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A new modified Krasnoselskii-Mann-type algorithm for an infinite family of multivalued quasi-nonexpansive mappings in unformly convexe spaces

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Abstract. The main objective of this paper is to introduce and study a new modified Krasnoselskii-Mann-type method for approximating a common fixed points of an infinite family of multivalued quasi-nonexpansive mappings in real Banach spaces. Under suitable assumptions, we prove the strong convergence of this algorithm in uniformly convex real Banach spaces without imposing any compactness assumption. Application to minimum-norm fixed point problem is provided to support our main results. Futhermore, numerical example is given to demonstrate the implementability of our algorithm. Finally, our algorithm generalize and extend of the existing results in the literature and moreover, its computationaly effort is less per each iteration compared with related works.

1. Introduction

Let E be a Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0, +\infty) \to [0, \infty)$ be a stricly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}, x \in E. \tag{1}$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), x \neq 0.$$

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Following Browder [4], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is a single-valued and is weak-to-weak* sequentially continous, i.e., if $(x_n) \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$. It is know that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see [6] fore more detais on dualitymaps).

Remark 1.1. *Note also that a duality mapping exists in each Banach space. We recall from [2] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.*
(i) $l_p: \ Jx = \|x\|_{l_p}^{2-p} y \in l_q, \ x = (x_1, x_2, \cdots, x_n, \cdots)$,
$y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots)$,
(ii) $L_p: \ Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q$,
(iii) $W^{m,p}: \ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha \Big(|D^\alpha u|^{p-2} D^\alpha u \Big) \in W^{-m,q}$,
where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Recall that a Banach space E satisfies Opial property (see, e.g., [20]) if $\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property. Let (X, d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$. For several years, the study of fixed point theory for multi-valued nonlinear mappings has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [3], Kakutani [13], Nash [18, 19], Geanakoplos [12], Nadla [17], Downing and Kirk [9]). Interest in the study of fixed point theory for multivalued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as *Game Theory* and *Non-Smooth Differential Equations*. We describe briefly the connection of fixed point theory for multivalued mappings with these applications.

1.1. Optimization problems with constraints

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $A : H \to 2^H$ be a set-valued mapping. Consider the following constrained optimization problem:

$$(P) \quad \begin{cases} \min f(x) \\ 0 \in Ax. \end{cases}$$

It is known that the multivalued map, ∂f the subdifferential of f, is maximal monotone, where for $x, w \in H$,

$$w \in \partial f(x) \Leftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle, \ \forall \ y \in H$$

$$\Leftrightarrow x \in \ \text{argmin}(f - \langle \cdot, w \rangle).$$

It is easily seen that, for $x \in H$ with x is a solution of (P) if and only if

$$x \in Fix(T_1) \cap Fix(T_2),$$

with $T_1 := I - \partial f$ and $T_2 := I - A$, where I where I is the identity map of H. Therefore, x is a solution of (P) if and only if x is a solution of common fixed point problem involving two multivalued. A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called L- *Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \ \forall x, y \in D(T). \tag{2}$$

When $L \in (0, 1)$ in (2), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.
A multivalued map T is called quasi-nonexpansive if

$$H(Tx, Tp) \leq \|x - p\|$$

holds for all $x \in D(T)$ and $p \in F(T)$. It is easy to see that the class of multivalued quasi-nonexpansive mappings properly includes that of multivalued nonexpansive maps with fixed points Existence theorems for fixed point of *multi-valued* contractions and nonexpansive mappings using the Hausdorff metric have been proved by several authors (see, e.g., Nadler [17], Markin [16], Lim [14]). Later, an interesting and rich fixed point theory for such maps and more general maps was developed which has applications in control theory, convex optimization, differential inclusion, and economics (see, Gorniewicz [10] and references cited therein). Recently, several theorems have been proved on the approximation of fixed points of multivalued nonexpansive mappings (see for example [1], [21], [23] , [24], and the references therein) and their generalizations (see e.g., [11], [8]). Sastry and Babu [23] introduced the following iterative scheme. Let $T : E \to P(E)$ be a multivalued mapping and x^* be a fixed point of T. The sequence of iterates is given for $x_1 \in E$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n \ \forall n \geq 0, \ y_n \in Tx_n, \tag{3}$$

$$\|y_n - x^*\| = d(Tx_n, x^*). \tag{4}$$

where α_n is real sequences in (0,1) satisfying the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\lim \alpha_n = 0$,
(iii) $\beta_n \in (0, 1)$.
They also introduced the following sequences:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n, \ z_n \in Tx_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \ u_n \in Ty_n, \end{cases} \tag{5}$$

$$\begin{cases} \|z_n - x^*\| = d(x^*, Tx_n), \\ \|u_n - x^*\| = d(Ty_n, x^*) \end{cases} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences satisfying the following conditions:

(i) $0 \leq \alpha_n, \ \beta_n < 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.
Sastry and Babu called the process defined by (3) a Mann iteration process and the process defined by (5) where the iteration parameters α_n, β_n satisfy conditions (i), (ii), and (iii) an Ishikawa iteration process. They proved in [23] that the Mann and Ishikawa iteration schemes for a multivalued map T with fixed point p converge to a fixed point of T under certain conditions. More precisely, they proved the following result for a multi-valued nonexpansive map with compact domain.

Theorem 1.2. *(Sastry and Babu [23]). Let H be real Hilbert space, K a nonempty compact convex subset of H, and $T : K \to P(K)$ a multivalued nonexpansive map with a fixed point p. Assume that (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n \beta_n = \infty$. Then, the sequence $\{x_n\}$ defined by (5) converges strongly to a fixed point of T.*

Recently, Sow et al. [25], motivated by the fact that Krasnoselskii-Mann algorithm method is remarkably useful for finding fixed points of single-valued nonexpansive mapping, proved the following theorem.

Theorem 1.3 (Sow et al. [25]). *Let E be a uniformly smooth real Banach space having a weakly continuous duality map and K a nonempty, closed and convex cone of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)Tx_n. \tag{7}$$

Suppose the following conditions hold:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$
(iii) $\lim_{n\to\infty} \lambda_n = 1$, $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$, *and* $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$
Then, the sequence $\{x_n\}$ *generated by* (7) *converges strongly to* $x^* \in F(T)$,

Motivated by the aforementioned results, we introduce a new iterative algorithm for finding a common fixed points of an infinite family of multivalued quasi-nonexpansive mappings in Banach spaces without compactness assumptions and rigid conditions like (4) and (6). We prove strong convergence by our scheme in real Banach space having a weakly continuous duality map. We apply, our main results to minimum-norm fixed point problem involving an infinite family of multivalued quasi-nonexpansive mappings. Finally, our method of proof is of independent interest.

2. Preliminaries

We start with the following demiclosedness principle for set-valued nonexpansive mappings.

Definition 2.1. *Let E be real Banach space and $T : D(T) \subset E \to 2^E$ be a multivalued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.*

Lemma 2.2 (Demi-closedness Principle, [4]). *Let E be a uniformly convex Banach space satisfying the Opial condition, K be a nonempty closed and convex subset of E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2.3 ([15]). *Let E be a real Banach spaces. Then, the following inequality holds*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle$$

for all x,y $\in E$.
In particular, for all x,y $\in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

Lemma 2.4 (Xu, [26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that* (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (Chang et al. [7]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$, a closed ball with center 0 and radius $r > 0$. For any given sequence $\{u_1, u_2, ..., u_n, ...\} \subset B(0)_r$ and any positive real numbers $\{\lambda_1, \lambda_2, ..., \lambda_n, ...\}$ with $\sum_{k=1}^{\infty} \lambda_k = 1$, then there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+, \; g(0) = 0,$$

such that for any integer i, j with $i < j$,

$$\|\sum_{k=1}^{\infty} \lambda_k u_k\|^2 \leq \sum_{k=1}^{\infty} \lambda_k \|u_k\|^2 - \lambda_i \lambda_j g(\|u_i - u_j\|).$$

Lemma 2.6. *[22] Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z)\rangle \leq 0 \tag{8}$$

for all $z \in C$ and $y \in D$.

It is noted that Lemma 2.6 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

3. Main results

In this section, we present and analyze our iterative method for finding a common fixed points of an infinite family of multivalued quasi-nonexpansive mappings in uniformly convex Banach spaces.
In what follows, we use the following explicit iteration scheme: let K a nonempty, closed and convex cone of E and $T_i : K \to CB(K),\ \ i \in \mathbb{N}$ be a multivalued quasi-nonexpansive mapping.
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in K$ by:

$$\begin{cases} y_n = \alpha_n(\lambda_n x_n) + (1-\alpha_n)x_n \\ x_{n+1} = \beta_{n,0}y_n + \displaystyle\sum_{i=1}^{\infty}\beta_{n,i}z_n^i,\ \ z_n^i \in T_i y_n, \end{cases} \tag{9}$$

$\{\beta_{n,i}\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be a real sequences in $(0, 1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

(ii) $\sum_{i=0}^{\infty}\beta_{n,i} = 1,\ \lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0,$ for all $i \in \mathbb{N}.$ We describe the differents steps of our algorithm for approximating fixed of an infinite family of multivalued mappings in real Banach spaces.

3.1. Algorithm

Step 0: Choose $\{\beta_{n,i}\}$ and $\{\alpha_n\}$ be a real sequence in $(0, 1)$ satisfying above conditions. Let $x_0 \in K$ be a given starting point. Set $n = 0$.
Step 1: Compute $y_n = \alpha_n(\lambda_n x_n) + (1-\alpha_n)x_n$. If $\{y_n\}$ is bounded and $d(y_n, T_i y_n) = 0$, STOP.
Step 2: Pick z_n^i in $T_i y_n$,
Step 3: Compute
$x_{n+1} = \beta_{n,0}y_n + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i,$
Step 4: Set $n \longleftarrow n + 1$, and go to **Step 1**.
Recall that $\{y_n\}$ is bounded and $d(y_n, T_i y_n) = 0$ implies that we are a solution of fixed points problem involving multivalued mapping. In our convergence theory, will implicitly assume that this does not occur after finitely many iterations, so that Algorithm 3.1 generates an infinite sequence satisfing $d(y_n, T_i y_n) \neq 0$ for all $n \in \mathbb{N}$.

Remark 3.1. *Algorithm 3.1 at each iteration requires onely four steps. This distinguishes our scheme from most other iterative methods for solving fixed points problem involving multivalued mappings, see, for exemple [1, 23, 24] where more than four steps per iteration is needed.*

Remark 3.2. *It is well known that if K is a closed and convex cone of a real Hilbert space H, we have $\lambda x \in K$ for all $\lambda \in (0, 1)$ and $x \in K$. Therefore, the sequence $\{x_n\}$ generated by (9) is well defined.*

We now prove the following results.

Theorem 3.3. *Let E be a uniformly convex real Banach space E having a weakly continuous duality map J_φ and K a nonempty, closed and convex cone of E. Let $T_i : K \to CB(K),\ i \in \mathbb{N}^*$ be a multivalued quasi-nonexpansive mapping such that $\Gamma = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $T_i p = \{p\}$ for all $p \in \Gamma$ and $I - T_i$ is demiclosed at the origine. Then, the sequence $\{x_n\}$ defined by (9) converges strongly to $x^* \in \Gamma$, where $x^* = Q_\Gamma(0)$.*

Proof. First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded. Take $p \in \Gamma$, using (9), the fact hat $T_i p = \{p\}$ and T_i is multivalued quasi-nonexpansive, we arrive at

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_{n,0} y_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i - p\| \\ &\leq \beta_{n,0}\|y_n - p\| + \sum_{i=1}^{\infty} \beta_{n,i}\|z_n^i - p\| \\ &\leq \beta_{n,0}\|y_n - p\| + \sum_{i=1}^{\infty} \beta_{n,i} H(T_i y_n, T_i p) \\ &\leq \|y_n - p\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)x_n - p\| \\ &\leq \alpha_n \lambda_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \alpha_n \lambda_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq [1 - (1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \max\{\|x_n - p\|,\ \|p\|\}. \end{aligned}$$

It immediately follows that

$$\|x_{n+1} - p\| \leq \max\{\|x_n - p\|,\ \|p\|\}.$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|,\ \|p\|\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded also are $\{y_n\}$ and $\{T_i y_n\}$.
Let $k \in \mathbb{N}^*$, from Lemma 2.5 and (9), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\beta_{n,0} y_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i - p\|^2 \\ &\leq \sum_{i=1}^{\infty} \beta_{n,i}\|z_n^i - p\|^2 + \beta_{n,0}\|y_n - p\|^2 - \beta_{n,0}\beta_{n,k} g(\|z_n^k - y_n\|) \\ &\leq \sum_{i=1}^{\infty} \beta_{n,i} H(T_i y_n, T_i p)^2 + \beta_{n,0}\|y_n - p\|^2 - \beta_{n,0}\beta_{n,k} g(\|z_n^k - y_n\|) \\ &\leq \|y_n - p\|^2 - \beta_{n,0}\beta_{n,k} g(\|z_n^k - y_n\|). \end{aligned}$$

Therefore, by Lemma 2.3, we have

$$\begin{aligned} \beta_{n,0}\beta_{n,k} g(\|z_n^k - y_n\|) &\leq \|y_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n(1-\lambda_n)\langle x_n, J(p - y_n)\rangle. \end{aligned}$$

which implies that,

$$\beta_{n,0}\beta_{n,k}g(\|z_n^k - y_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n(1-\lambda_n)\langle x_n, J(p - y_n)\rangle. \quad (10)$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exists a constant $A > 0$ sucht that

$$(1-\lambda_n)\langle x_n, J(p - y_n)\rangle \leq A, \text{ for all, } n \geq 0.$$

So, from (10) we have

$$\beta_{n,0}\beta_{n,k}g(\|z_n^k - y_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n A. \quad (11)$$

To complete the proof, we consider the following two cases. **Case 1**. Assume there exists an $n_0 \in \mathbb{N}$ for which $\|x_{n+1} - p\| \leq \|x_n - p\|$ for all $n \geq n_0$.
Then, $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0.$$

It then implies from (11) that

$$\lim_{n\to\infty} \beta_{n,0}\beta_{n,k}g(\|z_n^k - y_n\|) = 0. \quad (12)$$

Since $\lim_{n\to\infty}\inf \beta_{n,0}\beta_{n,k} > 0$ and property of g, we have

$$\lim_{n\to\infty} \|z_n^k - y_n\| = 0. \quad (13)$$

Hence,

$$\lim_{n\to\infty} d(y_n, T_k y_n) = 0. \quad (14)$$

From (9), we have

$$\|y_n - x_n\| = \alpha_n\|(\lambda_n x_n) - x_n\| \to 0, \text{ as } n \to \infty. \quad (15)$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - y_n)\rangle \leq 0$. Since E is reflexive and $\{y_n\}_{n\geq 0}$ is bounded there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that y_{n_k} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - y_n)\rangle = \lim_{k\to+\infty}\langle x^*, J_\varphi(x^* - y_{n_k})\rangle.$$

From (14) and $I - T_k$ are demiclosed, we obtain $a \in \Gamma$. On other hand, the assumption that the duality mapping J_φ is weakly continuous and the fact that $x^* = Q_\Gamma(0)$, we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - y_n)\rangle &= \lim_{k\to+\infty}\langle x^*, J_\varphi(x^* - y_{n_k})\rangle \\ &= \langle x^*, J_\varphi(x^* - a)\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. In fact, since $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma,\ \forall t \geq 0$, and φ is a gauge function, then for

$1 \geq k \geq 0,\ \Phi(kt) \leq k\Phi(t)$. From (9) and Lemma 2.3, we get that

$$\begin{aligned}
\Phi(\|x_{n+1} - x^*\|) &= \Phi(\|\beta_{n,0}y_n + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i - x^*\|) \\
&\leq \Phi(\beta_{n,0}\|y_n - x^*\| + \sum_{i=1}^{\infty}\beta_{n,i}\|z_n^i - x^*\|) \\
&\leq \Phi(\beta_{n,0}\|y_n - x^*\| + \sum_{i=1}^{\infty}\beta_{n,i}H(T_iy_n, T_ix^*)) \\
&\leq \Phi(\|y_n - x^*\|) \\
&= \Phi(\|\alpha_n(\lambda_n x_n) + (1-\alpha_n)x_n - x^*\|) \\
&\leq \Phi(\|\alpha_n\lambda_n(x_n - x^*) + (1-\alpha_n)(y_n - x^*)\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - y_n)\rangle \\
&\leq \Phi(\alpha_n\lambda_n\|x_n - x^*\| + \|(1-\alpha_n)(y_n - x^*)\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - y_n)\rangle \\
&\leq \Phi(\alpha_n\lambda_n\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - y_n)\rangle \\
&\leq \Phi((1-(1-\lambda_n)\alpha_n)\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - y_n)\rangle \\
&\leq [1-(1-\lambda_n)\alpha_n]\Phi(\|x_n - x^*\|) + (1-\lambda_n)\alpha_n\langle x^*, J_\varphi(x^* - y_n)\rangle.
\end{aligned}$$

From Lemma 2.4, its follows that $x_n \to x^*$.

Case 2. Suppose that Case 1 fails. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \ B_k \leq B_{k+1}\}$. We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. From (10), we have

$$\beta_{\tau(n),0}\beta_{\tau(n),i}g(\|z^i_{\tau(n)} - y_{\tau(n)}\|) \leq 2\alpha_{\tau(n)}A \to 0 \ \text{ as } \ n \to \infty.$$

Furthermore, we have

$$\|y_{\tau(n)} - z^i_{\tau(n)}\| \to 0 \ \text{ as } \ n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d\big(y_{\tau(n)}, T_iy_{\tau(n)+1}\big) = 0. \tag{16}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ is bounded in K and $\limsup\limits_{\tau(n)\to+\infty} \langle x^*, J_\varphi(x^* - y_{\tau(n)})\rangle \leq 0$.

We have for all $n \geq n_0$,

$$\Phi(\|x_{\tau(n)} - x^*\|) \leq \langle x^*, J_\varphi(x^* - y_{\tau(n)})\rangle.$$

Then, we obtain

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As consequence, we have for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim\limits_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. $\square$

Remark 3.4. *In our theorem, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$*

We now apply Theorem 3.3 for finding a common fixed points of an infinite family of multivalued nonexpansive mappings without demiclosedness assumption.

Theorem 3.5. *Let K be a nonempty, closed convex subset of a uniformly convex real Banach space E having a weakly continuous duality mapJ_φ. Let $T_i : K \to CB(K),\ i \in \mathbb{N}^*$ be a multivalued nonexpansive mapping with convex-values such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and such that $T_i p = \{p\}$ for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Then, the sequences $\{x_n\}$ defined by (9) converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. Since every nonexpansive mappings is quasi-nonexpansive mappings. It suffices to prove $I - T_i$ is demiclosed at the origine. Using the fact that E satisfies Opial's property and Lemma 2.2, we have demiclosedness assumption is satisfied. This completes the proof of Theorem 3.3. □

Corollary 3.6. *Assume that $E = l_q,\ 1 < q < \infty$. Let K be a nonempty, closed convex cone of E and $T_i : K \to CB(K),\ i \in \mathbb{N}^*$ be a multivalued quasi-nonexpansive mapping such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and such that $T_i p = \{p\}$ for all $p \in \bigcap_{i=1}^{\infty} F(T_i)$ and $I - T_i$ is demiclosed at the origine.*

$$\begin{cases} y_n = \alpha_n(\lambda_n x_n) + (1-\alpha_n)x_n \\ x_{n+1} = \beta_{n,0} y_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \quad z_n^i \in T_i y_n, \end{cases} \tag{17}$$

$\{\beta_{n,i}\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be a real sequences in $(0,1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; $\quad (ii)$ $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty} (1-\lambda_n)\alpha_n = \infty$.

(ii) $\sum_{i=0}^{\infty} \beta_{n,i} = 1,\ \lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0,$ for all $i \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ strongly to a common fixed point of T_i.

Proof. Since $E = l_q,\ 1 < q < \infty$ are uniformly convex and has a weakly continuous duality map. The proof follows from Theorem 3.3. □

4. Application

In this section, we apply our main results for finding minimum-norm fixed point of an infinite family of multivalued quasi-nonexpansive mappings in Hilbert spaces. Let H be a real Hilbert space. Let K be a nonempty, closed convex cone of H and $T_i : K \to CB(K),\ i \in \mathbb{N}^*$ be a multivalued quasi-nonexpansive mapping such that $\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. We consider the following convex minimization problem:

$$\begin{aligned} &\textit{Minimize } \|x\|^2 \\ &\textit{subject to } x \in \Gamma. \end{aligned} \tag{18}$$

Finding an optimal point over the set of fixed points of quasi-nonexpansive mapping is one that occurs frequently in various areas of mathematical sciences, engineering,time-optimal control, optimization, mathematical programming, mechanics.

Remark 4.1. *We have x^* is a solution of (18) if and only if $x^* \in \Gamma$ and x^* solves the following variational inequality problem :*

$$\langle x^*, x^* - p \rangle \leq 0, \quad \forall p \in \Gamma.$$

We also notice that it is quite usual to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset K of a Hilbert space H_1 and a bounded linear operator $A : H_1 \to H_2$, where H_2 is another Hilbert space. The K-constrained pseudoinverse of A, ${A_K}^+$ is then defined as the minimum-norm solution of the constrained minimization problem:

$$ {A_K}^+(b) = \text{argmin}_{x\in K} \|Ax - b\| \tag{19} $$

which is equivalent to the fixed point problem

$$ x = P_K(x - \lambda A^*(Ax - b)), \tag{20} $$

where P_K is the metric projection from H onto K, A^* is the adjoint of A, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{A(K)}}(b) \in A(K)$. From the listed references, there exist a largenumber of problems which need to find the minimum norm solution, see, e.g., [27, 28].A useful path to circumvent this problem is to use projection. The main difficult is in computation. Hence, it is an interesting problem of finding the minimum norm element without using the projection.

Hence, one has the following result

Theorem 4.2. *Let B be the closed unit ball of a real Hilbert space H. Let $T_i : B \to CB(B)$, $i \in \mathbb{N}^*$ be a multivalued quasi-nonexpansive mapping such that $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $T_ip = \{p\}$ for all $p \in F$ and $I - T_i$ is demiclosed at the origine. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined iteratively from arbitrary $x_0 \in B$ by:*

$$ \begin{cases} y_n = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)x_n \\ x_{n+1} = \beta_{n,0}y_n + \sum_{i=1}^{\infty} \beta_{n,i}z_n^i, \;\; z_n^i \in T_iy_n, \end{cases} \tag{21} $$

$\{\beta_{n,i}\}$, $\{\lambda_n\}$ and $\{\alpha_n\}$ be a real sequences in $(0, 1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.

(ii) $\sum_{i=0}^{\infty} \beta_{n,i} = 1$, $\lim_{n\to\infty} \inf \beta_{n,0}\beta_{n,i} > 0$, *for all* $i \in \mathbb{N}$.

Then, the sequence $\{x_n\}$ defined by (21) converges strongly to a solution of (18).

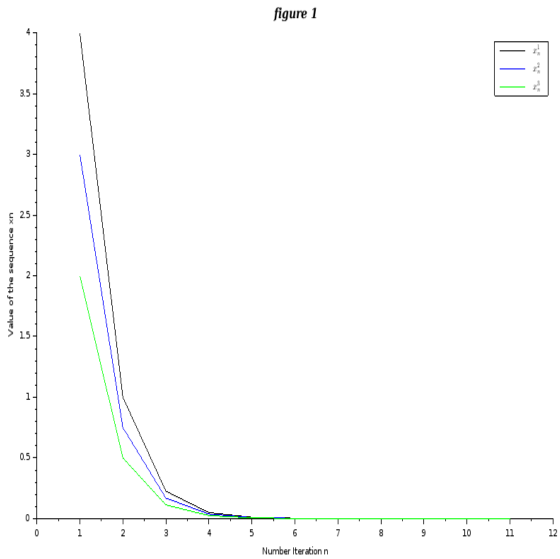
Proof. The proof follows from Theorem 3.3 and Remark 4.1. □

5. NUMERICAL ILLUSTRATION

In this last section, we present a numerical example to illustrate the convergence behavior of our iteration scheme (9). In our computations, we choose $\beta_{n,i} = \frac{1}{3}$, $i = 0, 2$, $\alpha_n = \frac{1}{2\sqrt{n}}$ and $\lambda_n = 1 - \frac{1}{\sqrt{n}}$. We consider the family of (T_i) of mappings defined by $T_j : [0, \gamma] \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, $T_jx = [0, \frac{x}{j}]$ $i = 1, 2$, where γ is a fixed vector in $\mathbb{R}^3$ and $[x, y]$ denotes the set $\{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$. This is a family of multivalued quasi-nonexpansive mappings having a common fixed point of 0. Taking the initial point $x_1 = (2, 3, 4)$, the result of the numerical example obtained by using MATLAB is given in figure 1 where it is shown that the sequence of iterates $\{x_n\}$ strongly converges to common fixed of T_1 and T_2.

$$ \begin{cases} y_n = \dfrac{2n-1}{2n}x_n \\ x_{n+1} = \dfrac{2n-1}{6n}x_n + \dfrac{z_n^1 + z_n^2}{3}, \quad z_n^1 \in \Big[0, \dfrac{2n-1}{2n}x_n\Big], \; z_n^2 \in \Big[0, \dfrac{2n-1}{4n}x_n\Big]. \end{cases} \tag{22} $$

Take the initial point $x_1 = (2, 3, 4)$ the numerical experiment result using MATLAB is given by Figure 1,which show the iteration process of the sequence x_n converges strongly to common fixed point of T_2 and T_2.



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