Functional Analysis,
Approximation and
Computation
16 (1) (2024), 49–54

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

A simple empirical inquiry concerning tail risk

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Abstract. This research study aims to compare and evaluate the performance of Hill's tail index estimator and the estimator proposed in [1]. The analysis was performed on GARCH (1,1) data which are widely used for modeling processes with time-varying volatility. These include financial time series, which can be particularly heavy-tailed. The tail index is a key parameter for quantifying the extreme tail behaviour of financial time series, which is crucial for the risk management and decision-making. The work is an empirical continuation of the results obtained in [12] and it tracks the behaviour of the tail index estimators in the simulated GARCH (1,1) samples and also in the case of the GBP/CAD exchange rate between 1st May 2007 and 18th October 2010. The results highlight the strengths and limitations of each estimator and thus provide the possibility of certain improvements to the risk assessment and decision-making processes in various financial applications.

1. Introduction

Understanding and analyzing extreme events and using them in modeling and managing financial risk is of great interest in financial markets. Specifically, the examination of the extreme value theory holds significant importance when studying the long-term characteristics of distribution tails in stationary return series.

Extreme value theory allows researchers to make inferences about the behaviour of return distributions beyond the observed range of sample returns, particularly focusing on extreme events such as market crashes or individual financial distress. One of the key aspects of extreme value theory is the estimation of the tail index, which characterizes the degree of extremeness of the distribution tails.

Several studies have explored the tail index of the return distributions, such as [4], [6] and [16]. These research endeavors explored varied approaches and methodologies aimed at estimating and analyzing the tail index to improve strategies for managing financial risk.

This paper is also built upon earlier studies focused on weekly and daily exchange rate data (see [14] and [15]). Also see [17] for a thorough examination of the generalized autoregressive conditional heteroskedastic (GARCH) models with student-t innovations in financial returns.

2020 *Mathematics Subject Classification*. 62G32
Keywords. Tail risk, Extreme events, Tail index estimation, Hill estimator, Pareto index
Received: 2 February 2024; Accepted: 25 April 2024
Communicated by Dragan S. Djordjević
The authors would like to thank the Ministry of Education, Science and Technological Development of Republic of Serbia (Grant No: 451-03-9/2021-14/200113) for financial support.
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Based on the experimental data sets, the paper examines the behaviour of the tail index estimators involving two phases. In the first one, the GARCH (1,1) model is considered a model for the simulated data in various structures, while in the second step, the paper focuses on the evaluation of the estimates of the tail index for exchange rate changes in the real-life example.

Overall, the paper aims to contribute to the understanding and evaluation of fat-tailed distributions that incorporate GARCH models by calculating certain tail index estimators and then comparing their characteristics.

The structure of the article is as follows:

Section 2: This section provides a review of the Hill estimator, which is commonly used to estimate the tail index. The theoretical reasons for analyzing and investigating the median estimator proposed in [1] are also presented in this section. The focus here is on introducing the main goal and the motivation for comparing the median estimator with Hills.

Section 3: In this section, we present the results from the simulation study. The simulations are designed to test the estimators in different scenarios involving GARCH (Generalized Autoregressive Conditional Heteroscedasticity) processes. The goal is to demonstrate the effectiveness of the median estimator in reducing the bias in tail-index estimates.

Section 4: This section provides tail-index estimates for returns on the GBP/CAD exchange rate (the value of the British Pound against the Canadian Dollar) between 1st May 2007 and 18th October 2010. This analysis aims to evaluate the performance of the new so-called median estimator in real-world financial data and compare it with the conventional Hill estimator.

Section 5: The article concludes in this section.

Overall, the article follows a structured approach, starting with a theoretical explanation of the adjusted methodology, validating it through the simulation studies, applying it to real-world exchange rate data, and finally concluding with a summary of the results and potential future directions.

2. Foundational concept and notation

The prediction of rare events and conclusively the estimation of the tail index are fundamental roles of extreme statistics. This estimation primarily involves the use of the k largest order statistics or exceedances above a certain threshold value. Of course, one of the most challenging and difficult questions in extreme value theory is how to determine the value of the top order statistics k, which is typically considered in practical applications. In the literature, there are many papers dealing with the optimal sequences of the largest sample values that should be used, such as the well-known bootstrap methodology and the Hill plot for choosing the optimal k. These techniques help us conduct the appropriate data analysis and estimate the tail index. The time series that appear in various economic fields are typically thick-tailed and can be well modeled by using distributions with the tails satisfying:

$$1 - F(x) = x^{-\alpha}L(x), \quad x > 0, \tag{1}$$

where $\alpha > 0$ denotes the index of regular variation and L is slowly varying at infinity, that is:

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} \to 1, \quad \text{for } x > 0, \tag{2}$$

where the parameter α is the tail-thickness index also known as the Pareto index or the index of regular variation.

Suppose $\{X_t\} = \{X_t : 1 \leq t \leq n\}$ is a sample of random variables equally distributed with the heavy-tailed distribution function F and $X_{(1)} > X_{(2)} > ... > X_{(n)}$ are the order statistics. Probably the most popular estimator of α, based on the extreme portion of the sample and obtained as the reciprocal value of the arithmetic mean of certain logarithmic differences is by all means the Hill estimator $\widehat{\alpha}_H$, defined as follows ([10]):

$$\widehat{\alpha}_H = \left(\frac{1}{k} \sum_{i=1}^{k} \ln X_{(i)} - \ln X_{(k+1)} \right)^{-1}, \tag{3}$$

where $k = k_n$ is a sequence of positive integers satisfying:

$$1 \leq k_n < n, \lim_{n\to\infty} k_n = \infty \text{ and } \lim_{n\to\infty} k_n/n = 0. \tag{4}$$

Among numerous papers devoted to the Hill's estimator behaviour see for example [8] and [3]. Also see [11] for the proof of the asymptotic normality of the Hill estimator under the assumption of the incompleteness of the sample consisting of heterogeneous and dependent data.

It is well known that $\widehat{\alpha}_H$ is a strong consistent estimator of α, asymptotically normally distributed.

On the other hand, in [1] has been defined and thoroughly studied an estimator formulated as:

$$\widehat{\alpha}_{B,p} = \left(-\frac{1}{\ln p} \ln \frac{X_{(\lceil p k_n \rceil)}}{X_{(\lceil k_n \rceil)}} \right)^{-1}, \tag{5}$$

where $0 < p < 1$ is a fixed constant and $\lceil x \rceil$ is a ceiling function and denotes the smallest integer greater than or equal to x and k_n is a nondecreasing sequence of positive real numbers satisfying (1.4). For the results on the strong asymptotic behaviour of B_p see [1].

In this paper we observe together with Hill's the empirical behaviour of the so-called median estimator $\widehat{\alpha}_M$:

$$\widehat{\alpha}_M = \left(\frac{1}{\ln 2} \ln \frac{X_{(\lceil \frac{k_n}{2} \rceil)}}{X_{(k_n)}} \right)^{-1}, \tag{6}$$

where k_n is a sequence of positive integers satisfying (4). At the end of this section, it is crucial to mention that this research paper is the natural continuation of the results obtained in [12] and it tends to reveal the behaviour of the $\widehat{\alpha}_M$ estimator when real data are involved.

3. Simulation study

A simulation study was conducted to investigate the performances of Hill's and $\widehat{\alpha}_M$ estimators. The simulation study via Monte Carlo method is performed on GARCH(1,1) model, assuming that the innovation terms follow a standard normal distribution. The GARCH model with normal distributed innovations has been analyzed for instance in [5] and [9]. The GARCH coefficients are estimated by the MLE outlined in [2]. With the estimated GARCH coefficients, we calculate the implied Hill's and $\widehat{\alpha}_M$ tail indices according to the fitted models. Also, we obtain the Hill estimates by choosing an optimal sample fraction k (the optimal k is chosen from the first stable region of the tail index estimates in the Hill plots, which is the plot of the estimates against various potential k levels, see [7]). The same value of k was used in the calculation of the $\widehat{\alpha}_M$ estimator.

Let us suppose that the returns X_t satisfy the following model:

$$X_t = \varepsilon_t \sigma_t, \tag{7}$$

$$\sigma_t^2 = \lambda_0 + \lambda_1 X_{t-1}^2 + \lambda_2 \sigma_{t-1}^2, \tag{8}$$

where ε_t are independent and identically distributed innovations with zero mean and unit variance and the parameters $\lambda_0, \lambda_1, \lambda_2$ are positive. Moreover, in order to have a stationary solution of the GARCH model, we assume $\lambda_1 + \lambda_2 < 1$ (see the results in [13]).

By combining the equations 7 and 8, we derive the following stochastic difference equation on the stochastic variance series σ_t^2 as

$$\sigma_t^2 = \lambda_0 + (\lambda_1 \varepsilon_{t-1}^2 + \lambda_2)\sigma_{t-1}^2.$$

Denote $\varphi = \varepsilon_{t-1}^2$. Suppose α is the solution to the equation

$$E((\lambda_1 \varphi + \lambda_2)^{\frac{\alpha}{2}}) = 1. \tag{9}$$

The stationary solution of σ_t^2 follows a heavy-tailed distribution with tail index $\frac{\alpha}{2}$. Hence, σ_t follows a heavy-tailed distribution with the tail index α. From [17], we can derive the following relation:

$$P\{|X_t| > x\} = P\{|\sigma_t \varepsilon_t| > x\} \sim E(|\varepsilon_t|)^{\frac{\alpha}{2}} P\{\sigma_t > x\}, \tag{10}$$

as $x \to \infty$.

Thus $|X_t|$ has a similar tail behaviour as σ_t: the tail index of $|X_t|$ equals to α. For the GARCH coefficients, we fixed $\lambda_0 = 0.5$ and analyzed the next pairs of coefficients: $\lambda_1 = 0.15$ and $\lambda_2 = 0.8$, $\lambda_1 = 0.07$ and $\lambda_2 = 0.88$, $\lambda_1 = 0.08$ and $\lambda_2 = 0.87$, $\lambda_1 = 0.08$ and $\lambda_2 = 0.9$ and $\lambda_1 = 0.11$ and $\lambda_2 = 0.88$. The reason for choosing the last pair of coefficients is for the real-life analysis that will be performed in the next section, having particularly these estimated coefficients. And for the rest of the pairs, we tried to maintain the condition $\lambda_1 + \lambda_2 < 1$, for the sake of the stability of the simulated process. For each pair λ_1 and λ_2 the value of true α is calculated from the equation 9. Next,we performed the simulation of 100 replications of 5000 observations long paths for the model. We set the threshold at $k = 500$ and then we calculated the tail index using the $\widehat{\alpha}_M$ estimator and the Hill estimator. The optimal k is chosen from the first stable region of the tail index estimates in the Hill plot, which is the plot of the estimates against various potential k levels [7]. For each model we repeat it 100 times to obtain an average estimate for each tail index estimator. Next, we calculated the corresponding ME and MSE, for both estimators. The results are presented in Table 1 and Table 2. Also, Fig.1 shows the behaviour of the estimators for various coefficients applied in the simulated Garch(1,1) models. In the concluding section, we gave a thorough analysis of the obtained values and also a comparison of the applied tail index estimators underlying their advantages and disadvantages.

Table 1: $\widehat{\alpha}_M$ of GARCH(1,1) normal series-complete sample

			$\lambda_0 = 0.5$		
λ_1	λ_2	α	$\widehat{\alpha}_M$	ME($\widehat{\alpha}_M$)	MSE($\widehat{\alpha}_M$)
0.15	0.8	5.78	5.45	−0.44	0.11
0.07	0.88	6.48	6.76	0.32	0.83
0.08	0.87	5.94	5.96	1.5	2.4
0.08	0.9	4.42	3.82	−0.63	0.23
0.11	0.88	3.7	3.69	−0.02	0.005

Table 2: $\widehat{\alpha}_H$ of GARCH(1,1) normal series-complete sample

			$\lambda_0 = 0.5$		
λ_1	λ_2	α	$\widehat{\alpha}_H$	ME($\widehat{\alpha}_H$)	MSE($\widehat{\alpha}_H$)
0.15	0.8	5.78	3.95	−1.83	3.35
0.07	0.88	6.48	2.65	−3.82	14.7
0.08	0.87	5.94	3.40	−2.56	6.64
0.08	0.9	4.42	3.10	−1.5	2.46
0.11	0.88	3.7	2.10	−1.89	3.24

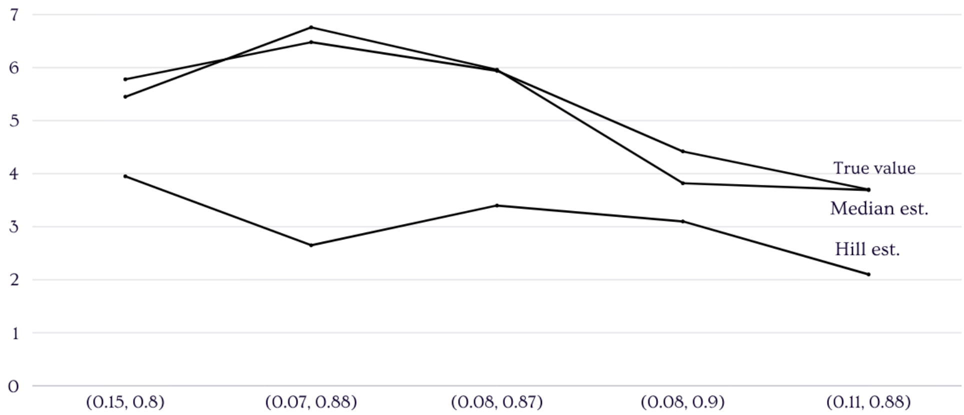

Figure 1: Comparison of the tail index estimators

Fig. 1 presents the values of the obtained $\widehat{\alpha}_H$ and $\widehat{\alpha}_M$ compared with the true value of α. For the fixed value of $\lambda_0 = 0.5$ we observed and calculated the estimators for different pairs of λ_1 and λ_2. It is obvious that $\widehat{\alpha}_M$ produced much lower bias and significantly superior results than $\widehat{\alpha}_H$ (see Fig.1).

4. Results on the exchange-rate returns

In this section, we consider the empirical data and analyze two estimators of the tail index. To illustrate the relevance of the median estimator in real-world cases, we now apply both the median and the Hill estimator to obtain tail estimates for exchange rates of the GBP (Great Britain pound) against the CAD (Canadian dollar) over the period from 1st may 2007 to 18th october 2010. Using daily returns, the sample size n equals 1260. The exchange rate between the British pound (GBP) and the Canadian dollar (CAD) can experience significant and rapid changes, showing high volatility. Investors who monitor these fluctuations can take advantage of changes in the values, so they use various tools to forecast price movements and make trading decisions. Both GBP and CAD are associated with strong economies. Therefore, this currency pair is highly sensitive to central bank monetary policies, political changes, or geopolitical factors.

The results for both estimators are given in Table 3.

Table 3: $\widehat{\alpha}_H$ and $\widehat{\alpha}_M$ of exchange rate between GBP and CAD

$\alpha = 5.94$					
$\widehat{\alpha}_H$	$\widehat{\alpha}_M$	ME($\widehat{\alpha}_H$)	ME($\widehat{\alpha}_M$)	MSE($\widehat{\alpha}_H$)	MSE($\widehat{\alpha}_M$)
4.75	5.04	-1.19	1.41	-0.89	0.8

Again, as we can see the conventional Hill estimator produces significantly lower estimates than the median $\widehat{\alpha}_M$ estimator. Although the Hill estimator is by all means the benchmark in the literature for calculating the degree of the tail fatness, it is biased in relatively small samples, as we can confirm from our study. So, the alternative estimators such as $\widehat{\alpha}_M$ need to be applied.

5. Conclusion

Note that this estimator represents, like Hill's, an important statistical measure, i.e $(\widehat{\alpha}_H)^{-1}$ may be pointed to as a mathematical expectation of a certain sequence of logarithmic differences and the $(\widehat{\alpha}_M)^{-1}$ as their median. To summarize the findings from the studies, the Hill estimator is not robust to outliers or extreme values in the tail of the distribution. It heavily relies on the largest observations, making it sensitive to extreme values and prone to a substantial impact from outliers. On the other hand, the median-based estimator is more robust to outliers compared to the Hill estimator. By using the median of top-order statistics, it mitigates the influence of extreme values, providing a more robust estimate of the tail index. The Hill estimator is consistent, meaning it converges to the true tail index as the sample size increases. However, its efficiency can vary depending on the characteristics of the data and the underlying distribution.

In summary, the Hill estimator and the median-based estimator exhibit the following characteristics:

The Hill estimator is less robust to outliers, can be biased, and its efficiency depends on the distribution and sample size. It can be highly variable in small samples. The median-based estimator is more robust to outliers, tends to be less biased, and can be efficient under certain conditions. It demonstrates better stability in small samples.

By conducting this research, we have hopefully contributed to the existing body of knowledge on tail risk and enhanced the understanding of extreme events in various domains. The findings from this study can potentially inform risk management strategies, asset allocation decisions, and the development of more robust financial models. Additionally, the empirical insights gained from this inquiry may offer valuable insights into the dynamics of rare events and aid in improving the accuracy of risk assessments and forecasting models. The implications of the presented research can potentially provide recommendations for the most suitable technique for further studies dealing with tail thickness evaluation, especially in the field of finance and economics.

Overall, this paper seeks to shed light on the nature of tail risk, providing a foundation for further research and practical applications. By investigating the empirical aspects of extreme events, we aim to provide valuable insights that can help individuals and institutions navigate the challenges posed by tail risk and enhance their ability to manage and mitigate potential losses.

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