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On convergence of continued fractions in a Banach algebra

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Abstract. In this paper, we present applications of various criteria for the convergence of continued fractions in a Banach algebra. We use them to inspect the convergence of power series and continued fractions in a special form.

1. Introduction and preliminaries

A complex continued fraction is given in the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}, \tag{1}$$

where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are the sequences of complex numbers. It is of great importance to find the conditions for the convergence of continued fractions. One of the important convergence criteria is given by Śleszyński–Pringsheim theorem. It says that the sufficient condition for the convergence of continued fraction (1) is that $|b_n| \geq |a_n| + 1$ for all $n \in \mathbb{N}$.

We consider an analogous situation in Banach algebras.

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1 (with $\|1\| = 1$). The sets of all invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$.

In this paper, we will consider continued fractions of elements of Banach algebra $\mathcal{A}$ in the following form

$$b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}. \tag{2}$$

The formal definition of continued fractions in Banach algebras is given by Baumann [1].

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Definition 1.1. [1] Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ be the sequences of Banach algebra elements. Define

$$\begin{aligned} K_n^{(n)} &= b_n, & n \in \mathbb{N}_0, \\ K_m^{(n)} &= b_m + a_{m+1}\left(K_{m+1}^{(n)}\right)^{-1}, & m, n \in \mathbb{N}_0, m < n. \end{aligned}$$

If $K_0^{(n)}$ exists for all $n \in \mathbb{N}_0$ and if $K = \lim_{n\to\infty} K_0^{(n)}$ exists, then K is convergent continued fraction.

The element $K_0^{(n)}$ is called nth approximant. The definition is using the backward recurrence algorithm for defining the nth approximant.

The nth approximant can be expressed as fraction in the form $K_0^{(n)} = A_n B_n^{-1}$, where the numerator A_n and denominator B_n are given by forward recurrence algorithm as in the following lemma.

Lemma 1.1. *[1] If all approximants $K_0^{(n)}$, $n \in \mathbb{N}_0$, exist, then $K_0^{(n)} = A_n B_n^{-1}$ holds for all $n \in \mathbb{N}_0$, where A_n and B_n are defined by*

$$\begin{aligned} A_{-1} &= 1, \quad A_0 = b_0, \quad A_n = A_{n-1}b_n + A_{n-2}a_n, \ n \in \mathbb{N} \\ B_{-1} &= 0, \quad B_0 = 1, \quad B_n = B_{n-1}b_n + B_{n-2}a_n, \ n \in \mathbb{N} \end{aligned}$$

Baumann [1] proved the generalized Śleszyński–Pringsheim theorem for continued fractions of Banach algebra elements. The Śleszyński–Pringsheim–criterion for convergence is given in the following theorem.

Theorem 1.1. *[1] If the conditions*

$$b_n \in \mathcal{A}^{-1} \text{ and } \|b_n^{-1}\| + \|a_n b_n^{-1}\| \le 1, \tag{3}$$

hold for all $n \in \mathbb{N}$, then the continued fraction K in (2) is convergent and we have $\|K - b_0\| \le 1$. If there exists $n_0 \in \mathbb{N}$ such that $\|b_{n_0}^{-1}\| + \|a_{n_0} b_{n_0}^{-1}\| < 1$, then $\|K - b_0\| < 1$.

Remark 1.1. *If the conditions of Theorem 1.1 hold, then we have $\|K_0^{(n)} - b_0\| < 1$ for all $n \in \mathbb{N}_0$. Indeed, we have $0 = \|K_n^{(n)} - b_n\| < 1$. Let $\|K_m^{(n)} - b_m\| < 1$ holds for some $m \in \{1, 2, \dots, n\}$. Then, we have*

$$\begin{aligned} \|K_{m-1}^{(n)} - b_{m-1}\| &= \|a_m (K_m^{(n)})^{-1}\| = \|a_m b_m^{-1} (K_m^{(n)} b_m^{-1})^{-1}\| \\ &= \|a_m b_m^{-1} (K_m^{(n)} b_m^{-1} - b_m b_m^{-1} + 1)^{-1}\| \\ &\le \|a_m b_m^{-1}\| \cdot \|((K_m^{(n)} - b_m) b_m^{-1} + 1)^{-1}\| \\ &\le \|a_m b_m^{-1}\| \cdot (\|K_m^{(n)} - b_m\| \cdot \|b_m^{-1}\| + 1) \\ &< (1 - \|b_m^{-1}\|) \cdot (\|b_m^{-1}\| + 1) = 1 - \|b_m^{-1}\|^2 < 1. \end{aligned}$$

So, we have proved that $\|K_m^{(n)} - b_m\| < 1$ holds for all $m \in \{0, 1, \dots, n\}$.

2. Transformation of continued fractions

The transformations of continued fractions are of great use, either as transformations between equivalent continued fractions or not.

In [1], it is given the transformation between the equivalent continued fractions. The proof was proposed using the forward recurrence algorithm of continued fractions. In the following lemma, we will show the relation between the nth approximants of continued fractions connected with this transformation. We will use the backward recurrence algorithm in the proof.

Lemma 2.1. *Let $K = b_0 + a_1(b_1 + a_2(b_2 + \cdots)^{-1})^{-1}$ be a continued fraction in the Banach algebra $\mathcal{A}$. Let $\{\rho_n\}_{n\in\mathbb{N}_0\cup\{-1\}}$ such that $\rho_n \in \mathcal{A}^{-1}$ for all $n \in \mathbb{N}_0 \cup \{-1\}$ and*

$$\tilde{a}_n = \rho_{n-2}^{-1} a_n \rho_n, \ n \in \mathbb{N}, \qquad \tilde{b}_n = \rho_{n-1}^{-1} b_n \rho_n, \qquad n \in \mathbb{N}_0, \tag{4}$$

Then, the continued fraction $\tilde{K} = \tilde{b}_0 + \tilde{a}_1(\tilde{b}_1 + \tilde{a}_2(\tilde{b}_2 + \cdots)^{-1})^{-1}$ satisfies $\tilde{K}_0^{(n)} = \rho_{-1}^{-1} K_0^{(n)} \rho_0$ for all $n \in \mathbb{N}_0$. Also, K is convergent if and only if $\tilde{K}$ is convergent and it holds $\tilde{K} = \rho_{-1}^{-1} K \rho_0$.

Proof. Firstly, we have $\tilde{K}_n^{(n)} = \tilde{b}_n = \rho_{n-1}^{-1} b_n \rho_n = \rho_{n-1}^{-1} K_n^{(n)} \rho_n$ for $n \in \mathbb{N}_0$. Now, suppose that $\tilde{K}_m^{(n)} = \rho_{m-1}^{-1} K_m^{(n)} \rho_m$ holds for some $m \in \{1, 2, \ldots, n\}$. Further, it holds

$$\begin{aligned}
\tilde{K}_{m-1}^{(n)} &= \tilde{b}_{m-1} + \tilde{a}_m (\tilde{K}_m^{(n)})^{-1} \\
&= \rho_{m-2}^{-1} b_{m-1} \rho_{m-1} + \rho_{m-2}^{-1} a_m \rho_m (\rho_{m-1}^{-1} K_m^{(n)} \rho_m)^{-1} \\
&= \rho_{m-2}^{-1} b_{m-1} \rho_{m-1} + \rho_{m-2}^{-1} a_m \rho_m \rho_m^{-1} (K_m^{(n)})^{-1} \rho_{m-1} \\
&= \rho_{m-2}^{-1} (b_{m-1} + a_m (K_m^{(n)})^{-1}) \rho_{m-1} \\
&= \rho_{m-2}^{-1} K_{m-1}^{(n)} \rho_{m-1}.
\end{aligned}$$

So, for all $m \in \{0, 1, \ldots, n\}$ we have that $\tilde{K}_m^{(n)} = \rho_{m-1}^{-1} K_m^{(n)} \rho_m$ holds.

Therefore, $\tilde{K}_0^{(n)} = \rho_{-1}^{-1} K_0^{(n)} \rho_0$ has been proved. Also, we have $\tilde{K} = \rho_{-1}^{-1} K \rho_0$. □

Note that for $\rho_{-1} = \rho_0 = 1$, the transformation (4) gives the equivalent continued fractions.

Remark 2.1. *It is interesting to see what sequence $\{\rho_n\}_{n \in \mathbb{N} \cup \{-1\}}$ will transform the continued fraction (2) to the continued fraction*

$$1 + a_1 (1 + a_2 (1 + \cdots)^{-1})^{-1}. \tag{5}$$

Using simple calculations, with the assumptions $b_n \in \mathcal{A}^{-1}$, $n \in \mathbb{N}_0$, it can be proved that this transformation is given by

$$\rho_n = b_n^{-1} b_{n-1}^{-1} \ldots b_0^{-1}, \ n \in \mathbb{N}_0 \quad \textit{and} \quad \rho_{-1} = 1.$$

Remark 2.2. *For the transformation of the continued fraction (2) into the continued fraction in the form*

$$b_0 + (b_1 + (b_2 + \cdots)^{-1})^{-1} \tag{6}$$

with assumptions $a_n \in \mathcal{A}^{-1}$, $n \in \mathbb{N}$, the sequence $\{\rho_n\}_{n \in \mathbb{N}_0}$ is given by

$$\rho_n = \begin{cases} a_n^{-1} a_{n-2}^{-1} \ldots a_2^{-1}, & n \text{ is even} \\ a_n^{-1} a_{n-2}^{-1} \ldots a_1^{-1}, & n \text{ is odd} \end{cases}, \ n \in \mathbb{N} \quad \textit{and} \quad \rho_0 = 1, \quad \rho_{-1} = 1.$$

Transformations of continued fractions can be helpful in getting another convergence criterion. The following theorem is proved in the paper [3] for continued fractions in a Banach space. It also holds for continued fractions in a Banach algebra, as stated in [1].

If we use the transformation $\rho_n = q_1 q_2 \ldots q_n \|b_1^{-1}\| \|b_2^{-1}\| \ldots \|b_n^{-1}\| \cdot 1$, where $q_i > 0, i = 1, \ldots n$, and we get the following convergence criterion.

Theorem 2.1. *Let $\{q_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers. If*

$$q_1 > 1 \quad \textit{and} \quad \|b_{n-1}^{-1}\| \|a_n b_n^{-1}\| \le \frac{q_n - 1}{q_{n-1} q_n}, \quad n = 2, 3, 4, \ldots,$$

then the continued fraction (2) converges.

3. Power series and continued fractions

When we want to connect power series and continued fractions, we can use the famous Euler's continued fraction formula, which is proved for continued fractions of complex numbers. Euler's formula can be, also, used for representing the power series in the form of continued fraction for the elements of a Banach algebra.

For $t \in \mathcal{A}$, we can define the sequences $\{a_n\}_n$ and $\{b_n\}_n$ as follows:

$$\begin{array}{llll} b_0 = 0, & b_1 = 1, & b_n = 1 + t, & n = 2, 3, \dots \\ & a_1 = 1, & a_n = -t, & n = 2, 3, \dots . \end{array} \tag{7}$$

The continued fraction that is defined by the sequences in (7) is

$$(1 - t(1 + t - t(1 + t - t(...)^{-1})^{-1})^{-1})^{-1}.$$

Theorem 3.1. *The continued fraction* $(1 - t(1 + t - t(1 + t - t(...)^{-1})^{-1})^{-1})^{-1}$ *converges if and only if the power series* $\sum\limits_{n=0}^{+\infty} t^n$ *converges.*

Proof. We will prove that the nth approximant of the given continued fraction is equal to the nth partial sum of the given power series. The required equivalence is then obvious.

Let $K_0^{(n)} = A_n B_n^{-1}$ be the nth approximant of the given continued fraction, and let S_n be the nth partial sum of the power series. If we have the sequences $\{a_n\}_n$ and $\{b_n\}_n$ as in (7), then we have

$$A_{-1} = 1,\ A_0 = 0,\ A_1 = 1,\ A_n = A_{n-1}(1 + t) - A_{n-2}t,\ n = 2, 3, ...$$
$$B_{-1} = 0,\ B_0 = 1,\ B_1 = 1,\ B_n = B_{n-1}(1 + t) - B_{n-2}t,\ n = 2, 3, ...$$

Note that, for $n \in \mathbb{N}\backslash\{1\}$, we have

$$A_n - A_{n-1} = (A_{n-1} - A_{n-2})t = (A_{n-2} - A_{n-3})t^2 = \cdots = (A_1 - A_0)t^{n-1} = t^{n-1}$$
$$B_n - B_{n-1} = (B_{n-1} - B_{n-2})t = (B_{n-2} - B_{n-3})t^2 = \cdots = (B_1 - B_0)t^{n-1} = 0$$

so,

$$A_n = A_{n-1} + t^{n-1} = A_{n-2} + t^{n-2} + t^{n-1} = \cdots = 1 + t + \cdots + t^{n-1} = S_n$$
$$B_n = 1,$$

and then $K_0^{(n)} = S_n$. □

Now, if we use the generalized Śleszyński–Pringsheim Theorem 1.1, we get the following statement.

Theorem 3.2. *If* $1 + t \in \mathcal{A}^{-1}$ *and* $\|(1 + t)^{-1}\| + \|t(1 + t)^{-1}\| = 1$, *then continued fraction* $(1 - t(1 + t - t(1 + t - t(...)^{-1})^{-1})^{-1})^{-1}$ *converges.*

Proof. Let $1 + t$ be an invertible element in $\mathcal{A}$ and let the sequences $\{a_n\}_n$ and $\{b_n\}_n$ be defined as in (7). It holds from Theorem 1.1 that if we have

$$\|b_n^{-1}\| + \|a_n b_n^{-1}\| = \|(1 + t)^{-1}\| + \|t(1 + t)^{-1}\| \leq 1,$$

then continued fraction $(1 - t(1 + t - t(1 + t - t(...)^{-1})^{-1})^{-1})^{-1}$ converges.

Since $1 = (1 + t)(1 + t)^{-1} = (1 + t)^{-1} + t(1 + t)^{-1}$, then

$$1 = \|1\| = \|(1 + t)(1 + t)^{-1}\| \leq \|(1 + t)^{-1}\| + \|t(1 + t)^{-1}\|.$$

So, the condition $\|b_n^{-1}\| + \|a_n b_n^{-1}\| \leq 1$ is equivalent to the condition $\|(1 + t)^{-1}\| + \|t(1 + t)^{-1}\| = 1$. □

Similarly, we can represent the alternating power series using continued fractions. If $t \in \mathcal{A}$, we use the following

$$\begin{array}{llll} b_0 = 0, & b_1 = 1, & b_n = 1 - t, & n = 2, 3, \dots \\ & a_1 = 1, & a_n = t, & n = 2, 3, \dots \end{array} \tag{8}$$

to get the continued fraction

$$(1 + t(1 - t + t(1 - t + t(...)^{-1})^{-1})^{-1})^{-1}.$$

Theorem 3.3. *The continued fraction* $(1+t(1-t+t(1-t+t(\dots)^{-1})^{-1})^{-1})^{-1}$ *converges if and only if the alternating power series* $\sum_{n=0}^{+\infty}(-t)^n$ *converges.*

Proof. The proof is very similar to the proof of the Theorem 3.1. We will prove that the nth approximant of the given continued fraction is equal to the nth partial sum of the alternating power series.

Let $K_0^{(n)} = A_n B_n^{-1}$ be the nth approximant of given continued fraction, and S_n be the nth partial sum of the alternating power series. If we have the sequences $\{a_n\}_n$ and $\{b_n\}_n$ as in (8), then we have

$$A_{-1} = 1,\ A_0 = 0,\ A_1 = 1,\ A_n = A_{n-1}(1-t) + A_{n-2}t,\ \ n = 2,3,\dots$$
$$B_{-1} = 0,\ B_0 = 1,\ B_1 = 1,\ B_n = B_{n-1}(1-t) + B_{n-2}t,\ \ n = 2,3,\dots$$

For $n \in \mathbb{N}\backslash\{1\}$, we have

$$A_n - A_{n-1} = (A_{n-1} - A_{n-2})(-t) = (A_{n-2} - A_{n-3})(-t)^2 = \cdots = (A_1 - A_0)(-t)^{n-1} = (-t)^{n-1}$$
$$B_n - B_{n-1} = (B_{n-1} - B_{n-2})(-t) = (B_{n-2} - B_{n-3})(-t)^2 = \cdots = (B_1 - B_0)(-t)^{n-1} = 0$$

so,

$$A_n = A_{n-1} + (-t)^{n-1} = A_{n-2} + (-t)^{n-2} + (-t)^{n-1} = \cdots = 1 + (-t) + \cdots + (-t)^{n-1} = S_n$$
$$B_n = 1,$$

and then $K_0^{(n)} = S_n$. □

4. Convergence of continued fractions in a special form

Firstly, we will discuss the continued fractions in the form (5), i.e. $1 + a_1(1 + a_2(1+\cdots)^{-1})^{-1}$.

Theorem 4.1. *Let* $\lambda \geq 2$ *and* $\|a_n\| \leq \frac{1}{\lambda^2}$ *for all* $n \in \mathbb{N}$*. Then the continued fraction (5) converges. Also, it holds* $\|K_0^{(n)} - 1\| \leq \frac{1}{\lambda}$ *and* $\|K - 1\| < \frac{1}{\lambda}$.

Proof. Let $\tilde{K}$ be the continued fraction obtain by the transformation (4) where $\rho_n = \lambda^n$, $n \in \mathbb{N}_0 \cup \{-1\}$ and $\lambda \geq 2$. So, we have $\tilde{a}_n = \lambda^2 a_n$ for all $n \in \mathbb{N}$ and $\tilde{b}_n = \lambda$ for all $n \in \mathbb{N}_0$. Also, we obtain $\tilde{K}_0^{(n)} = \lambda K_0^{(n)}$, $n \in \mathbb{N}_0$ and $\tilde{K} = \lambda K$.

Since it holds $\|\tilde{b}_n^{-1}\| + \|\tilde{a}_n\tilde{b}_n^{-1}\| \leq \frac{1}{\lambda} + \lambda\|a_n\| \leq \frac{2}{\lambda} \leq 1$, we can apply Theorem 1.1 (and Remark (1.1)) to the continued fraction $\tilde{K}$. Then, we get $\|\tilde{K}_0^{(n)} - \lambda\| < 1$, $n \in \mathbb{N}$ and $\|\tilde{K} - \lambda\| \leq 1$. Therefore, for all $n \in \mathbb{N}$ we have $\|\tilde{K}_0^{(n)} - \lambda\| = \|\lambda K_0^{(n)} - \lambda\| = \lambda\|K_0^{(n)} - 1\| < 1$, which implies $\|K_0^{(n)} - 1\| < \frac{1}{\lambda}$. Also, $\|\tilde{K} - \lambda\| = \|\lambda K - \lambda\| = \lambda\|K - 1\| \leq 1$, which implies $\|K - 1\| \leq \frac{1}{\lambda}$. □

Notice that λ cannot be lower then 2 and for $\lambda = 2$ we obtain Worpitzky's Theorem [2].

On the other hand, for $\lambda > 1$, if we apply Theorem 2.1 to the continued fraction (5) with $q_n = \lambda$, for all $n \in \mathbb{N}$, we get the following criterion.

Theorem 4.2. *Let* $\lambda > 1$ *and* $\|a_n\| \leq \frac{\lambda - 1}{\lambda^2}$, *for all* $n \in \mathbb{N}$*. Then the continued fraction (5) converges.*

Note that, for $\lambda \geq 2$, if the conditions in Theorem 4.1 hold, then the conditions in Theorem 4.2 are also satisfied. For $1 < \lambda < 2$, we cannot use Theorem 4.1, but we can use Theorem 4.2.

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