R_2}{1-R_1}\Big) < 1$, since $R_1 + R_2 < 1$.

Then for each $n \in \mathbb{N}$, we have

$$\lambda_n \leq \tau\lambda_{n-1} \leq \tau^2\lambda_{n-2} \leq \cdots \leq \tau^n\lambda_0. \tag{23}$$

If $\lambda_0 = 0$, then $|\mathcal{P}_c(p_0, p_1)| + |\mathcal{P}_c(q_0, q_1)| = 0$. Hence, we get $p_0 = p_1 = F(p_0, q_0)$ and $q_0 = q_1 = F(q_0, p_0)$, which shows that (p_0, q_0) is a coupled fixed point of F. Now, we assume that $\lambda_0 > 0$. For each $n \geq m$, where $n, m \in \mathbb{N}$, we have, by using condition ($CPM4$)

$$\begin{aligned}\mathcal{P}_c(p_n, p_m) &\precsim \mathcal{P}_c(p_n, p_{n-1}) + \mathcal{P}_c(p_{n-1}, p_{n-2}) + \dots\\ &\quad + \mathcal{P}_c(p_{m+1}, p_m) - \mathcal{P}_c(p_{n-1}, p_{n-1}) - \mathcal{P}_c(p_{n-2}, p_{n-2})\\ &\quad - \cdots - \mathcal{P}_c(p_{m+1}, p_{m+1})\\ &\precsim \mathcal{P}_c(p_n, p_{n-1}) + \mathcal{P}_c(p_{n-1}, p_{n-2}) + \cdots + \mathcal{P}_c(p_{m+1}, p_m),\end{aligned}$$

which implies that

$$|\mathcal{P}_c(p_n, p_m)| \leq |\mathcal{P}_c(p_n, p_{n-1})| + |\mathcal{P}_c(p_{n-1}, p_{n-2})| + \cdots + |\mathcal{P}_c(p_{m+1}, p_m)|. \tag{24}$$

Similarly, we have

$$\begin{aligned}\mathcal{P}_c(q_n, q_m) &\precsim \mathcal{P}_c(q_n, q_{n-1}) + \mathcal{P}_c(q_{n-1}, q_{n-2}) + \dots\\ &\quad + \mathcal{P}_c(q_{m+1}, q_m) - \mathcal{P}_c(q_{n-1}, q_{n-1}) - \mathcal{P}_c(q_{n-2}, q_{n-2})\\ &\quad - \cdots - \mathcal{P}_c(q_{m+1}, q_{m+1})\\ &\precsim \mathcal{P}_c(q_n, q_{n-1}) + \mathcal{P}_c(q_{n-1}, q_{n-2}) + \cdots + \mathcal{P}_c(q_{m+1}, q_m),\end{aligned}$$

which implies that

$$|\mathcal{P}_c(q_n, q_m)| \leq |\mathcal{P}_c(q_n, q_{n-1})| + |\mathcal{P}_c(q_{n-1}, q_{n-2})| + \cdots + |\mathcal{P}_c(q_{m+1}, q_m)|. \tag{25}$$

Thus,

$$\begin{aligned}
\lambda_n &= |\mathcal{P}_c(p_n, p_m)| + |\mathcal{P}_c(q_n, q_m)| \\
&\leq \lambda_{n-1} + \lambda_{n-2} + \lambda_{n-3} + \cdots + \lambda_m \\
&\leq (\tau^{n-1} + \tau^{n-2} + \cdots + \tau^m)\lambda_0 \\
&\leq \Big(\frac{\tau^m}{1-\tau}\Big)\lambda_0 \\
&\leq \Big(\frac{\tau^n}{1-\tau}\Big)\lambda_0 \to 0 \text{ as } n \to +\infty, \qquad (26)
\end{aligned}$$

which implies that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in $(\mathcal{X}, \mathcal{P}_c)$ because $0 \leq \tau < 1$. Since the complex partial metric space $(\mathcal{X}, \mathcal{P}_c)$ is complete, so there exist $L, M \in \mathcal{X}$ such that $p_n \to L$, $q_n \to M$ as $n \to +\infty$ and

$$\mathcal{P}_c(L, L) = \lim_{n\to\infty} \mathcal{P}_c(p_n, L) = \lim_{n,m\to\infty} \mathcal{P}_c(p_n, p_m) = 0, \qquad (27)$$

$$\mathcal{P}_c(M, M) = \lim_{n\to\infty} \mathcal{P}_c(q_n, M) = \lim_{n,m\to\infty} \mathcal{P}_c(q_n, q_m) = 0. \qquad (28)$$

We now show that $L = F(L, M)$. Suppose on the contrary that $L \neq F(L, M)$ and $M \neq F(M, L)$ so that $0 \prec \mathcal{P}_c(L, F(L, M)) = W_1$ and $0 \prec \mathcal{P}_c(M, F(M, L)) = W_2$, then

$$\begin{aligned}
W_1 &= \mathcal{P}_c(L, F(L, M)) \\
&\precsim \mathcal{P}_c(L, p_{n+1}) + \mathcal{P}_c(p_{n+1}, F(L, M)) \\
&\quad -\mathcal{P}_c(p_{n+1}, p_{n+1}) \\
&\precsim \mathcal{P}_c(L, p_{n+1}) + \mathcal{P}_c(p_{n+1}, F(L, M)) \\
&= \mathcal{P}_c(p_{n+1}, F(L, M)) + \mathcal{P}_c(L, p_{n+1}) \\
&= \mathcal{P}_c(F(p_n, q_n), F(L, M)) + \mathcal{P}_c(L, p_{n+1}) \\
&\precsim R_1\, \Delta_1(p_n, q_n, L, M) + R_2\, \Delta_2(p_n, q_n, L, M) \\
&\quad +\mathcal{P}_c(L, p_{n+1}), \qquad (29)
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1(p_n, q_n, L, M) &= \omega\Big(\mathcal{P}_c(F(L, M), L)\frac{1 + \mathcal{P}_c(F(p_n, q_n), p_n)}{1 + \mathcal{P}_c(p_n, L)}\Big) \\
&= \omega\Big(\mathcal{P}_c(F(L, M), L)\frac{1 + \mathcal{P}_c(p_{n+1}, p_n)}{1 + \mathcal{P}_c(p_n, L)}\Big), \qquad (30)
\end{aligned}$$

passing to the limit as $n \to +\infty$ in equation (30) and using equation (27), we obtain

$$\Delta_1(p_n, q_n, L, M) \to \omega\Big(\mathcal{P}_c(F(L, M), L)\Big), \qquad (31)$$

and

$$\begin{aligned}
\Delta_2(p_n, q_n, L, M) &= \max\Big\{\omega(\mathcal{P}_c(p_n, L)), \omega(\mathcal{P}_c(q_n, M)), \omega(\mathcal{P}_c(F(p_n, q_n), p_n)), \\
&\qquad \omega(\mathcal{P}_c(F(L, M), L)), \omega(\mathcal{P}_c(F(p_n, q_n), L)), \\
&\qquad \omega(\mathcal{P}_c(F(q_n, p_n), M))\Big\} \\
&= \max\Big\{\omega(\mathcal{P}_c(p_n, L)), \omega(\mathcal{P}_c(q_n, M)), \omega(\mathcal{P}_c(p_{n+1}, p_n)), \\
&\qquad \omega(\mathcal{P}_c(F(L, M), L)), \omega(\mathcal{P}_c(p_{n+1}, L)), \\
&\qquad \omega(\mathcal{P}_c(q_{n+1}, M))\Big\}, \qquad (32)
\end{aligned}$$

passing to the limit as $n \to +\infty$ in equation (32), using equations (27), (28) and the property of ω, that is, $\omega(0) = 0$, we obtain

$$\Delta_2(p_n, q_n, L, M) \to \omega\Big(\mathcal{P}_c(F(L,M), L)\Big). \tag{33}$$

Now from equations (29), (31) and (33), we obtain

$$\begin{aligned} W_1 &= \mathcal{P}_c(L, F(L,M)) \\ &\precsim R_1\, \omega\Big(\mathcal{P}_c(F(L,M), L)\Big) + R_2\, \omega\Big(\mathcal{P}_c(F(L,M), L)\Big) \\ &\quad + \mathcal{P}_c(L, p_{n+1}). \end{aligned} \tag{34}$$

Passing to the limit as $n \to +\infty$ in equation (34) and using equation (27), we obtain

$$\begin{aligned} W_1 &= \mathcal{P}_c(L, F(L,M)) \\ &\precsim R_1\, \omega\Big(\mathcal{P}_c(F(L,M), L)\Big) + R_2\, \omega\Big(\mathcal{P}_c(F(L,M), L)\Big) \\ &= (R_1 + R_2)\, \omega\Big(\mathcal{P}_c(F(L,M), L)\Big), \end{aligned}$$

which implies that

$$\begin{aligned} |W_1| &= |\mathcal{P}_c(L, F(L,M))| \\ &\leq (R_1 + R_2)\, |\omega\Big(\mathcal{P}_c(L, F(L,M))\Big)| \\ &= (R_1 + R_2)\, |\omega\Big(W_1\Big)|. \end{aligned} \tag{35}$$

Similarly, one can obtain

$$\begin{aligned} |W_2| &= |\mathcal{P}_c(M, F(M,L))| \\ &\leq (R_1 + R_2)\, |\omega\Big(\mathcal{P}_c(M, F(M,L))\Big)| \\ &= (R_1 + R_2)\, |\omega\Big(W_2\Big)|. \end{aligned} \tag{36}$$

Hence from equations (35) and (36), we obtain

$$|W_1| + |W_2| \leq (R_1 + R_2)\Big(|\omega\Big(W_1\Big)| + |\omega\Big(W_2\Big)|\Big),$$

using the fact that $\omega(t) < t$ for all $t > 0$, we obtain

$$\begin{aligned} |W_1| + |W_2| &\leq (R_1 + R_2)\,(|W_1| + |W_2|) \\ &< |W_1| + |W_2|, \end{aligned}$$

which is a contradiction, since $R_1 + R_2 < 1$. Hence, we conclude that $|W_1| + |W_2| = 0$, that is, $|\mathcal{P}_c(L, F(L,M))| + |\mathcal{P}_c(M, F(M,L))| = 0$ and hence $\mathcal{P}_c(L, F(L,M)) = 0$ and $\mathcal{P}_c(M, F(M,L)) = 0$. Thus, $F(L,M) = L$ and $F(M,L) = M$. This shows that (L,M) is a coupled fixed point of F.

Now, we show the uniqueness. Suppose that (L_1, M_1) is another coupled fixed point of F such that $(L,M) \neq (L_1, M_1)$, then from equations (3)-(5) and using equations (27), (28) and $(CPM2)$, we have

$$\begin{aligned} \mathcal{P}_c(L, L_1) &= \mathcal{P}_c(F(L,M), F(L_1, M_1)) \\ &\precsim R_1\, \Delta_1(L, M, L_1, M_1) + R_2\, \Delta_2(L, M, L_1, M_1), \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Delta_1(L, M, L_1, M_1) &= \omega\Big(\mathcal{P}_c(F(L_1, M_1), L_1)\frac{1 + \mathcal{P}_c(F(L,M), L)}{1 + \mathcal{P}_c(L, L_1)}\Big) \\ &= \omega\Big(\mathcal{P}_c(L_1, L_1)\frac{1 + \mathcal{P}_c(L, L)}{1 + \mathcal{P}_c(L, L_1)}\Big), \end{aligned}$$

using equation (27) and the property of ω, that is, $\omega(0) = 0$, we obtain

$$\Delta_1(L, M, L_1, M_1) \to 0, \tag{38}$$

and

$$\begin{aligned}\Delta_2(L, M, L_1, M_1) &= \max\Big\{\omega(\mathcal{P}_c(L, L_1)), \omega(\mathcal{P}_c(M, M_1)), \omega(\mathcal{P}_c(F(L, M), L)),\\ &\quad \omega(\mathcal{P}_c(F(L_1, M_1), L_1)), \omega(\mathcal{P}_c(F(L, M), L1)),\\ &\quad \omega(\mathcal{P}_c(F(M, L), M_1))\Big\}\\ &= \max\Big\{\omega(\mathcal{P}_c(L, L_1)), \omega(\mathcal{P}_c(M, M_1)), \omega(\mathcal{P}_c(L, L)),\\ &\quad \omega(\mathcal{P}_c(L_1, L_1)), \omega(\mathcal{P}_c(L, L1)), \omega(\mathcal{P}_c(M, M_1))\Big\},\end{aligned}$$

using equation (27) and the property of ω, that is, $\omega(0) = 0$, we obtain

$$\Delta_2(L, M, L_1, M_1) \to \max\Big\{\omega(\mathcal{P}_c(L, L_1)), \omega(\mathcal{P}_c(M, M_1)), 0\Big\}. \tag{39}$$

From equations (37)-(39), we obtain

$$\begin{aligned}\mathcal{P}_c(L, L_1) &= \mathcal{P}_c(F(L, M), F(L_1, M_1))\\ &\precsim R_2 \max\Big\{\omega(\mathcal{P}_c(L, L_1)), \omega(\mathcal{P}_c(M, M_1)), 0\Big\},\end{aligned}$$

which implies that

$$\begin{aligned}|\mathcal{P}_c(L, L_1)| &= |\mathcal{P}_c(F(L, M), F(L_1, M_1))|\\ &\le R_2 \Big|\max\Big\{\omega(\mathcal{P}_c(L, L_1)), \omega(\mathcal{P}_c(M, M_1)), 0\Big\}\Big|.\end{aligned} \tag{40}$$

Similarly, one can obtain

$$\begin{aligned}|\mathcal{P}_c(M, M_1)| &= |\mathcal{P}_c(F(M, L), F(M_1, L_1))|\\ &\le R_2 \Big|\max\Big\{\omega(\mathcal{P}_c(L, L_1)), \omega(\mathcal{P}_c(M, M_1)), 0\Big\}\Big|.\end{aligned} \tag{41}$$

Set

$$F = \mathcal{P}_c(L, L_1),\ G = \mathcal{P}_c(M, M_1),\ H = |F| + |G|. \tag{42}$$

Now, we consider the following **possible cases**.

Case 1^0. If $\max\Big\{\omega(F), \omega(G), 0\Big\} = \omega(F)$, then from equations (40)-(42), we obtain

$$\begin{aligned}H &= |\mathcal{P}_c(L, L_1)| + |\mathcal{P}_c(M, M_1)|\\ &\le 2R_2\,|\omega(F)|.\end{aligned} \tag{43}$$

Case 2^0. If $\max\Big\{\omega(F), \omega(G), 0\Big\} = \omega(G)$, then from equations (40)-(42), we obtain

$$\begin{aligned}H &= |\mathcal{P}_c(L, L_1)| + |\mathcal{P}_c(M, M_1)|\\ &\le 2R_2\,|\omega(G)|.\end{aligned} \tag{44}$$

From equations (43) and (44), we obtain

$$2H \le 2R_2\,(|\omega(F)| + |\omega(G)|),$$

using the fact that $\omega(t) < t$ for all $t > 0$, then we obtain

$$2H \le 2R_2\,(|F| + |G|),$$

or

$$\begin{aligned} H &\leq R_2\,(|F| + |G|) = R_2\, H \\ &\leq (R_1 + R_2)\, H < H, \end{aligned}$$

which is a contradiction, since $R_1 + R_2 < 1$. Hence, we conclude that $H = 0$, that is, $|\mathcal{P}_c(L, L_1)| + |\mathcal{P}_c(M, M_1)| = 0$ or $\mathcal{P}_c(L, L_1) = 0$ and $\mathcal{P}_c(M, M_1) = 0$ and hence $L = L_1$ and $M = M_1$.

Case 3^0. If $\max\Big\{\omega(F), \omega(G), 0\Big\} = 0$, then from equations (40)-(42), we obtain

$$\begin{aligned} H &= |\mathcal{P}_c(L, L_1)| + |\mathcal{P}_c(M, M_1)| \\ &\leq R_2\,(0 + 0) = 0. \end{aligned}$$

Hence we conclude that $H = 0$, that is, $|\mathcal{P}_c(L, L_1)| + |\mathcal{P}_c(M, M_1)| = 0$ or $\mathcal{P}_c(L, L_1) = 0$ and $\mathcal{P}_c(M, M_1) = 0$ and hence $L = L_1$ and $M = M_1$.

Thus in both the above cases, we obtain $L = L_1$ and $M = M_1$. This shows that the coupled fixed point of F is unique. This completes the proof. □

4. Consequences of Theorem 3.1

By taking $R_1 = k$ and $R_2 = 0$ in Theorem 3.1, then we obtain the following result.

Corollary 4.1. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $p, q, r, s \in \mathcal{X}$:*

$$\mathcal{P}_c(F(p, q), F(r, s)) \quad \precsim \quad k\,\omega\Big(\mathcal{P}_c(F(r, s), r)\frac{1 + \mathcal{P}_c(F(p, q), p)}{1 + \mathcal{P}_c(p, r)}\Big),$$

where $k \in [0, 1)$ is a constant and $\omega \in \Omega$. Then F has a unique coupled fixed point.

By taking $R_1 = 0$ and $R_2 = h$ in Theorem 3.1, then we obtain the following result.

Corollary 4.2. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $p, q, r, s \in \mathcal{X}$:*

$$\begin{aligned} \mathcal{P}_c(F(p, q), F(r, s)) \quad \precsim \quad h\,\max\Big\{&\omega(\mathcal{P}_c(p, r)), \omega(\mathcal{P}_c(q, s)), \omega(\mathcal{P}_c(F(p, q), p)), \\ &\omega(\mathcal{P}_c(F(r, s), r)), \omega(\mathcal{P}_c(F(p, q), r)), \\ &\omega(\mathcal{P}_c(F(q, p), s))\Big\}, \end{aligned}$$

where $h \in [0, 1)$ is a constant and $\omega \in \Omega$. Then F has a unique coupled fixed point.

Corollary 4.3. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying all the conditions of Theorem 3.1, except that the condition (3) is replaced by the following contractive condition:*

$$\begin{aligned} \mathcal{P}_c(F(p, q), F(r, s)) \quad \precsim \quad & a_1\,\omega(\mathcal{P}_c(p, r)) + a_2\,\omega(\mathcal{P}_c(q, s)) + a_3\,\omega(\mathcal{P}_c(F(p, q), p)) \\ & + a_4\,\omega(\mathcal{P}_c(F(r, s), r)) + a_5\,\omega(\mathcal{P}_c(F(p, q), r)) \\ & + a_6\,\omega(\mathcal{P}_c(F(q, p), s)) \\ & + a_7\,\omega\Big(\mathcal{P}_c(F(r, s), r)\frac{1 + \mathcal{P}_c(F(p, q), p)}{1 + \mathcal{P}_c(p, r)}\Big), \end{aligned}$$

for all $p, q, r, s \in \mathcal{X}$, where $\omega \in \Omega$ and some nonnegative constants a_i $(i = 1, 2, \ldots, 7)$ with $a_1 + a_2 + \cdots + a_7 < 1$. Then F has a unique coupled fixed point.

Corollary 4.4. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying all the conditions of Theorem 3.1, except that the condition (3) is replaced by the following contractive condition:*

$$\begin{aligned}\mathcal{P}_c(F(p,q),F(r,s)) &\precsim a_1\,\omega(\mathcal{P}_c(p,r)) + a_2\,\omega(\mathcal{P}_c(q,s)) + a_3\,\omega(\mathcal{P}_c(F(p,q),p)) \\ &\quad + a_4\,\omega(\mathcal{P}_c(F(r,s),r)) + a_5\,\omega(\mathcal{P}_c(F(p,q),r)) \\ &\quad + a_6\,\omega(\mathcal{P}_c(F(q,p),s)),\end{aligned}$$

for all $p,q,r,s \in \mathcal{X}$, where $\omega \in \Omega$ and some nonnegative constants a_i $(i = 1,2,\dots,6)$ with $a_1 + a_2 + \cdots + a_6 < 1$. Then F has a unique coupled fixed point.

Proof. Follows from Corollary 4.2 by taking

$$\begin{aligned}& a_1\,\omega(\mathcal{P}_c(p,r)) + a_2\,\omega(\mathcal{P}_c(q,s)) + a_3\,\omega(\mathcal{P}_c(F(p,q),p)) + a_4\,\omega(\mathcal{P}_c(F(r,s),r)) \\ & + a_5\,\omega(\mathcal{P}_c(F(p,q),r)) + a_6\,\omega(\mathcal{P}_c(F(q,p),s)) \\ &\quad \precsim \gamma \max\Big\{\omega(\mathcal{P}_c(p,r)), \omega(\mathcal{P}_c(q,s)), \omega(\mathcal{P}_c(F(p,q),p)), \omega(\mathcal{P}_c(F(r,s),r)), \\ &\qquad\qquad \omega(\mathcal{P}_c(F(p,q),r)), \omega(\mathcal{P}_c(F(q,p),s))\Big\},\end{aligned}$$

where $\gamma = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$. □

By taking $a_3 = a_4 = \cdots = a_6 = 0$, $\omega(t) = kt$ for all $t > 0$, where $0 < k < 1$ and $ka_1 \to l$, $ka_2 \to m$, where $0 < l, m < 1$ in Corollary 4.4, then we obtain the following result.

Corollary 4.5. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $p,q,r,s \in \mathcal{X}$:*

$$\mathcal{P}_c(F(p,q),F(r,s)) \le l\,\mathcal{P}_c(p,r) + m\,\mathcal{P}_c(q,s),$$

where l, m are nonnegative constants such that $l + m < 1$. Then F has a unique coupled fixed point.

By taking $a_1 = a_2 = a_5 = a_6 = 0$, $\omega(t) = kt$ for all $t > 0$, where $0 < k < 1$ and $ka_3 \to l$, $ka_4 \to m$, where $0 < l, m < 1$ in Corollary 4.4, then we obtain the following result.

Corollary 4.6. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition for all $p,q,r,s \in \mathcal{X}$:*

$$\mathcal{P}_c(F(p,q),F(r,s)) \le l\,\mathcal{P}_c(F(p,q),p) + m\,\mathcal{P}_c(F(r,s),r),$$

where l, m are nonnegative constants such that $l + m < 1$. Then F has a unique coupled fixed point.

Theorem 4.7. *Let $(\mathcal{X}, \mathcal{P}_c)$ be a complete complex partial metric space and the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 3.1, except that condition (3) is replaced by the following contractive condition for all $p,q,r,s \in \mathcal{X}$:*

$$\mathcal{P}_c(F(p,q),F(r,s)) \precsim R_1\,\Delta_1(p,q,r,s) + R_2\,\Delta_\omega(p,q,r,s),$$

where

$$\begin{aligned}\Delta_1(p,q,r,s) &= \omega\Big(\mathcal{P}_c(F(r,s),r)\frac{1+\mathcal{P}_c(F(p,q),p)}{1+\mathcal{P}_c(p,r)}\Big), \\ \Delta_\omega(p,q,r,s) &= \omega\Big(\max\Big\{\mathcal{P}_c(p,r), \mathcal{P}_c(q,s), \mathcal{P}_c(F(p,q),p), \mathcal{P}_c(F(r,s),r), \\ &\qquad \mathcal{P}_c(F(p,q),r), \mathcal{P}_c(F(q,p),s)\Big\}\Big),\end{aligned}$$

R_1, R_2 are nonnegative constants with $R_1 + R_2 < 1$ and $\omega \in \Omega$ is nondecreasing. Then the same conclusions hold as in Theorem 3.1.

Proof. It follows from Theorem 3.1 by observing that if ω is nondecreasing, we have

$$\begin{aligned}\Delta_2(p,q,r,s) &= \omega\Big(\max\Big\{\mathcal{P}_c(p,r),\mathcal{P}_c(q,s),\mathcal{P}_c(F(p,q),p),\mathcal{P}_c(F(r,s),r),\\ &\qquad \mathcal{P}_c(F(p,q),r),\mathcal{P}_c(F(q,p),s)\Big\}\Big).\end{aligned}$$

□

Remark 4.8. *(1) Theorem 3.1 extends the results of Aydi [4] from partial metric space to the setting of complex partial metric space.*

(2) Theorem 3.1 also extends the results of Sabetghadam et al. [21] from cone metric space to the setting of complex partial metric space.

(3) Corollary 4.5 and Corollary 4.6 extend Theorem 2.1 and Theorem 2.4 respectively of Aydi [4] from partial metric space to the setting of complex partial metric space.

Now, we give an example to validate the result.

Example 4.9. *Let $\mathcal{X} = [0,+\infty)$ endowed with the usual complex partial metric $\mathcal{P}_c$ defined by $\mathcal{P}_c\colon \mathcal{X}\times\mathcal{X} \to [0,+\infty)$ with $\mathcal{P}_c(p,q) = \max\{p,q\}(1+i)$. The complex partial metric space $(\mathcal{X},\mathcal{P}_c)$ is complete because $(\mathcal{X},\mathcal{P}_c^t)$ is complete. Indeed, for any $p,q \in \mathcal{X}$,*

$$\begin{aligned}\mathcal{P}_c^t(p,q) &= 2\mathcal{P}_c(p,q) - \mathcal{P}_c(p,p) - \mathcal{P}_c(q,q)\\ &= 2\max\{p,q\}(1+i) - (p+ip) - (q+iq)\\ &= |p-q| + i|p-q| = |p-q|(1+i). \qquad (45)\end{aligned}$$

Thus, $(\mathcal{X},\mathcal{P}_c^t)$ is the Euclidean complex metric space which is complete. Consider the mapping $F\colon \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ defined by $F(p,q) = \frac{p+q}{12}$. Now, for any $p,q,r,s \in \mathcal{X}$, we have

(1)

$$\begin{aligned}\mathcal{P}_c(F(p,q),F(r,s)) &= \frac{1}{12}\max\{p+q,r+s\}(1+i)\\ &\le \frac{1}{12}[\max\{p,r\}(1+i) + \max\{q,s\}(1+i)]\\ &= \frac{1}{12}[\mathcal{P}_c(p,r) + \mathcal{P}_c(q,s)],\end{aligned}$$

which is the contractive condition of Corollary 4.5 for $l = 1/6 < 1$ (if $l = m$). Therefore, by Corollary 4.5, F has a unique coupled fixed point, which is $(0,0)$.

(2)

$$\begin{aligned}\mathcal{P}_c(F(p,q),F(r,s)) &= \frac{1}{12}\max\{p+q,r+s\}(1+i)\\ &\le \frac{1}{12}[\max\{p+q,p\}(1+i) + \max\{r+s,s\}(1+i)]\\ &= \frac{1}{12}[\mathcal{P}_c(F(p,q),p) + \mathcal{P}_c(F(r,s),r)],\end{aligned}$$

which is the contractive condition of Corollary 4.6 for $l = 1/6 < 1$ (if $l = m$). Therefore, by Corollary 4.6, F has a unique coupled fixed point, which is $(0,0)$.

Note that if the mapping $F\colon \mathcal{X}\times\mathcal{X}\to\mathcal{X}$ *is given by* $F(p,q)=\frac{p+q}{2}$*, then F satisfies contractive condition of Corollary 4.5, 4.6 for* $l=1$ *(if* $l=m$*),*

$$\begin{aligned}\mathcal{P}_c(F(p,q),F(r,s)) &= \frac{1}{2}\max\{p+q,r+s\}(1+i)\\ &\leq \frac{1}{2}[\max\{p,r\}(1+i)+\max\{q,s\}(1+i)]\\ &= \frac{1}{2}[\mathcal{P}_c(p,r)+\mathcal{P}_c(q,s)].\end{aligned}$$

In this case $(0,0)$ *and* $(1,1)$ *are both coupled fixed points of F, and hence, the coupled fixed point of F is not unique. This shows that the condition* $l<1$ *in Corollary 4.5, 4.6 and hence* $l+m<1$ *cannot be omitted in the statement of the aforesaid results.*

5. Conclusion

In this article, we prove some unique coupled fixed point theorems in the setting of complex partial metric spaces based on control function and provide some repercussions of the main result. Also we give an example to support the result. The results obtained in this article extend and generalize several previously published results from the existing literature.

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