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The narrow recurrence of continuous-time Markov chains

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Abstract. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a Polish space equipped with its Borel σ-algebra $\mathcal{B}$. We consider a transition function probability $\{P_t, t \in \mathbb{R}^+\}$ of a continuous Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in X. This transition function defines a semi group acting on $Pr(X)$, the set of all probability measures on X, which is also a Polish space endowed with the narrow topology. In this paper, we introduce and study the notion of the narrow recurrence of transition functions of continuous Markov chains and provide some properties, which can be considered as an initiation of applications of properties of topological dynamics on stochastic process theory and random dynamical systems.

1. Introduction

Let X be a complex Banach space. In the following, by an operator, we mean a linear and continuous map acting on X.
A very central notion in topological dynamics that has a long story is that of recurrence, which goes back to Poincaré [14], and it refers to the existence of points in the space for which parts of their orbits under a continuous map return to themselves, in other words, a vector $x \in X$ is called a **recurrent vector** for an operator T acting on X if there exists a strictly increasing sequence (n_k) of positive integers such that

$$T^{n_k}x \longrightarrow x.$$

The purpose of this note is the study of the notion of recurrence, together with its variations, in the context of topological dynamics. Some examples and characterizations of recurrence for special classes of operators have appeared in [6, 7, 12] and a systematic study of this notion goes back to the works of Furstenberg [10], and Gottschalk and Hedlund [11].

Instead of the norm topology, by taking density in the weak topology, we can consider the notions of weak hypercyclicity, weak recurrence and weak orbits in general.
The study of weak orbits was began in 1996 by J. Van Neerven in [18]. Importation contribution to weak hypercyclicity is due to J. Bès, K. Chan and R. Sanders [3, 4, 13, 17]. Moreover, The notion of weak recurrence was studied in [1] by M. Amouch et al.
In [19], We have studied the weak recurrence also known as narrow recurrence of a new class of operators

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called Markov kernels, along with their characteristics, with the goal of exploiting this concept to study the distributional stability Markov chains with discrete time on general state spaces.

In the present paper, we apply the properties of the recurrence on transition functions of continuous Markov chains, and consequently investigate the stability in distribution of continuous-time Markov chains. More precisely, the contribution of the paper is two-fold. First, studying the recurrence in the context of topological dynamics of a new class of semi group, which are transition functions of continuous Markov chains, and examining their characteristics. Second, exploiting the concept of the recurrence to study the stability in distribution of continuous Markov chains on general state spaces.

The paper is organized as follows. In section 1, we will give a summary of some notions and results concerning Markov chains that we will need in next paragraphs. In Section 2, we introduce the notion of narrow recurrence of transition functions and provide some interesting examples. We explain that the narrow recurrence of Markov kernels makes it possible to study the stability in distribution of continuous Markov chains. In Section 3, we give some properties that characterizes the narrow recurrence of transition functions. In particular, we characterize the narrow recurrence of transition functions in terms of the convergence of measures of sets. We also give a result which establishes the relationship between the narrow recurrence of two transition functions.

Throughout in what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a probability space, X will denote a Polish space equipped with a complete metric d, $\mathcal{B}$ will denote the σ-algebra of Borel sets of X, $\Pr(X)$ denote the set of all probability measures on X, and $C_b(X)$ denote the space of all bounded continuous functions from X to $\mathbb{R}$.

2. The basic set-up

In this part, we will give a summary of some notions and results that we will need in the next paragraph. For a comprehensive exposition on this subject see [8] and [9].

Definition 2.1. *A Markov kernel on $X \times \mathcal{B}$ is a mapping $P : X \times \mathcal{B} \to [0, 1]$ satisfying the following conditions:*

(i) for each $x \in X$, the mapping $P(x, .) : A \mapsto P(x, A)$ is a probability measure on $\mathcal{B}$,

(ii) for each $A \in \mathcal{B}$, the mapping $P(., A) : x \mapsto P(x, A)$ is a measurable function from $(X, \mathcal{B})$ to $([0, 1], \mathcal{B}_{[0,1]})$.

Definition 2.2. *A family $(P_t)_{t\geq 0}$ of Markov kernels on $(X, \mathcal{B})$ is called a transition function if: For all $x \in X$ and all $A \in \mathcal{B}$,*

$$\begin{cases} P_0(x, A) = \delta_x(A) \\ P_{t+s}(x, A) = \int_X P_t(x, dy) P_s(y, A), \quad \forall s, t \geq 0. \end{cases}$$

Remark 2.3. *The equality*

$$P_{t+s}(x, A) = \int_X P_t(x, dy) P_s(y, A), \quad \forall s, t \geq 0, \ \forall A \in \mathcal{B}.$$

is called the Chapman-Kolmogorov equation.

Definition 2.4. *A process stochastic $\{X_t \colon t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in X is called a Markov chain with continuous time if there exists a transition function $(P_t)_{t\geq 0}$ such that, for all $t_0, t_1, ..., t_n \in \mathbb{R}^+$, with $t_0 \leq t_1 \leq t_2 \ldots t_n$, we have:*

$$\mathbb{P}(X_{t_{n+1}} \in A / (X_{t_0}, X_{t_1}, ..., X_{t_n})) = P_{t_{n+1}}(X_{t_n}, A) \ \ \mathbb{P}\text{-}a.s. \ \ \forall A \in \mathcal{B}.$$

The distribution of X_0 is called the initial distribution.

The following result ensures that any transition function can be considered as a semigroup acting on the left on Pr(X) and on the right on $C_b(X)$.

Proposition 2.5. 1. *Any transition function $(P_t)_{t\geq 0}$ on $(X, \mathcal{B})$ defines a semigroup of linear operators P_t, $t \geq 0$ on the space $C_b(X)$, by the formula:*

$$P_t\varphi(x) = \int_X P_t(x, dy)\varphi(y), \;\; t \geq 0, \; x \in X, \; \varphi \in C_b(X).$$

2. *Any transition function $(P_t)_{t\in\mathbb{R}^+}$ on $(X, \mathcal{B})$ acts on measures by the relationship:*

$$\mu P_t(A) = \int_X \mu(dx)P_t(x, A), \;\; \mu \in Pr(X), \; A \in \mathcal{B}, \; t \geq 0.$$

Furthermore, for any probability measure $\mu \in Pr(X)$, we have $\mu P_{t+s} = \mu P_t P_s$.

The following result is due to Prohorov, see [15], is the key result of this paper.

Theorem 2.6. *We may define a topology on Pr(X), called the narrow topology, which is the smallest topology on Pr(X), such that $\nu \longmapsto \int_X f d\nu$ is continuous for every f of $C_b(X)$. Furthermore, the space* Pr(X) *is a Polish equipped with this topology.*

3. The narrow recurrence of transition functions

In this section, we introduce the notion of narrow recurrence for transition functions and provide some interesting examples.

Definition 3.1. *Let $(P_t)_{t\geq 0}$ be a transition function on $(X, \mathcal{B})$. The orbit of probability measure $\mu \in Pr(X)$ under $(P_t)_{t\geq 0}$ is defined as follows:*

$$\mathrm{Orb}(P_t, \mu) := \{\mu P_t : \; t \in \mathbb{R}^+\}.$$

Remark 3.2. *Let $(P_t)_{t\geq 0}$ be a transition function on $(X, \mathcal{B})$, and μ be a probability measure on X. Let $\{X_t : t \geq 0\}$ be a continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with transition function $(P_t)_{t\geq 0}$, and initial distribution μ.*
Then, for each $t \geq 0$, the random variable X_t is distributed according to μP_t; that is, the law of X_t is μP_t.
Hence, the orbit of μ under $(P_t)_{t\geq 0}$ represents the laws of random variables X_t, $t > 0$.
Consequently, studying these orbits is equivalent to studying, in distribution, the trajectories of the Markov chain $\{X_t : t \geq 0\}$.

Definition 3.3. *A probability measure μ is said to be a **Narrow recurrent measure** for $(P_t)_{t\geq 0}$ if there exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that*

$$\mu P_{t_{n_k}} \xrightarrow[n \longrightarrow +\infty]{N} \mu.$$

We denotes by (P_t) the set of all weakly recurrent measures for $(P_t)_{t\geq 0}$

Remark 3.4. *Let $(P_t)_{t\geq 0}$ be a transition function on $(X, \mathcal{B})$, and μ be a probability measure on X. Let $\{X_t : t \geq 0\}$ be a continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with transition function $(P_t)_{t\geq 0}$, and initial distribution μ.*
Suppose that μ is a narrow recurrent probability measure of $(P_t)_{t\geq 0}$. Then there exists a sub-sequence $(X_{t_{n_k}})$ of $\{X_t : t \geq 0\}$ which converges in distribution to a random variable ζ, distributed according to μ. Consequently, we can extract a sub-process of $\{X_t : t \geq 0\}$ which converges in distribution to μ.

Example 3.5. *Let $\{T_t\}_{t\in\mathbb{R}^+}$ be a semi group acting on X. For each $x \in X$ and $A \in \mathcal{B}$, we get*

$$P_t(x, A) = 1_A \circ T_t(x)$$

Then $(P_t)_{t\in\mathbb{R}^+}$ defines a transition function on $(X, \mathcal{B})$.

Let μ a probability measure on X. For any integer $t \geq 0$, we get

$$\mu P_t(A) = \int_X \mu(dx) P_t(x, A) = \mu(T_t^{-1}(A)),$$

If μ is invariant under T_t; that is $T_t\mu = \mu$, for all $t \in \mathbb{R}^+$, then

$$\mu P_t = \mu$$

Hence

$$\mu P_{t_{n_k}} \underset{n\longrightarrow+\infty}{\overset{N}{\longrightarrow}} \mu.$$

Thus μ is a Narrow recurrent probability measure of P. furthermore,

$$Inv(T_t) \subset (P_t)$$

where, $Inv(T_t)$ denote the set of all invariant probability measures under $\{T_t\}_{t\in\mathbb{R}^+}$.

Remark 3.6. *Let $(P_t)_{t\geq 0}$ be a transition function on $(X, \mathcal{B})$, and μ a probability measure on X. Suppose that $(P_t)_{t\geq 0}$ admits an invariant probability measure μ; that is $\mu P_t = \mu$, for all $t \geq 0$. Then μ is a Narrow recurrent probability measure of $(P_t)_{t\geq 0}$. Moreover,*

$$Inv(P_t) \subset (P_t),$$

where $Inv(P_t)$ is the set of all invariant probability measures under $(P_t)_{t\geq 0}$.

4. Characterization of the narrow recurrence of transition functions

In this section, we provide some properties that characterize the narrow recurrence of transition functions, which are extensions of the narrow recurrence characterizations for Markov chains proven in [19], to continuous-time Markov chains.
The following result is based on the Portmanteau theorem, see [5], provides useful conditions equivalent to the narrow recurrence of transition functions. In particular, it characterizes the narrow recurrence of transition functions in terms of the convergence of measures of sets.

Theorem 4.1. *Let $(P_t)_{t\geq 0}$ be a transition function on $(X, \mathcal{B})$, and μ be a probability measure on X. The following statement are equivalent :*

1. $\mu \in (P_t)$
2. *There exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that*

   $$\int_X f(x)\, \mu P_{t_{n_k}}(dx) \underset{k\longrightarrow+\infty}{\longrightarrow} \int_X f(x)\mu(dx),$$

   for all f bounded real valued uniformly continuous function.
3. *There exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that*

   $$\overline{\lim_k}\, \mu P_{t_{n_k}}(F) \leqslant \mu(F), \text{ for every closed set } F.$$

4. *There exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that*

   $$\underline{\lim_k}\, \mu P_{t_{n_k}}(U) \geqslant \mu(U), \text{ for every open set } U.$$

5. *There exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that*

$$\lim_k \mu P_{t_{n_k}}(A) = \mu(A),$$

for every Borel set A whose boundary has μ-measure 0.

Proof. (1) $\Longrightarrow$ (2): Suppose that $\mu \in (P_t)$, then there exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that

$$\int_X f(x)\, \mu P_{t_{n_k}}(dx) \underset{k\longrightarrow+\infty}{\longrightarrow} \int_X f(x)\mu(dx), \quad \forall f \in C_b(X),$$

particularly,

$$\int_X f(x)\, \mu P_{t_{n_k}}(dx) \underset{k\longrightarrow+\infty}{\longrightarrow} \int_X f(x)\mu(dx),$$

for all f bounded real valued uniformly continuous function.

(2) $\Longrightarrow$ (3): Suppose that there exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that

$$\int_X f(x)\, \mu P_{t_{n_k}}(dx) \underset{k\longrightarrow+\infty}{\longrightarrow} \int_X f(x)\mu(dx),$$

for all f bounded real valued uniformly continuous function. Let F be any closed set and $F_p = \{x \in X \colon d(x,F) < \frac{1}{p}\}$, for all $p \in \mathbb{N}^*$. Then F and F_p^C, where F_p^C denotes the complement of F_p, are disjoint closed sets such that $\underset{(x,y)\in F\times F_p^C}{inf} d(x,y) \geqslant \frac{1}{p}$. Now, we put

$$f_p(x) = \frac{d(x,F_p^C)}{d(x,F_p^C) + d(x,F)}, \quad \text{for any } p \in \mathbb{N}^*.$$

It is clear that $0 \leqslant f_p \leqslant 1$, $f_p(x) = 1$ for all $x \in F$, $f_p(x) = 0$ for all $x \in F_p^C$, and f_p is a bounded uniformly continuous function. Further, the sequence $(F_p)_{p\in\mathbb{N}^*}$ is decreasing and $F = \bigcap_{p\in\mathbb{N}^*} F_p$. Thus

$$\overline{\lim_k}\, \mu P_{t_{n_k}}(F) \leqslant \overline{\lim_k} \int_X f_p d(\mu P_{t_{n_k}}) \leqslant \mu(F_p).$$

Letting $p \to \infty$, we obtain $\overline{\lim_k}\, \mu P_{t_{n_k}}(F) \leqslant \mu(F)$.

(3) $\Leftrightarrow$ (4): A simple complementation argument proves this equivalence.

(4) $\Longrightarrow$ (5): Suppose that there exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that

$$\underline{\lim_k}\, \mu P_{t_{n_k}}(U) \geqslant \mu(U) \text{ for every open set } U.$$

Let $A \in \mathcal{B}$ whose boundary has μ-measure 0. Then

$$\overline{\lim_k}\, \mu P_{t_{n_k}}(A) \leqslant \overline{\lim_k}\, \mu P_{t_{n_k}}(\bar{A}) \leqslant \mu(\bar{A}),$$

$$\underline{\lim_k}\, P_{t_{n_k}}(A) \geqslant \underline{\lim_k}\, \mu P_{t_{n_k}}(\mathring{A}) \geqslant \mu(\mathring{A}).$$

Since $\mu(\bar{A} - \mathring{A}) = 0$, it follows that $\lim_k \mu P_{t_{n_k}}(A)$ exists and equals $\mu(A)$.

(5) $\Longrightarrow$ (1): Suppose that there exists a strictly increasing sequence (t_{n_k}) of positive real numbers such that

$$\lim_k \mu P_{t_{n_k}}(A) = \mu(A),$$

for every Borel set A whose boundary has μ-measure 0.
Let $f \in C_b(X)$, a and b be two numbers such that $a < f(x) < b$, for all $x \in X$. Given any $\varepsilon > 0$, we can find numbers $h_1, h_2, ..., h_m$, such that $a = h_0 < h_1 < ... < h_m = b$, $h_i - h_{i-1} < \frac{\varepsilon}{2}$ and $\mu(\{x \in X : f(x) = h_i\}) = 0$, for all $i = 1, 2, ..., m$, see [2]. For all $i = 1, 2, ..., m$, we put

$$A_i = \{x \in X : \; h_{i-1} \leqslant f(x) < h_i\},$$

then $A_1, A_2, ..., A_m$ are disjoint Borel sets with $X = \bigcup_{i=1}^{m} A_i$. Moreover

$$(\bar{A}_i - \mathring{A}_i) \subset \{x \in X : \; f(x) = h_{i-1}\} \cup \{x \in X : \; f(x) = h_i\},$$

then $\mu(\bar{A}_i - \mathring{A}_i) = 0$, hence $\lim_k \mu P_{t_{n_k}}(A_i) = \mu(A_i)$, for all $i = 1, 2, \ldots, m$.
Now, we put $g = \Sigma_{i=1}^{m} h_{i-1} 1_{A_i}$, then

$$|\int_X f \mathrm{d}(P_{t_{n_k}}) - \int_X f \mathrm{d}\mu| \leqslant \int_X |f - g| \mathrm{d}(\mu P_{t_{n_k}}) + |\int_X f \mathrm{d}(\mu P_{t_{n_k}}) - \int_X g \mathrm{d}\mu| + \int_X |f - g| \mathrm{d}\mu.$$

On the other hand it is clear that $|f(x) - g(x)| < \frac{\varepsilon}{2}$ for all $x \in X$, thus

$$|\int_X f \mathrm{d}(\mu P_{t_{n_k}}) - \int_X f \mathrm{d}\mu| \leqslant \varepsilon + \Sigma_{i=1}^{m} |\mu P_{t_{n_k}}(A_i) - \mu(A_i)| h_{i-1}.|$$

Hence,

$$\overline{\lim_k} |\int_X f \mathrm{d}(\mu P_{t_{n_k}}) - \int_X f \mathrm{d}\mu| \leqslant \varepsilon.$$

□

As an application of theorem (4.1) and the remark (3.4) we deduce the following important corollary:

Corollary 4.2. *Let $(P_t)_{t \geq 0}$ be a transition function on $(X, \mathcal{B})$, and μ be a probability measure on X. Let $\{X_t : t \geq 0\}$ be a continuous-time Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with transition function $(P_t)_{t \geq 0}$, and initial distribution μ. Suppose that μ is a narrow recurrent probability measure of $(P_t)_{t \geq 0}$, then there exists a subsequence $(X_{t_{n_k}})_{k \in \mathbb{N}}$ of $\{X_t : t \geq 0\}$ and a random variable ξ, with values in X, distributed according to μ, such that the following five statements are valid :*

1. *$(X_{t_{n_k}})$ converges in distribution to ξ.*
2. *$E(f(X_{t_{n_k}})) \longrightarrow E(f(\xi))$ for all bounded, uniformly continuous function f.*
3. *$\lim sup_k \mathbb{P}(X_{t_{n_k}} \in F) \leqslant \mathbb{P}(\xi \in F)$, for all closed F.*
4. *$\lim inf_k \mathbb{P}(X_{t_{n_k}} \in U) \geqslant \mathbb{P}(\xi \in U)$, for all open U.*
5. *$\mathbb{P}(X_{t_{n_k}} \in A) \longrightarrow \mathbb{P}(\xi \in A)$ for any Borel set A whose boundary has μ-measure 0.*

The following theorem gives, under conditions, the relation between the narrow recurrence of two transition functions.

Theorem 4.3. *Let $(P_t)_{t \geq 0}$ and $(Q_t)_{t \geq 0}$ be two transition functions on $(X, \mathcal{B})$. Let $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ be two continuous-time Markov chains on $(\Omega, \mathcal{F}, \mathbb{P})$ with transition functions $(P_t)_{t \geq 0}$ and $(Q_t)_{t \geq 0}$ respectively and same initial distribution μ. Suppose that $d(X_t, Y_t)$ converges to 0 in probability, then*

$$\mu \in (P_t) \textit{ if only if } \mu \in (Q_t).$$

Proof. Suppose that $\mu \in (P_t)$. Then there exists a sub-sequence $(X_{t_{n_k}})_k$ of $(X_t)_{t \geq 0}$ which converges in distribution to a random variable ζ, distributed according to μ.
Let F be a closed set and $\varepsilon > 0$. we consider the closed set,

$$F_\varepsilon = \{x \in X : d(x, F) \leqslant \varepsilon\}.$$

Then,

$$\mathbb{P}(Y_{t_{n_k}} \in F) \leqslant \mathbb{P}(d(X_{t_{n_k}}, Y_{t_{n_k}}) \geqslant \varepsilon) + \mathbb{P}(X_{t_{n_k}} \in F_\varepsilon).$$

Then, by the corollary 4.2,

$$\limsup_k \mathbb{P}(Y_{t_{n_k}} \in F) \leqslant \limsup_k \mathbb{P}(X_{t_{n_k}} \in F_\varepsilon) \leqslant \mathbb{P}(\zeta \in F_\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_k \mathbb{P}(Y_{t_{n_k}} \in F) \leqslant \mathbb{P}(\zeta \in F).$$

Thus $(Y_{t_{n_k}})$ converges in distribution to ϕ, and hence

$$\mu Q_{t_{n_k}} \xrightarrow[k \longrightarrow +\infty]{N} \mu.$$

□

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Conflict of interest

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