Functional Analysis,
Approximation and
Computation
16 (1) (2024), 55–68

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Some common coupled fixed point theorems for a pair of occasionally weakly compatible mappings in S-metric spaces

Gurucharan Singh Saluja [a]

[a] H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India

Abstract. The aim of this paper is to prove some common coupled fixed point theorems for a pair of occasionally weakly compatible mappings and for a pair of (CLR_G) property in the framework of S-metric spaces by using quadratic inequality. Also, we illustrate an example to validate the result. The results presented in this paper generalize, extend and enrich several previous works from the existing literature.

1. Introduction

Banach contraction principle [7] in metric spaces is one of the most important results in the theory of fixed points and non-linear analysis. Fixed point problem for contractive mappings in metric spaces with a partial order have been studied by many authors. In literature, there are many generalizations of the metric spaces and generalized metric spaces are exist. The more generalized form of metric space named as S-metric space, was first proposed by *Sedghi et al.* [29] as a generalization of G-metric (*Mustafa and Sims* [23]) and D^*-metric (*Sedghi et al.* [28]) in 2012. They studied its some properties and they also stated that S-metric space is a generalization of G-metric space. But *Dung et al.* [11] in 2014 showed by an example that an S-metric space is not a generalization of G-metric space and conversely. Consequently, the class of S-metric spaces and the class of G-metric spaces are different. Many results which were proved earlier in metric space are valid in the framework of S-metric spaces.

Bhashkar and Lakshmikantham in [8] (see, also [13]) introduced the notion of coupled fixed points and proved some coupled fixed point theorems in partially ordered complete metric spaces. They also proved mixed monotone property for the first time and gave their classical coupled fixed point theorem for mapping which satisfy the mixed monotone property. As, an application, they studied the existence and uniqueness of the solution for a periodic boundary value problem associated with first order differential equation. These results were further extended and generalized by *Ćirić and Lakshmikantham* [10] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces (see, also [22], [24], [26]).

Several authors have introduced various conditions, known as compatible conditions in order to establish the presence of common fixed points. If the two mappings commute (*Jungck* [16]), it is the simplest

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25.
Keywords. Common coupled fixed point, contractive type inequality, occasionally weakly compatible condition, S-metric space.
Received: 15 May 2023; Accepted: 9 July 2023
Communicated by Dragan S. Djordjević
Email address: saluja1963@gmail.com (Gurucharan Singh Saluja)

technique to find common fixed points. However, this condition is the strongest one, it is quite common to look for weaker one. In 1986, *Jungck* [17] proposed the property of compatibility between two mappings. After that, the idea of weak compatibility was first introduced by *Jungck and Rhoades* [19] in the year 1998. *Thagafi and Shahazad* [4] presented occasional weak compatibility (owc) between two mappings which is a weaker condition than weak compatibility. *Aamri and Moutawakil* [1] introduced the new concept called ($E.A$)-property, which is widely used by authors to verify common fixed points. In 2011, *Sintunavarat and Kumam* [32] introduced a new property known as (CLR) property. The importance of (CLR) property ensures that one does not require the closedness of range subspaces. In 2012, *Imdad et al.* [15] introduced the new concept called (CLR)-property for two pairs of self mappings and proved some common fixed point theorems using this new concept. Recently, some researchers employed this notion to obtain some new fixed point results in various metric spaces (see, for example, [3], [6], [9], [21], [25], [32] and many others).

Aydi [5] proved some coupled fixed point theorems for various contractive type conditions in the setting of partial metric spaces and give some corollaries of the established results. Recently, Kim et al. [21] proved some common coupled fixed point results satisfying some contractive condition for a pair of weakly compatible mappings in the framework of complete partial metric spaces.

In this paper, we give the concept of (E.A) property, (CLR)-property, occasionally weakly compatibility and weakly compatibility condition for coupled mappings and prove some common coupled fixed point theorems for a pair of occasionally weakly compatible (owc) mappings in the setting of S-metric spaces. The results presented in this paper extend and generalize several previous works from the existing literature.

2. Preliminaries

In this section, we need the following definitions, lemmas and auxiliary results to prove our main results (see, [29]).

Definition 2.1. *([29]) Let X be a nonempty set and let $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $a \in X$:*

(S1) $0 < S(x,y,z)$ for all $x,y,z \in X$ with $x \neq y \neq z$;

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$;

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space.

Example 2.2. *([29])*

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Example 2.3. *([30]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.*

Example 2.4. *([20]) Let X be a non-empty set and d be an ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X.*

Example 2.5. *([31]) Let X be a non-empty set and d_1, d_2 be two ordinary metrics on X. Then $S(x,y,z) = d_1(x,z) + d_2(y,z)$ for all $x,y,z \in X$ is an S-metric on X.*

Example 2.6. *([29]) Let $X = \mathbb{R}^2$ and d an ordinary metric on X. Put $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given x,y,z. Then S is an S-metric on X.*

Definition 2.7. *Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $\mathcal{B}_S(x,r)$ and closed ball $\mathcal{B}_S[x,r]$ with center x and radius r as follows, respectively:*

$$\mathcal{B}_S(x,r) = \{y \in X : S(y,y,x) < r\},$$

$$\mathcal{B}_S[x,r] = \{y \in X : S(y,y,x) \leq r\}.$$

Example 2.8. *([30]) Let $X = \mathbb{R}$. Denote by $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Then*

$$\begin{aligned} \mathcal{B}_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_S[2,4] &= \{y \in \mathbb{R} : S(y,y,2) \le 4\} = \{y \in \mathbb{R} : |y-2| \le 2\} \\ &= \{y \in \mathbb{R} : 0 \le y \le 4\} = [0,4]. \end{aligned}$$

Definition 2.9. *([29], [30]) Let (X,S) be an S-metric space and $A \subset X$.*

(Δ_1) *The subset A is said to be an open subset of X, if for every $x \in A$ there exists $r > 0$ such that $\mathcal{B}_S(x,r) \subset A$.*

(Δ_2) *A sequence $\{y_n\}$ in X converges to $y \in X$ if $S(y_n,y_n,y) \to 0$ as $n \to +\infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $S(y_n,y_n,y) < \varepsilon$. We denote this by $\lim_{n\to+\infty} y_n = y$ or $y_n \to y$ as $n \to +\infty$.*

(Δ_3) *A sequence $\{y_n\}$ in X is called a Cauchy sequence if $S(y_n,y_n,y_m) \to 0$ as $n,m \to +\infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \ge n_0$ we have $S(y_n,y_n,y_m) < \varepsilon$.*

(Δ_4) *The S-metric space (X,S) is called complete if every Cauchy sequence in X is convergent in X.*

(Δ_5) *Let τ be the set of all $A \subset X$ with the property that for each $x \in A$ and there exists $r > 0$ such that $\mathcal{B}_S(x,r) \subset A$. Then τ is a topology on X (induced by the S-metric space).*

(Δ_6) *A nonempty subset A of X is S-closed if closure of A coincides with A.*

Definition 2.10. *([29]) Let (X,S) be an S-metric space. A mapping $\mathcal{T} \colon X \to X$ is said to be a contraction if there exists a constant $0 \le h < 1$ such that*

$$S(\mathcal{T}x,\mathcal{T}y,\mathcal{T}z) \le h\, S(x,y,z), \tag{1}$$

for all $x,y,z \in X$.

Remark 2.11. *If the S-metric space (X,S) is complete then the mapping defined as above has a unique fixed point (see [29], Theorem 3.1).*

Definition 2.12. *([29]) Let (X,S) and (Y,S') be two S-metric spaces. A function $F \colon X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every sequence $\{x_n\}$ in X with $S(x_n,x_n,x_0) \to 0$, $S'(F(x_n),F(x_n),F(x_0)) \to 0$ as $n \to +\infty$. We say that F is continuous on X if F is continuous at every point $x_0 \in X$.*

Definition 2.13. *Let X be a non-empty set and let $F,G \colon X \to X$ be two self mappings of X. Then a point $z \in X$ is called a*

- *fixed point of operator F if $F(z) = z$;*
- *common fixed point of F and G if $F(z) = G(z) = z$.*

Definition 2.14. *([2]) Let F and G be single valued self-mappings on a set X. If $z = Fv = Gv$ for some $v \in X$, then v is called a coincidence point of F and G, and z is called a point of coincidence of F and G. We denote the coincidence point of F and G by $C(F,G)$, that is, $C(F,G) = \{v \in X : Fv = Gv\}$.*

Definition 2.15. *([17]) Let F and G be single valued self-mappings on a set X. Mappings F and G are said to be commuting if $FGz = GFz$ for all $z \in X$.*

Example 2.16. *Let $X = [0, \frac{3}{4}]$ and define $F,G \colon X \to X$ defined by $F(x) = \frac{x^3}{4}$ and $G(x) = x^4$ for all $x,y \in X$. Then the mappings F and G have two coincidence points 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.*

Definition 2.17. *([18]) Let F and G be single valued self-mappings on a set X. Mappings F and G are said to be weakly compatible if they commute at their coincidence points, i.e., if $Fz = Gz$ for some $z \in X$ implies $FGz = GFz$.*

Definition 2.18. *([1]) Let (X,S) be an S-metric space and let $A,S \colon X \to X$ be two self mappings of X. The pair (A,S) is said to satisfies the $(E.A)$-property if there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to+\infty} Au_n = \lim_{n\to+\infty} Su_n = z$ for some $z \in X$.*

Definition 2.19. *([15]) Let (X, S) be an S-metric space and $A, S, R, T\colon X \to X$ be four self mappings of X. We say that the pairs (A, R) and (S, T) satisfy the common limit range property with respect to R and T if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$\lim_{n\to+\infty} Ru_n = \lim_{n\to+\infty} Au_n = \lim_{n\to+\infty} Sv_n = \lim_{n\to+\infty} Tv_n = z,$$

for some $z \in R(X) \cap T(X)$ and it is denoted by (CLR_{RT}).

Definition 2.20. *([5], [21]) An element $(x, y) \in X \times X$ is called:*

- *a coupled fixed point of the mapping $F\colon X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$;*
- *a coupled coincidence point of the mappings $F\colon X\times X \to X$ and $G\colon X \to X$ if $F(x, y) = G(x)$ and $F(y, x) = G(y)$;*
- *a common coupled fixed point of the mappings $F\colon X \times X \to X$ and $G\colon X \to X$ if $x = F(x, y) = G(x)$ and $y = F(y, x) = G(y)$.*

Example 2.21. *Let $X = [0, +\infty)$ and $F\colon X \times X \to X$ defined by $F(x, y) = \frac{x+y}{3}$ for all $x, y \in X$. One can easily see that F has a unique coupled fixed point $(0, 0)$.*

Example 2.22. *Let $X = [0, +\infty)$ and $F\colon X \times X \to X$ be defined by $F(x, y) = \frac{x+y}{2}$ for all $x, y \in X$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.*

Definition 2.23. *([10]) Let X be a nonempty set. Then we say that the mappings $F\colon X\times X \to X$ and $A\colon X \to X$ are commutative if $A(F(x, y)) = F(Ax, Ay)$.*

Definition 2.24. *([3]) The mappings $F\colon X \times X \to X$ and $A\colon X \to X$ are called weakly compatible if $A(F(x, y)) = F(Ax, Ay)$ and $A(F(y, x)) = F(Ay, Ax)$ for all $x, y \in X$, whenever $A(x) = F(x, y)$ and $A(y) = F(y, x)$.*

Example 2.25. *Let $X = [0, 3]$ endowed with $S(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Define $F\colon X \times X \to X$ and $A\colon X \to X$ by*

$$F(x, y) = \begin{cases} x + y, & \text{if } x, y \in [0, 1), \\ 3, & \text{otherwise,} \end{cases}$$

for all $x, y \in X$ and

$$A(x) = \begin{cases} x, & \text{if } x \in [0, 1), \\ 3, & \text{if } x \in [1, 3], \end{cases}$$

for all $x \in X$. Then for any $x, y \in [1, 3]$,

$$F(Ax, Ay) = F(3, 3) = 3 = A(F(x, y)) = A(3) = 3.$$

Similarly, we have

$$F(Ay, Ax) = F(3, 3) = 3 = A(F(y, x)) = A(3) = 3.$$

Thus,

$$F(Ax, Ay) = A(F(x, y)) \text{ and } F(Ay, Ax) = A(F(y, x)).$$

This shows that the mappings F and A are weakly compatible on $[0, 3]$.

Example 2.26. *Let $X = \mathbb{R}$ endowed with the metric $S(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Define $F\colon X\times X \to X$ and $A\colon X \to X$ by $F(x, y) = x + y$ and $A(x) = x^2$ for all $x, y \in X$. Then F and A are not weakly compatible maps on $\mathbb{R}$, since*

$$F(Ax, Ay) = F(x^2, y^2) = x^2 + y^2, \text{ but } A(F(x, y)) = A(x + y) = (x + y)^2.$$

Therefore,

$$F(Ax, Ay) \neq A(F(x, y)).$$

Hence the mappings F and A are not weakly compatible on $\mathbb{R}$.

Now, we define the following concept.

Definition 2.27. *Let (X, S) be an S-metric space and let mappings $F\colon X \times X \to X$ and $G\colon X \to X$ are said to satisfy the $(E.A)$ property if there exist sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$\lim_{n\to+\infty} F(u_n, v_n) = \lim_{n\to+\infty} Gu_n = u,$$

and

$$\lim_{n\to+\infty} F(v_n, u_n) = \lim_{n\to+\infty} Gv_n = v,$$

for some $u, v \in X$.

Definition 2.28. *Let (X, S) be an S-metric space and let $F\colon X \times X \to X$, $G\colon X \times X \to X$, $A\colon X \to X$, $S\colon X \to X$ be four mappings. Then the pairs (G, A) and (F, S) are said to satisfy the common $(E.A)$ property if there exist sequences $\{u_n\}$, $\{v_n\}$, $\{u'_n\}$, $\{v'_n\}$ in X such that*

$$\lim_{n\to+\infty} F(u_n, v_n) = \lim_{n\to+\infty} Su_n = \lim_{n\to+\infty} G(u'_n, v'_n) = \lim_{n\to+\infty} Au'_n = u,$$

and

$$\lim_{n\to+\infty} F(v_n, u_n) = \lim_{n\to+\infty} Sv_n = \lim_{n\to+\infty} G(v'_n, u'_n) = \lim_{n\to+\infty} Av'_n = v,$$

for some $u, v \in X$.

Definition 2.29. *Let (X, S) be an S-metric space. The mappings $F\colon X \times X \to X$ and $G\colon X \to X$ are said to satisfy the (CLR_G) property if there exist sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$\lim_{n\to+\infty} F(u_n, v_n) = \lim_{n\to+\infty} Gu_n = Gu,$$

and

$$\lim_{n\to+\infty} F(v_n, u_n) = \lim_{n\to+\infty} Gv_n = Gv,$$

for some $u, v \in X$.

Definition 2.30. *Let (X, S) be an S-metric space and let $F\colon X \times X \to X$, $G\colon X \times X \to X$, $T\colon X \to X$, $S\colon X \to X$ be four mappings. Then the pairs (G, T) and (F, S) are said to satisfy the common (CLR_{ST}) property if there exist sequences $\{u_n\}$, $\{v_n\}$, $\{u'_n\}$, $\{v'_n\}$ in X such that*

$$\lim_{n\to+\infty} F(u_n, v_n) = \lim_{n\to+\infty} Su_n = \lim_{n\to+\infty} G(u'_n, v'_n) = \lim_{n\to+\infty} Tu'_n = u,$$

and

$$\lim_{n\to+\infty} F(v_n, u_n) = \lim_{n\to+\infty} Sv_n = \lim_{n\to+\infty} G(v'_n, u'_n) = \lim_{n\to+\infty} Tv'_n = v,$$

where $u, v \in S(X) \cap T(X)$.

Definition 2.31. *Let (X, S) be an S-metric space and let $F\colon X \times X \to X$ and $G\colon X \to X$ be two mappings. Then the pair (F, G) is called occasionally weakly compatible (owc) if $G(F(x, y)) = F(Gx, Gy)$ and $G(F(y, x)) = F(Gy, Gx)$ for some $x, y \in X$, whenever $G(x) = F(x, y)$ and $G(y) = F(y, x)$.*

Example 2.32. *Let $X = [0,1]$ endowed with $S(x,y,z) = \max\{x,y,z\}$ for all $x,y,z \in X$. Define $F\colon X \times X \to X$ and $G\colon X \to X$ by*

$$F(x,y) = \begin{cases} \frac{1}{2}(x+y), & \text{if } x,y \in [0,\frac{1}{2}], \\ \frac{1}{4}, & \text{if } x,y \in (\frac{1}{2},1], \end{cases}$$

for all $x,y \in X$ and

$$G(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0,\frac{1}{2}], \\ 0, & \text{if } x \in (\frac{1}{2},1], \end{cases}$$

for all $x \in X$. Then for $x = y = \frac{1}{2}$, we have

$$F(Gx,Gy) = \frac{1}{2}[G(x)+G(y)] = \frac{1}{2}\Big[\frac{1}{2}+\frac{1}{2}\Big] = \frac{1}{2}$$

$$= G(F(x,y)) = G\Big[\frac{1}{2}(x+y)\Big] = \frac{1}{2}.$$

Similarly, we have

$$F(Gy,Gx) = \frac{1}{2}\Big[\frac{1}{2}+\frac{1}{2}\Big] = \frac{1}{2} = G(F(y,x)) = G\Big[\frac{1}{2}(x+y)\Big] = \frac{1}{2}.$$

Thus,

$$F(Gx,Gy) = G(F(x,y)) \ \text{ and } \ F(Gy,Gx) = G(F(y,x)).$$

This shows that the mappings F and G are occasionally weakly compatible on $[0,1]$.

Lemma 2.33. *([29], Lemma 2.5) Let (X,S) be an S-metric space. Then, we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Lemma 2.34. *([29], Lemma 2.12) Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to +\infty$ then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to +\infty$.*

Lemma 2.35. *([12], Lemma 8) Let (X,S) be an S-metric space and A is a nonempty subset of X. Then A is said to be S-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to +\infty$, then $x \in A$.*

Lemma 2.36. *([29]) Let (X,S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $\mathcal{B}_S(x,r)$ is an open subset of X.*

Lemma 2.37. *([30]) The limit of a sequence $\{x_n\}$ in an S-metric space (X,S) is unique.*

Lemma 2.38. *([29]) Let (X,S) be an S-metric space. Then any convergent sequence $\{x_n\}$ in X is Cauchy.*

In the following lemma we see the relationship between a metric and S-metric.

Lemma 2.39. *([14]) Let (X,d) be a metric space. Then the following properties are satisfied:*
(*i*) *$S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$ is an S-metric on X.*
(*ii*) *$x_n \to x$ in (X,d) if and only if $x_n \to x$ in (X,S_d).*
(*iii*) *$\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X,S_d).*
(*iv*) *(X,d) is complete if and only if (X,S_d) is complete.*

We call the function S_d defined in Lemma 2.39 (*i*) as the S-metric generated by the metric d. It can be found an example of an S-metric which is not generated by any metric in [14, 27].

Example 2.40. *([14]) Let $X = \mathbb{R}$ and the function $S\colon X^3 \to [0,\infty)$ be defined as*

$$S(x,y,z) = |x-z| + |x+z-2y|,$$

for all $x,y,z \in \mathbb{R}$. Then the function S is an S-metric on X and (X,S) is an S-metric space. Now, we prove that there does not exists any metric d such that $S = S_d$. On the contrary, suppose that there exists a metric d such that

$$S(x,y,z) = d(x,z) + d(y,z),$$

for all $x,y,z \in \mathbb{R}$. Hence, we obtain

$$S(x,x,z) = 2d(x,z) = 2|x-z|,$$

and

$$d(x,z) = |x-z|.$$

Similarly, we get

$$S(y,y,z) = 2d(y,z) = 2|y-z|,$$

and

$$d(y,z) = |y-z|,$$

for all $x,y,z \in \mathbb{R}$. Hence, we have

$$|x-z| + |x+z-2y| = |x-z| + |y-z|,$$

which is a contradiction. Therefore, $S \neq S_d$ and $(\mathbb{R},S)$ is a complete S-metric space.

3. Main Results

In this section, we prove some unique common coupled fixed point theorems for a pair of occasionally weakly compatible (owc) mappings in the framework of S-metric spaces.

Theorem 3.1. *Let (X,S) be a complete S-metric space. Let $F\colon X\times X \to X$ and $G\colon X \to X$ be two mappings satisfying the following conditions:*

(1) *for all $x,y,u,v,z,w \in X$:*

$$\begin{aligned}[S(F(x,y),F(u,v),F(z,w))]^2 \leq\ & a_1\,[S(Gx,Gu,Gz)]^2 + a_2\,[S(Gy,Gv,Gw)]^2 \\ & + a_3\,[S(F(x,y),F(x,y),Gx)]^2 \\ & + a_4\,[S(F(u,v),F(u,v),Gu)]^2 \\ & + a_5\,[S(F(z,w),F(z,w),Gz)]^2, \end{aligned} \tag{2}$$

where a_1,a_2,a_3,a_4,a_5 are nonnegative constants such that $a_1+a_2+a_5<1$;

(2) *the pair (F,G) is occasionally weakly compatible;*

(3) *the pair (F,G) satisfies (CLR_G) property.*

Then F and G have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = F(x,x) = G(x)$.

Proof. Since F and G satisfy (CLR_G) property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} &\lim_{n\to+\infty} F(x_n,y_n) = \lim_{n\to+\infty} G(x_n) = G(p), \\ &\lim_{n\to+\infty} F(y_n,x_n) = \lim_{n\to+\infty} G(y_n) = G(q), \end{aligned} \tag{3}$$

for some $p, q \in X$.

Step I. We show that F and G have a coupled coincidence point. Then from equation (2), we have

$$\begin{aligned}
&[S(F(x_n, y_n), F(x_n, y_n), F(p, q))]^2 \\
&\qquad \leq a_1 [S(Gx_n, Gx_n, Gp)]^2 + a_2 [S(Gy_n, Gy_n, Gq)]^2 \\
&\qquad\quad +a_3 [S(F(x_n, y_n), F(x_n, y_n), Gx_n)]^2 \\
&\qquad\quad +a_4 [S(F(x_n, y_n), F(x_n, y_n), Gx_n)]^2 \\
&\qquad\quad +a_5 [S(F(p, q), F(p, q), Gp)]^2. \qquad (4)
\end{aligned}$$

Letting $n \to +\infty$ in equation (4), using equation (3), $(S2)$ and Lemma 2.33, we obtain

$$[S(F(p, q), F(p, q), Gp)]^2 \leq a_5 [S(F(p, q), F(p, q), Gp)]^2,$$

which is a contradiction, since $a_5 < 1$. Hence, we conclude that $S(F(p, q), F(p, q), Gp) = 0$ and so $F(p, q) = Gp$. Similarly, we can show that $F(q, p) = Gq$.

Let

$$F(p, q) = Gp = x \quad \text{and} \quad F(q, p) = Gq = y. \qquad (5)$$

Since the pair (F, G) is occasionally weakly compatible (owc), it follows that

$$Gx = G(F(p, q)) = F(Gp, Gq) = F(x, y), \qquad (6)$$

and

$$Gy = G(F(q, p)) = F(Gq, Gp) = F(y, x). \qquad (7)$$

Hence F and G have a coupled coincidence point.

Step II. To show that $G(x) = y$, $G(y) = x$. For this, using equation (2), we have

$$\begin{aligned}
[S(Gx_n, Gx_n, Gy)]^2 &= [S(F(x_n, y_n), F(x_n, y_n), F(y, x))]^2 \\
&\leq a_1 [S(Gx_n, Gx_n, Gy)]^2 + a_2 [S(Gy_n, Gy_n, Gx)]^2 \\
&\quad +a_3 [S(F(x_n, y_n), F(x_n, y_n), Gx_n)]^2 \\
&\quad +a_4 [S(F(x_n, y_n), F(x_n, y_n), Gx_n)]^2 \\
&\quad +a_5 [S(F(y, x), F(y, x), Gy)]^2. \qquad (8)
\end{aligned}$$

Letting $n \to +\infty$ in equation (8), using equations (3), (5), $(S2)$ and Lemma 2.33, we obtain

$$[S(x, x, Gy)]^2 \leq (a_1 + a_2) [S(x, x, Gy)]^2,$$

which is a contradiction, since $a_1 + a_2 < 1$. Hence, we conclude that $S(x, x, Gy) = 0$ and so $Gy = x$. Similarly, we can show that $Gx = y$.

Step III. Now, we show that $x = y$. From equations (2), (6) and (7), we have

$$\begin{aligned}
[S(x,x,y)]^2 &= [S(F(p,q),F(p,q),F(q,p))]^2 \\
&\le a_1\,[S(Gp,Gp,Gq)]^2 + a_2\,[S(Gq,Gq,Gp)]^2 \\
&\quad + a_3\,[S(F(p,q),F(p,q),Gp)]^2 \\
&\quad + a_4\,[S(F(p,q),F(p,q),Gp)]^2 \\
&\quad + a_5\,[S(F(q,p),F(q,p),Gq)]^2 \\
&= a_1\,[S(x,x,y)]^2 + a_2\,[S(y,y,x)]^2 \\
&\quad + a_3\,[S(x,x,x)]^2 + a_4\,[S(x,x,x)]^2 \\
&\quad + a_5\,[S(y,y,y)]^2. \qquad (9)
\end{aligned}$$

Using condition $(S2)$ and Lemma 2.33 in equation (9), we obtain

$$[S(x,x,y)]^2 \le (a_1 + a_2)\,[S(x,x,y)]^2,$$

which is a contradiction, since $a_1 + a_2 < 1$. Hence, we conclude that $S(x,x,y) = 0$ and so $x = y$. From equation (6) and **Step II**, we conclude that $x = F(x,x) = G(x)$. This completes the proof.

□

Theorem 3.2. *Let (X,S) be a complete S-metric space. Let $F\colon X\times X \to X$ and $G\colon X \to X$ be two mappings satisfying the following conditions:*

(1) *for all $x,y,u,v,z,w \in X$:*

$$\begin{aligned}
&[S(F(x,y),F(u,v),F(z,w))]^2 \\
&\quad \le k \max\Big\{[S(Gx,Gu,Gz)]^2, [S(Gy,Gv,Gw)]^2, [S(F(x,y),F(x,y),Gx)]^2, \\
&\qquad\qquad [S(F(u,v),F(u,v),Gu)]^2, [S(F(z,w),F(z,w),Gz)]^2\Big\}, \qquad (10)
\end{aligned}$$

where $k \in [0,1)$ is a constant;

(2) *the pair (F,G) is occasionally weakly compatible;*

(3) *the pair (F,G) satisfies (CLR_G) property.*

Then F and G have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = F(x,x) = G(x)$.

Proof. Since F and G satisfy (CLR_G) property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}
&\lim_{n\to+\infty} F(x_n,y_n) = \lim_{n\to+\infty} G(x_n) = G(p), \\
&\lim_{n\to+\infty} F(y_n,x_n) = \lim_{n\to+\infty} G(y_n) = G(q), \qquad (11)
\end{aligned}$$

for some $p,q \in X$.

Now, we show that F and G have a coupled coincidence point. Then from equation (10), we have

$$\begin{aligned}
&[S(F(x_n,y_n),F(x_n,y_n),F(p,q))]^2 \\
&\quad \le k \max\Big\{[S(Gx_n,Gx_n,Gp)]^2, [S(Gy_n,Gy_n,Gq)]^2, \\
&\qquad\qquad [S(F(x_n,y_n),F(x_n,y_n),Gx_n)]^2, \\
&\qquad\qquad [S(F(x_n,y_n),F(x_n,y_n),Gx_n)]^2, \\
&\qquad\qquad [S(F(p,q),F(p,q),Gp)]^2\Big\}. \qquad (12)
\end{aligned}$$

Letting $n \to +\infty$ in equation (12), using equation (11), ($S2$) and Lemma 2.33, we obtain

$$\begin{aligned}[S(F(p,q),F(p,q),Gp)]^2 &\leq k\max\Big\{0,0,0,0,0,[S(F(p,q),F(p,q),Gp)]^2\Big\}\\ &= k\,[S(F(p,q),F(p,q),Gp)]^2,\end{aligned}$$

which is a contradiction, since $k < 1$. Hence, we conclude that $S(F(p,q),F(p,q),Gp) = 0$ and so $F(p,q) = Gp$. Similarly, we can show that $F(q,p) = Gq$.

Let

$$F(p,q) = Gp = x \quad \text{and} \quad F(q,p) = Gq = y. \tag{13}$$

Since the pair (F,G) is occasionally weakly compatible (owc), it follows that

$$Gx = G(F(p,q)) = F(Gp,Gq) = F(x,y), \tag{14}$$

and

$$Gy = G(F(q,p)) = F(Gq,Gp) = F(y,x). \tag{15}$$

Hence F and G have a coupled coincidence point. The rest of the proof follows from Theorem 3.1, so we omit it. This completes the proof. □

Theorem 3.3. *Let (X,S) be a complete S-metric space. Let $F\colon X\times X \to X$ and $G\colon X \to X$ be two mappings satisfying the following conditions:*

(1) *for all $x,y,u,v,z,w \in X$:*

$$\begin{aligned}[S(F(x,y),F(u,v),F(z,w))]^2 \leq\ & r_1\max\Big\{[S(Gx,Gu,Gz)]^2,[S(F(x,y),F(x,y),Gx)]^2,\\ &\qquad [S(F(x,y),F(x,y),Gu)]^2\Big\}\\ &+r_2\max\Big\{[S(F(u,v),F(u,v),Gz)]^2,\\ &\qquad [S(F(z,w),F(z,w),Gz)]^2\Big\}\\ &+r_3\,S(F(x,y),F(x,y),Gz)S(F(z,w),F(z,w),Gu),\end{aligned} \tag{16}$$

where r_1, r_2, r_3 are nonnegative constants with $r_1 + r_2 + r_3 < 1$;

(2) *the pair (F,G) is occasionally weakly compatible;*

(3) *the pair (F,G) satisfies (CLR_G) property.*

Then F and G have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = F(x,x) = G(x)$.

Proof. Since F and G satisfy (CLR_G) property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} F(x_n,y_n) = \lim_{n\to+\infty} G(x_n) = G(p), \tag{17}$$

$$\lim_{n\to+\infty} F(y_n,x_n) = \lim_{n\to+\infty} G(y_n) = G(q), \tag{18}$$

for some $p,q \in X$.

Now, we show that F and G have a coupled coincidence point. Then from equation (16), we have

$$\begin{aligned}
[S(F(x_n,y_n),F(x_n,y_n),F(p,q))]^2 \leq\ & r_1 \max\Big\{[S(Gx_n,Gx_n,Gp)]^2, \\
& [S(F(x_n,y_n),F(x_n,y_n),Gx_n)]^2, \\
& [S(F(x_n,y_n),F(x_n,y_n),Gx_n)]^2\Big\} \\
& +r_2 \max\Big\{[S(F(x_n,y_n),F(x_n,y_n),Gp)]^2, \\
& [S(F(p,q),F(p,q),Gp)]^2\Big\} \\
& +r_3\, S(F(x_n,y_n),F(x_n,y_n),Gp) \times \\
& S(F(p,q),F(p,q),Gx_n). \qquad (19)
\end{aligned}$$

Letting $n \to +\infty$ in equation (19), using equations (17)-(18), $(S2)$ and Lemma 2.33, we obtain

$$\begin{aligned}
[S(F(p,q),F(p,q),Gp)]^2 &\leq r_1 .0 + r_2 [S(F(p,q),F(p,q),Gp)]^2 + r_3 .0 \\
&= r_2 [S(F(p,q),F(p,q),Gp)]^2,
\end{aligned}$$

which is a contradiction, since $r_2 < 1$. Hence, we conclude that $S(F(p,q),F(p,q),Gp) = 0$ and so $F(p,q) = Gp$. Similarly, we can show that $F(q,p) = Gq$.

Let

$$F(p,q) = Gp = x \quad \text{and} \quad F(q,p) = Gq = y. \qquad (20)$$

Since the pair (F,G) is occasionally weakly compatible (owc), it follows that

$$Gx = G(F(p,q)) = F(Gp,Gq) = F(x,y), \qquad (21)$$

and

$$Gy = G(F(q,p)) = F(Gq,Gp) = F(y,x). \qquad (22)$$

Hence F and G have a coupled coincidence point. The rest of the proof follows from Theorem 3.1, so we omit it. This completes the proof. □

Remark 3.4. *Our results extend and generalize the corresponding results of Aydi [5] from partial metric spaces to the setting of S-metric spaces.*

Now, we give an example to justify the result.

Example 3.5. *Let $X = [0,1]$ and the function $S\colon X^3 \to [0,\infty)$ be defined as $S(x,y,z) = |y-z| + |y+z-2x|$ for all $x,y,z \in X$. Then the function S is an S-metric on X and (X,S) is an S-metric space. Define mappings $F\colon X \times X \to X$ and $G\colon X \to X$ by*

$$F(x,y) = \begin{cases} \frac{1}{25}(x+y), & \text{if } x,y \in [0,\frac{1}{2}], \\ \frac{1}{4}, & \text{if } x,y \in (\frac{1}{2},1], \end{cases}$$

for all $x,y \in X$ and

$$G(x) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0,\frac{1}{2}], \\ 0, & \text{if } x \in (\frac{1}{2},1], \end{cases}$$

for all $x \in X$.

(1) Now, for $x = y = u = v = z = w = \frac{1}{2}$, we have

$$
\begin{aligned}
[S(F(x,y),F(u,v),F(z,w))]^2 &= [|F(u,v)+F(z,w)-2F(x,y)| \\
&\quad +|F(u,v)-F(z,w)|]^2 \\
&= \Big[\Big|\frac{u+v}{25}+\frac{z+w}{25}-\frac{2(x+y)}{25}\Big| \\
&\quad +\Big|\frac{u+v}{25}-\frac{z+w}{25}\Big|\Big]^2 \\
&= \Big[\frac{1}{25}|u+z-2x|+\frac{1}{25}|v+w-2y| \\
&\quad +\frac{1}{25}|u-z|+\frac{1}{25}|v-w|\Big]^2 \\
&= \Big[\frac{1}{25}\Big(|u+z-2x|+|u-z|\Big) \\
&\quad +\frac{1}{25}\Big(|v+w-2y|+|v-w|\Big)\Big]^2 \\
&= \Big[\frac{1}{5}\Big(\Big|\frac{u}{5}+\frac{z}{5}-\frac{2x}{5}\Big|+\Big|\frac{u}{5}-\frac{z}{5}\Big|\Big) \\
&\quad +\frac{1}{5}\Big(\Big|\frac{v}{5}+\frac{w}{5}-\frac{2y}{5}\Big|+\Big|\frac{v}{5}-\frac{w}{5}\Big|\Big)\Big]^2 \\
&= \frac{1}{25}\Big[S(Gx,Gu,Gz)+S(Gy,Gv,Gw)\Big]^2 \\
&\le \frac{1}{12}\Big([(S(Gx,Gu,Gz)]^2+[S(Gy,Gv,Gw)]^2\Big) \\
&\le \frac{1}{12}\Big([S(Gx,Gu,Gz)]^2+[S(Gy,Gv,Gw)]^2 \\
&\qquad +[S(F(x,y),F(x,y),Gx)]^2 \\
&\qquad +[S(F(u,v),F(u,v),Gu)]^2 \\
&\qquad +[S(F(z,w),F(z,w),Gz)]^2\Big),
\end{aligned}
$$

holds for all $x,y,z,u,v,w \in X$, where $a_1 = a_2 = a_3 = a_4 = a_5 = \frac{1}{12}$ with $a_1 + a_2 + a_5 = \frac{3}{12} < 1$.

(2) To show that the pair (F, G) satisfies the occasionally weakly compatible (owc) property.

Case (i): *If $x = y = 0$, then we have*

$$G(F(0,0)) = G(0) = 0 \quad and \quad F(G(0),G(0)) = F(0,0) = 0.$$

Thus, $G(F(0,0)) = F(G(0),G(0)) = 0$.

Case (ii): *If $x = y = \frac{1}{2}$, then we have*

$$G(F(1/2,1/2)) = G(1/25) = 1/125 \text{ and } F(G(1/2),G(1/2)) = F(1/10,1/10) = 1/125.$$

Thus, $G(F(1/2,1/2)) = F(G(1/2),G(1/2)) = 1/125$.

Case (iii): *If $x = y = 1$, then we have*

$$G(F(1,1)) = G(1/4) = 1/20 \quad and \quad F(G(1),G(1)) = F(0,0) = 0.$$

Thus, $G(F(1,1)) \neq F(G(1),G(1))$. This shows that $G(F(x,y)) = F(Gx,Gy)$ for some $x,y \in X$.

Hence, we see that the pair (F, G) satisfies the occasionally weakly compatible (owc) property.

(3) Now, we show that the pair (F, G) satisfies (CLR_G) property.

Consider two sequences $\{x_n\} = \{\frac{1}{n^2}\}$ *and* $\{y_n\} = \{\frac{1}{n^3}\}$ *for all* $n \geq 1$. *Then*

$$\begin{aligned} S(F(x_n, y_n), F(x_n, y_n), 0) &= S\Big(\frac{x_n}{25} + \frac{y_n}{25}, \frac{x_n}{25} + \frac{y_n}{25}, 0\Big) \\ &= \Big|\frac{x_n}{25} + \frac{y_n}{25} + 0 - \frac{2x_n}{25} - \frac{2y_n}{25}\Big| + \Big|\frac{x_n}{25} + \frac{y_n}{25} - 0\Big| \\ &= 2\Big|\frac{x_n}{25} + \frac{y_n}{25}\Big| \\ &\to 0 \text{ as } n \to +\infty, \end{aligned}$$

and

$$\begin{aligned} S(G(x_n), G(x_n), 0) &= S\Big(\frac{x_n}{5}, \frac{x_n}{5}, 0\Big) \\ &= \Big|\frac{x_n}{5} + 0 - \frac{2x_n}{5}\Big| + \Big|\frac{x_n}{5} - 0\Big| \\ &= 2\Big|\frac{x_n}{5}\Big| \to 0 \text{ as } n \to +\infty. \end{aligned}$$

This shows that

$$\lim_{n\to+\infty} S(F(x_n, y_n), F(x_n, y_n), 0) = \lim_{n\to+\infty} S(G(x_n), G(x_n), 0) = 0 = G(0),$$

where $0 \in X$.

Thus the pair (F, G) *satisfies* (CLR_G) *property.*

Hence the mappings F and G satisfy all the conditions of Theorem 3.1 and consequently F and G have a unique common coupled fixed point, namely $F(0, 0) = 0$ *and* $G(0) = 0$.

4. Conclusion

In this paper, we prove some unique common coupled fixed point theorems in the setting of S-metric spaces for a pair of occasionally weakly compatible (owc) mappings. Our results extend and generalize several previous findings from the existing literature.

5. Acknowledgement

The author is thankful to the learned referee for his careful reading and valuable suggestions that helped to improve this paper.

References

[1] M. Aamri and D. El. Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002), 181–188. (dx.doi.org/10.1016/S0022-247X(02)00059-8)
[2] M. Abbas and B. E. Rhoades, Common fixed point results for non-commuting mappings without continuity generalized metric spaces, Applied Mathematics and Computation 215(1) (2009), 262–269.
[3] M. Abbas, M. Ali Khan and S. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Applied Mathematics and Computation 217(1) (2010), 195–202.
[4] M. A. Al-Thagafi and N. Shahzad, Generalized I-nonexpansive self-maps and invariant approximations, Acta Math. Sinica 24(5) (2008), 867–876.
[5] H. Aydi, Some coupled fixed point results on partial metric spaces, International J. Math. Math. Sci. 2011, Article ID 647091, 11 pages. DOI: 10.1155/2011/647091
[6] G. V. R. Babu and M. V. R. Kameswari, Common fixed point theorems for weakly compatible maps using a contraction quadratic inequality, Adv. Stud. Contemp. Math. 9 (2004), 139–152.
[7] S. Banach, Sur les operation dans les ensembles abstraits et leur application aux equation integrals, Fund. Math. 3 (1922), 133–181.
[8] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: TMA, 65(7) (2006), 1379–1393.

[9] S. Chauhan, Fixed points of weakly compatible mappings in fuzzy metric spaces satisfying common limit in the range property, Indian J. Math. 54 (2012), 375–397.
[10] L. Ćirić and V. Lakshmikantham, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: (TMA), 70(12) (2009), 4341–4349.
[11] N. V. Dung, N. T. Hieu and S. Radojević, Fixed point theorems for g-monotone maps on partially ordered S-metric spaces, Filomat 28(9) (2014), 1885–1898.
[12] A. Gupta, Cyclic contraction on S-metric space, Int. J. Anal. Appl. 3(2) (2013), 119–130.
[13] D. Guo and V. Lakshmikantham, Coupled fixed point of nonlinear operator with application, Nonlinear Anal.: (TMA.), 11 (1987), 623–632.
[14] N. T. Hieu, N. T. Ly and N. V. Dung, A generalization of Ćirić quasi-contractions for maps on S-metric spaces, Thai J. Math. 13(2) (2015), 369–380.
[15] M. Imdad, B. D. Pant and S. Chauhan, Fixed point theorems in Menger spaces using ($CLRST$) property and applications, J. Nonlinear Anal. Optim. 3(2) (2012), 225–237.
[16] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83(4) (1976), 261–263.
[17] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986), 771–779.
[18] G. Jungck, Common fixed points for noncontinuous, nonself maps on nonnumetric spaces, Far East J. Math. Sci. 4(2) (1996), 195–215.
[19] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29 (1998), 227–238.
[20] J. K. Kim, S. Sedghi, A. Gholidahneh and M. M. Rezaee, Fixed point theorems in S-metric spaces, East Asian Math. J. 32(5) (2016), 677–684.
[21] J. K. Kim, G. A. Okeke and W. H. Lim, Common coupled fixed point theorems for w-compatible mappings in partial metric spaces, Global J. Pure Appl. Math. 13(2) (2017), 519–536.
[22] N. V. Luong and N. X. Thuan, Coupled fixed points theorems for mixed monotone mappings and an application to integral equations, Comput. Math. Appl. 62 (2011), 4238–4248.
[23] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297.
[24] H. K. Nashine, J. K. Kim, A. K. Sharma and G. S. Saluja, Some coupled fixed point without mixed monotone mappings, Nonlinear Funct. Anal. Appl. 21(2) (2016), 235–247.
[25] F. Nikbakhtasarvestani, S. M. Vaezpour and M. Asadi, Common fixed point theorems for weakly compatible mappings by (CLR) property on partial metric spaces, Indian J. Math. Sci. Inform. 14 (2019), 19–32.
[26] J. O. Olaleru, G. A. Okeke and H. Akewe, Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces, Int. J. Math. Model. Comput. 2(2) (2012), 87–98.
[27] N. Y. Özgür and N. Tas, Some new contractive mappings on S-metric spaces and their relationships with the mapping (**S25**), Math. Sci. 11(7) (2017), 7–16.
[28] S. Sedghi, N. Shobe and H. Zhou, A common fixed point theorem in D^*-metric spaces, Fixed Point Theory Appl. (2007), Article ID 27906, 13 pages. (doi: 10.1155/2007/27906)
[29] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik 64(3) (2012), 258–266.
[30] S. Sedghi and N. V. Dung, Fixed point theorems on S-metric spaces, Mat. Vesnik 66(1) (2014), 113–124.
[31] S. Sedghi, N. Shobkolaei, M. Shahraki and T. Došenović, Common fixed point of four maps in S-metric space, Math. Sci. 12 (2018), 137–143.
[32] W. Sintunavarat and P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. (2011), Article ID 637958, 2011.