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Strict isometric and strict symmetric commuting d-tuples of Banach space operators

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Abstract. Given commuting d-tuples $\mathbb{S}_i$ and $\mathbb{T}_i$, $1 \leq i \leq 2$, of Banach space operators such that the tensor products pair $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$ is strict m-isometric (resp., $\mathbb{S}_1$, $\mathbb{S}_2$ are invertible and $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$ is strict m-symmetric), there exist integers $m_i > 0$, and a non-zero scalar c, such that $m = m_1 + m_2 - 1$, $(\mathbb{S}_1, \frac{1}{c}\mathbb{T}_1)$ is strict m_1-isometric and $(\mathbb{S}_2, c\mathbb{T}_2)$ is strict m_2-isometric (resp., there exist integers $m_i > 0$, and a non-zero scalar c, such that $m = m_1 + m_2 - 1$, $(\mathbb{S}_1, \frac{1}{c}\mathbb{T}_1)$ is strict m_1-symmetric and $(\mathbb{S}_2, c\mathbb{T}_2)$ is strict m_2-symmetric). However, $(\mathbb{S}_i, \mathbb{T}_i)$ is strict m_i-isometric (resp., strict m_i-symmetric) for $1 \leq i \leq 2$ implies only that $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$ is m-isometric (resp., $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$ is m-symmetric).

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, i.e. bounded linear transformations, on an infinite dimensional complex Banach space $\mathcal{X}$ into itself (resp., on an infinite dimensional complex Hilbert space $\mathcal{H}$ into itself) , $\mathbb{C}$ denote the complex plane, $B(\mathcal{X})^d$ (resp., $B(\mathcal{H})^d$ and $\mathbb{C}^d$) the product of d copies of $B(\mathcal{X})$ (resp., $B(\mathcal{H})$ and $\mathbb{C}$) for some integer $d \geq 1$, $\overline{z}$ the conjugate of $z \in \mathbb{C}$ and $\mathbf{z} = (z_1, z_2, ..., z_d) \in \mathbb{C}^d$. A d-tuples $\mathbb{A} = (A_1, \cdots, A_d) \in B(\mathcal{X})^d$ is a commuting d-tuple if $[A_i, A_j] = A_iA_j - A_jA_i = 0$ for all $1 \leq i, j \leq d$. If $\mathbf{P}$ is a polynomial in $\mathbb{C}^d$ and $\mathbb{A}$ is a d-tuple of commuting operators in $B(\mathcal{H})^d$, then $\mathbb{A}$ is a hereditary root of $\mathbf{P}$ if $\mathbf{P}(\mathbb{A}) = 0$. Two particular operator classes of hereditary roots which have drawn a lot of attention in the recent past are those of m-isometric and m-symmetric (also called m-selfadjoint) operators, where $A \in B(\mathcal{H})$ is m-isometric, m some positive integer, if $\sum_{j=0}^{m}(-1)^j \binom{m}{j} A^{*j}A^j = 0$ and $A \in B(\mathcal{H})$ is m-symmetric if $\sum_{j=0}^{m}(-1)^j \binom{m}{j} A^{*(m-j)}A^j = 0$. Clearly, m-isometric operators arise as solutions of $P(z) = (\overline{z}z - 1)^m = 0$ and m-symmetric operators arise as solutions of $P(z) = (\overline{z} - z)^m = 0$. The class of m-isometric operators was introduced by Agler [1] and the class of m-symmetric operators was introduced by Helton [13] (*albeit* not as operator solutions of the polynomial equation $(\overline{z} - z)^m = 0$). These classes of operators, and their

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variants (of the *left m-invertible and m-symmetric Banach space pairs* (A, B) *type* [6]), have since been studied by a multitude of authors, amongst them Agler and Stankus [2], Bayart [3], Bermudez *et al* [4], Duggal and Kim [6, 7], Gu [11], Gu and Stankus [12] and Paul and Gu [14].

Generalising the m-isometric property of operators $A \in B(\mathcal{H})$ to commuting d-tuples $\mathbb{A} \in B(\mathcal{H})^d$, Gleeson and Richter [10] say that $\mathbb{A}$ is m-isometric if

$$\sum_{j=0}^{m}(-1)^j\left(\begin{array}{c} m \\ j \end{array}\right)\sum_{|\beta|=j}\frac{j!}{\beta!}\mathbb{A}^{*\beta}\mathbb{A}^{\beta}=0, \tag{1}$$

where

$$\beta=(\beta_1,\cdots,\beta_d),\ |\beta|=\sum_{i=1}^{d}\beta_i,\ \beta!=\Pi_{i=1}^{d}\beta_i!,\ \mathbb{A}^{\beta}=\Pi_{i=1}^{d}A_i^{\beta_i},\ \mathbb{A}^{*\beta}=\Pi_{i=1}^{d}A_i^{*\beta_i};$$

$\mathbb{A}$ is said to be m-symmetric if

$$\sum_{j=0}^{m}(-1)^j\left(\begin{array}{c} m \\ j \end{array}\right)(A_1^*+\cdots+A_d^*)^{m-j}(A_1+\cdots+A_d)^j=0. \tag{2}$$

These generalisations and certain of their variants, in particular left m-invertible Banach space operator pairs (A, B):

$$\triangle_{A,B}^{m}(I)=\sum_{j=0}^{m}(-1)^j\left(\begin{array}{c} m \\ j \end{array}\right)A^jB^j=0$$

and m-symmetric pairs (A, B):

$$\delta_{A,B}^{m}(I)=\sum_{j=0}^{m}(-1)^j\left(\begin{array}{c} m \\ j \end{array}\right)A^{m-j}B^j=0,$$

have recently been the subject matter of a number of studies, see [5, 6, 9, 11, 14, 15] for further references.

Recall that a pair (A, B) of Banach space operators is strict m-left invertible if $\triangle_{A,B}^{m}(I)=0$ and $\triangle_{A,B}^{m-1}(I)\neq 0$; similarly, the pair (A, B) is strict m-symmetric if $\delta_{A,B}^{m}(I)=0$ and $\delta_{A,B}^{m-1}(I)\neq 0$. Products (A_1A_2, B_1B_2) of m_i-isometric, similarly $m_i$symmetric, pairs (A_i, B_i), $1\le i\le 2$, such that A_1 commutes with A_2, and B_1 commutes with B_2, are (m_1+m_2-1)-isometric, respectively (m_1+m_2-1)-symmetric [4, 6, 9, 11]. The converse fails, even for strict m-isometric (and strict m-symmetric) operator pairs (A_1A_2, B_1B_2). A case where there is an answer in the positive is that of the tensor product pairs $(A_1\otimes A_2, B_1\otimes B_2)$: (i) $(A_1\otimes A_2, B_1\otimes B_2)$ is m-isometric if and only if there exist integers $m_i>0$, and a non-zero scalar c, such that $m=m_1+m_2-1$, $(A_1, \frac{1}{c}B_1)$ is m_1-isometric and (A_2, cB_2) is m_2-isometric [14, Theorem 1.1]; (ii) if A_i are left invertible and B_i are right invertible, $1\le i\le 2$, then $(A_1\otimes A_2, B_1\otimes B_2)$ is m-symmetric if and only if there exist integers $m_i>0$ and a non-zero scalar c such that $m=m_1+m_2-1$, $(A_1, \frac{1}{c}B_1)$ is m_1-symmetric and (A_2, cB_2) is m_2-symmetric [14, Theorem 5.2].

In this paper, we start by (equivalently) defining m-isometric and m-symmetric pairs $(\mathbb{A}, \mathbb{B})$ of commuting d-tuples of Banach space operators in terms of the elementary operators of left and right multiplication (see [7], where this is done for single linear operators). Alongwith introducing other relevant notations and terminology, this is done in Section 2. Section 3 considers the relationship between the m-isometric properties of $(\mathbb{A}_i, \mathbb{B}_i)$ (similarly, m-symmetric properties of $(\mathbb{A}_i, \mathbb{B}_i)$), $1\le i\le 2$, and their product $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$. A necessary and sufficient condition for product pairs $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ to be strict m-isometric (similarly, strict m-symmetric) is proved and its relationship with the strictness of m-isometric pairs $(\mathbb{A}_i, \mathbb{B}_i)$ is explained. Section 4, the penultimate section, proves that the results of Paul and Gu [14] extend to tensor products of commuting d-tuples. (We remark here that the conditional statement of Theorem 1.1 of [14] holds in one direction, thus opening of the bracket (*strict*) in statement (a) of the theorem implies the opening of the brackets (*strict*) in statement (b) of the theorem, but fails the other way: see [8] for an example.) The advantage of our defining m-isometric (and, similarly, m-symmetric) pairs $(\mathbb{A}, \mathbb{B})$ using the left/right multiplication operators over definition (1) (resp., (2)) lies in the fact that it provides us with a means to exploit familiar arguments used to prove 1-tuple (i.e., single linear operator) version of these results. Here it is seen that the invertibility of S_i, $1 \leq i \leq 2$, is a sufficient condition, a condition guaranteed by the left invertibility of S_i and the right invertibility of T_i ($1 \leq i \leq 2$), in [14, Theorem 5.2].

2. Definitions and introductory properties

For $A, B \in B(\mathcal{X})$, let L_A and $R_B \in B(B(\mathcal{X}))$ denote respectively the operators

$$L_A(X) = AX \text{ and } R_B(X) = XB$$

of left multiplication by A and right multiplication by B. Given commuting d-tuples $\mathbb{A} = (A_1, \cdots, A_d)$ and $\mathbb{B} = (B_1, \cdots, B_d) \in B(\mathcal{X})^d$, let $\mathbb{L}^\alpha_\mathbb{A}$ and $\mathbb{R}^\alpha_\mathbb{B}$,

$$\alpha = (\alpha_1, \cdots, \alpha_d),\ |\alpha| = \sum_{i=1}^{d} \alpha_i,\ \alpha_i \geq 0 \text{ for all } 1 \leq i \leq d,$$

be defined by

$$\mathbb{L}^\alpha_\mathbb{A} = \Pi^d_{i=1} L^{\alpha_i}_{A_i},\ \mathbb{R}^\alpha_\mathbb{B} = \Pi^d_{i=1} R^{\alpha_i}_{B_i}.$$

For an operator $X \in B(\mathcal{X})$, let convolution "$*$" and multiplication "$\times$" denote, respectively, the operations

$$(\mathbb{L}_\mathbb{A} * \mathbb{R}_\mathbb{B})^j(X) = \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!} \mathbb{L}^\alpha_\mathbb{A} \mathbb{R}^\alpha_\mathbb{B}\right)(X) = \left(\sum_{i=1}^{d} L_{A_i} R_{B_i}\right)^j (X)$$

(all integers $\mathrm{j} \geq 0$, $\alpha! = \alpha_1! \cdots \alpha_\mathrm{d}!$) and

$$(\mathbb{L}_\mathbb{A} \times \mathbb{R}_\mathbb{B})(X) = \left(\sum_{i=1}^{d} L_{A_i}\right)\left(\sum_{i=1}^{d} R_{B_i}\right)(X).$$

Define the operator $\lfloor \sum_{i=1}^d A_i X B_i \rfloor^n$ by

$$\lfloor \sum_{i=1}^{d} A_i X B_i \rfloor^n = \sum_{i=1}^{d} A_i \lfloor \sum_{i=1}^{d} A_i X B_i \rfloor^{n-1} B_i \text{ for all positive integers } n.$$

(Thus, $\lfloor A \rfloor^n = A \lfloor A \rfloor^{n-1} I = I \lfloor A \rfloor^{n-1} A = \cdots = A^n$, $\lfloor AB \rfloor^n = A \lfloor AB \rfloor^{n-1} B = \cdots = A^n B^n$ and $\lfloor \sum_{i=1}^d A_i B_i \rfloor^n = \sum_{i=1}^d A_i \lfloor \sum_{i=1}^d A_i B_i \rfloor^{n-1} B_i$.)

We say that the d-tuples $\mathbb{A}$ and $\mathbb{B}$ commute, $[\mathbb{A}, \mathbb{B}] = 0$, if

$$[A_i, B_j] = A_i B_j - B_j A_i = 0 \text{ for all } 1 \leq i, j \leq d.$$

Evidently,

$$[\mathbb{L}_\mathbb{A}, \mathbb{R}_\mathbb{B}] = 0$$

and if $[\mathbb{A}, \mathbb{B}] = 0$, then

$$[\mathbb{L}_\mathbb{A}, \mathbb{L}_\mathbb{B}] = [\mathbb{R}_\mathbb{A}, \mathbb{R}_\mathbb{B}] = 0.$$

A pair $(\mathbb{A},\mathbb{B})$ of commuting d-tuples $\mathbb{A}$ and $\mathbb{B}$ is said to be m-isometric, $(\mathbb{A},\mathbb{B}) \in m$-isometric, for some positive integer m, if

$$\begin{aligned}
\triangle^m_{\mathbb{A},\mathbb{B}}(I) &= (I - \mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^m(I) \\
&= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^j\,(I) \\
&= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \left(\sum_{i=1}^{d} L_{A_i}R_{B_i}\right)^j (I) \\
&= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \lfloor\sum_{i=1}^{d} A_iB_i\rfloor^j \\
&= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \left(\sum_{|\alpha|=j} \frac{j!}{\alpha!}\mathbb{A}^\alpha\mathbb{B}^\alpha\right) \\
&= 0;
\end{aligned}$$

$(\mathbb{A},\mathbb{B})$ is n-symmetric, for some positive integer n, if

$$\begin{aligned}
\delta^n_{\mathbb{A},\mathbb{B}}(X) &= (\mathbb{L}_{\mathbb{A}} - \mathbb{R}_{\mathbb{B}})^n(I) \\
&= \left(\sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \mathbb{L}^{n-j}_{\mathbb{A}} \times \mathbb{R}^j_{\mathbb{B}}\right)(I) \\
&= \left(\sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \left(\sum_{i=1}^{d} L_{A_i}\right)^{n-j} \left(\sum_{i=1}^{d} R_{B_i}\right)^j\right)(I) \\
&= \sum_{j=0}^{n}(-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \left(\sum_{i=1}^{d} A_i\right)^{n-j} \left(\sum_{i=1}^{d} B_i\right)^j \\
&= 0.
\end{aligned}$$

Commuting tuples of m-isometric, similarly n-symmetric operators, share a large number of properties with their single operator counterparts: for example, $\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0$ implies $\triangle^t_{\mathbb{A},\mathbb{B}}(I) = 0$, similarly $\delta^m_{\mathbb{A},\mathbb{B}}(I) = 0$ implies $\delta^t_{\mathbb{A},\mathbb{B}}(I) = 0$, for integers $t \geq m$ However, there are instances where a property holds for the single operator version but fails for the d-tuple version: for example, whereas

$$\triangle^m_{A,B}(I) = 0 \iff \triangle^m_{A^{-1},B^{-1}}(I) = 0 \text{ for all invertible } A \text{ and } B$$

(similarly, for m-symmetric (A,B)), this property fails for d-tuples. Consider, for example, 2-tuples $\mathbb{A} = \mathbb{B} = (\frac{1}{2}I, \frac{1}{2}I)$ and $\mathbb{A}^{-1} = \mathbb{B}^{-1} = (2I, 2I)$, when it is seen that $(\mathbb{A},\mathbb{B})$ is 1-isometric but $(\mathbb{A}^{-1},\mathbb{B}^{-1})$ is not 1-isometric.

If $(\mathbb{A},\mathbb{B}) \in (X,m)$-isometric, then

$$\begin{aligned}
& \triangle^m_{\mathbb{A},\mathbb{B}}(X) = 0 \iff (I - \mathbb{L}_{\mathbb{A}} * R_{\mathbb{B}})\left(\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X)\right) = 0 \\
\iff\ & (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X) \\
\Longrightarrow\ & \cdots \Longrightarrow (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})^t\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(X)
\end{aligned}$$

and if $(\mathbb{A},\mathbb{B}) \in (X,n)$-symmetric, then

$$\begin{aligned}
& \delta^n_{\mathbb{A},\mathbb{B}}(X) = 0 \iff (\mathbb{L}_{\mathbb{A}} - \mathbb{R}_{\mathbb{B}})(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X) = 0 \\
\iff\ & \mathbb{L}_{\mathbb{A}}\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X) = \mathbb{R}_{\mathbb{B}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) \\
\Longrightarrow\ & \cdots \Longrightarrow \mathbb{L}^t_{\mathbb{A}}\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X) = \mathbb{R}^t_{\mathbb{B}}\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X).
\end{aligned}$$

for all integers $t \geq 0$. Here

$$\begin{aligned}\mathbb{L}_{\mathbb{A}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) &= \mathbb{L}_{\mathbb{A}}\left(\sum_{j=0}^{n-1}(-1)^j\begin{pmatrix} n-1 \\ j\end{pmatrix}\mathbb{L}_{\mathbb{A}}^{n-1-j}\times\mathbb{R}_{\mathbb{B}}^{j}\right)(X)\\ &= \left(\sum_{j=0}^{n-1}(-1)^j\begin{pmatrix} n-1 \\ j\end{pmatrix}\mathbb{L}_{\mathbb{A}}^{n-j}\times\mathbb{R}_{\mathbb{B}}^{j}\right)(X)\end{aligned}$$

and

$$\mathbb{R}_{\mathbb{B}}(\delta^{n-1}_{\mathbb{A},\mathbb{B}}(X)) = \left(\sum_{j=0}^{n-1}(-1)^j\begin{pmatrix} n-1 \\ j\end{pmatrix}\mathbb{L}_{\mathbb{A}}^{n-1-j}\times\mathbb{R}_{\mathbb{B}}^{j+1}\right)(X).$$

3. Results: strictness of products

Let $\mathbb{A}.\mathbb{B} \in B(\mathcal{X})^d$ be commuting d-tuples, and let $\mathcal{E}_{\mathbb{A},\mathbb{B}}$ denote the operator

$$\mathcal{E}_{\mathbb{A},\mathbb{B}}(X) = (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})(X),\ X \in B(\mathcal{X}).$$

By definition, $(\mathbb{A},\mathbb{B})$ is strict m-isometric if $\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0$ and $\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) \neq 0$; similarly, $(\mathbb{A},\mathbb{B})$ is strict m-symmetric if $\delta^m_{\mathbb{A},\mathbb{B}}(I) = 0$ and $\delta^{m-1}_{\mathbb{A},\mathbb{B}}(I) \neq 0$. In the following, we give a necessary and sufficient condition for the products pair $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$, $\mathbb{A}_i$ and $\mathbb{B}_i$ commuting d-tuples, to be strict m-isometric (resp., strict m-symmetric), and explore its relationship with the strict m-isometric (resp., m_i-symmetric) property of $(\mathbb{A}_i,\mathbb{B}_i)$; $i = 1, 2$. We start with a technical lemma.

Lemma 3.1. *(i). If $(\mathbb{A},\mathbb{B})$ is strict m-isometric, then the sequence $\{\mathcal{E}^{t\pm r}_{\mathbb{A},\mathbb{B}}\triangle^r_{\mathbb{A},\mathbb{B}}(I)\}_{r=0}^{m-1}$ is linearly independent for all $t \geq m-1$.*

(ii). If $(\mathbb{A},\mathbb{B})$ is strict m-symmetric and $\mathbb{L}^{m-1}_{\mathbb{A}}\delta^{m-1}_{\mathbb{A},\mathbb{B}}(I) \neq 0$, then the sequences $\{\mathbb{L}^r_{\mathbb{A}}\delta^r_{\mathbb{A},\mathbb{B}}(I)\}_{r=0}^{m-1}$ and $\{\mathbb{R}^r_{\mathbb{B}}\delta^r_{\mathbb{A},\mathbb{B}}(I)\}_{r=0}^{m-1}$ are linearly independent.

Proof. The proof in both the cases is by contradiction.

(i). Assume that there exist scalars a_i, $0 \leq i \leq m-1$, not all zero such that $\sum_{r=0}^{m-1} a_r\mathcal{E}^{t\pm r}_{\mathbb{A},\mathbb{B}}\triangle^r_{\mathbb{A},\mathbb{B}}(I) = 0$. Then, since $\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0$ and $\mathcal{E}_{\mathbb{A},\mathbb{B}}$ commutes with $\triangle_{\mathbb{A},\mathbb{B}}$,

$$\begin{aligned}&\triangle^{m-1}_{\mathbb{A},\mathbb{B}}\left(\sum_{r=0}^{m-1} a_r\mathcal{E}^{t\pm r}_{\mathbb{A},\mathbb{B}}\triangle^r_{\mathbb{A},\mathbb{B}}(I)\right) = 0\\ \Longrightarrow\quad & a_0\mathcal{E}^t_{\mathbb{A},\mathbb{B}}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = 0\\ \Longrightarrow\quad & a_0 = 0,\end{aligned}$$

since

$$\triangle^m_{\mathbb{A},\mathbb{B}}(I) = 0 \Longleftrightarrow \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = \mathcal{E}_{\mathbb{A},\mathbb{B}}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I)$$

implies

$$\mathcal{E}^t_{\mathbb{A},\mathbb{B}}\triangle^{m-1}_{\mathbb{A}\mathbb{B}}(I) = \triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) \neq 0.$$

Again,

$$\begin{aligned}&\triangle^{m-2}_{\mathbb{A},\mathbb{B}}\left(\sum_{r=1}^{m-1} a_r\mathcal{E}^{t\pm r}_{\mathbb{A},\mathbb{B}}\triangle^r_{\mathbb{A},\mathbb{B}}(I)\right) = 0\\ \Longrightarrow\quad & a_1\mathcal{E}^{t\pm 1}_{\mathbb{A},\mathbb{B}}\triangle^{m-1}_{\mathbb{A},\mathbb{B}}(I) = 0\\ \Longrightarrow\quad & a_1 = 0,\end{aligned}$$

and hence repeating the argument

$$\begin{aligned} & \triangle_{\mathbb{A},\mathbb{B}}\left(\sum_{r=m-2}^{m-1} A_r \mathcal{E}_{\mathbb{A},\mathbb{B}}^{t\pm r}\triangle_{\mathbb{A},\mathbb{B}}^{r}(I)\right) = 0 \\ \Longrightarrow\quad & a_{m-2}\mathcal{E}_{\mathbb{A},\mathbb{B}}^{t\pm(m-2)}\triangle_{\mathbb{A},\mathbb{B}}^{m-1}(I) = 0 \\ \Longrightarrow\quad & a_{m-2} = 0 \Longrightarrow a_{m-1}\triangle_{\mathbb{A},\mathbb{B}}^{m-1}(I) = 0 \Longleftrightarrow a_{m-1} = 0. \end{aligned}$$

This is a contradiction.

(ii). We prove the linear independence of the sequence $\{\mathbb{L}_{\mathbb{A}}^{r}\delta_{\mathbb{A},\mathbb{B}}^{r}(\mathbb{I})\}_{r=0}^{m-1}$; since

$$\delta_{\mathbb{A},\mathbb{B}}^{m}(I) = 0 \Longleftrightarrow \mathbb{L}_{\mathbb{A}}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) = \mathbb{R}_{\mathbb{B}}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I),$$

the proof for the linear independence of the second sequence follows from that of the first. Suppose there exist scalars a_i, not all zero, such that $\sum_{r=0}^{m-1} a_r\mathbb{L}_{\mathbb{A}}^{r}\delta_{\mathbb{A},\mathbb{B}}^{r}(I) = 0$. Then, since $\delta_{\mathbb{A},\mathbb{B}}^{m}(I) = 0$, $\mathbb{L}_{\mathbb{A}}$ commutes with $\delta_{\mathbb{A},\mathbb{B}}$ and $\mathbb{L}_{\mathbb{A}}^{m-1}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) \neq 0$,

$$\begin{aligned} & \delta_{\mathbb{A},\mathbb{B}}^{m-1}\left(\sum_{r=0}^{m-1} a_r\mathbb{L}_{\mathbb{A}}^{r}\delta_{\mathbb{A},\mathbb{B}}^{r}(I)\right) = 0 \\ \Longrightarrow\quad & a_0\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) = 0 \Longleftrightarrow a_0 = 0, \\ & \delta_{\mathbb{A},\mathbb{B}}^{m-2}\left(\sum_{r=1}^{m-1} a_r\mathbb{L}_{\mathbb{A}}^{r}\delta_{\mathbb{A},\mathbb{B}}^{r}(I)\right) = 0 \\ \Longrightarrow\quad & a_1\mathbb{L}_{\mathbb{A}}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) = 0 \Longleftrightarrow a_1 = 0, \end{aligned}$$

and hence repeating the argument

$$\begin{aligned} & \delta_{\mathbb{A},\mathbb{B}}\left(\sum_{r=m-2}^{m-1} a_r\mathbb{L}_{\mathbb{A}}^{r}\delta_{\mathbb{A},\mathbb{B}}^{r}(I)\right) = 0 \\ \Longrightarrow\quad & a_{m-2}\mathbb{L}_{\mathbb{A}}^{m-2}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) = 0 \Longleftrightarrow a_{m-2} = 0 \\ \Longrightarrow\quad & a_{m-1}\mathbb{L}_{\mathbb{A}}^{m-1}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) = 0 \\ \Longleftrightarrow\quad & a_{m-1} = 0. \end{aligned}$$

This is a contradiction. □

Remark 3.2. Since $\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) \neq 0$ for a strict m-symmetric commuting d-tuple $(\mathbb{A},\mathbb{B})$, the left invertibility of $\mathbb{A}$ (and hence $\mathbb{L}_A$) is a sufficient condition for $\mathbb{L}_{\mathbb{A}}^{m-1}\delta_{\mathbb{A},\mathbb{B}}^{m-1}(I) \neq 0$.

Let $\mathcal{X}\overline{\otimes}\mathcal{X}$ denote the completion, endowed with a reasonable cross norm, of the algebraic tensor product of $\mathcal{X}$ with itself. Let $S \otimes T$ denote the tensor product of $S \in B(\mathcal{X})$ with $T \in B(\mathcal{X})$. The tensor product of the d-tuples $\mathbb{A} = (A_1, \cdots, A_d)$ and $\mathbb{B} = (B_1, \cdots, B_d)$ is the d^2-tuple

$$\mathbb{A}\otimes\mathbb{B} = (A_1\otimes B_1, \cdots, A_1\otimes B_d, A_2\otimes B_1, \cdots, A_2\otimes B_d, \cdots, A_d\otimes B_1, \cdots, A_d\otimes B_d).$$

Let $\mathbb{I} = I \otimes I$. (Recall that the operator $\mathcal{E}_{\mathbb{A},\mathbb{B}}$ is defined by $\mathcal{E}_{\mathbb{A},\mathbb{B}}(X) = (\mathbb{L}_{\mathbb{A}} * \mathbb{R}_{\mathbb{B}})(X)$.)

Theorem 3.3. *Given commuting d-tuples $\mathbb{A}_i, \mathbb{B}_i \in B(\mathcal{X})^d$, any two of the following conditions implies the third.*

(i) $(\mathbb{A}_1\otimes\mathbb{A}_2, \mathbb{B}_1\otimes\mathbb{B}_2)$ is m-isometric; $m = m_1 + m_2 - 1$.
(ii) $(\mathbb{A}_1, \mathbb{B}_1)$ is m_1-isometric.
(iii) $(\mathbb{A}_2, \mathbb{B}_2)$ is m_2-isometric.

Proof. It is well known, see [7], that (*ii*) and (*iii*) imply (*i*). We prove (*i*) and (*ii*) imply (*iii*); the proof of (*i*) and (*iii*) imply (*ii*) is similar.

Start by observing that if we let

$$\mathbb{S}_1 = \mathbb{A}_1 \otimes I, \mathbb{S}_2 = I \otimes \mathbb{A}_2, \mathbb{T}_1 = \mathbb{B}_1 \otimes I \text{ and } \mathbb{T}_2 = I \otimes \mathbb{B}_2,$$

then

$$\begin{aligned}
&[\mathbb{S}_1, \mathbb{S}_2] = [\mathbb{T}_1, \mathbb{T}_2] = 0 = [\mathbb{S}_1, \mathbb{T}_2] = [\mathbb{S}_2, \mathbb{T}_1] \\
&(\mathbb{A}_i, \mathbb{B}_i) \text{ is } m_i - \text{isometric} \iff (\mathbb{S}_i, \mathbb{T}_i) \text{ is } m_i - \text{isometric}, i = 1, 2, \\
&(\mathbb{A}_1 \otimes \mathbb{A}_2, \mathbb{B}_1 \otimes \mathbb{B}_2) \text{ is } m - \text{isometric} \iff (\mathbb{S}_1\mathbb{S}_2, \mathbb{T}_1\mathbb{T}_2) \in m - \text{isometric}
\end{aligned}$$

and

$$\begin{aligned}
&(i) \wedge (ii) \Longrightarrow (iii) \text{ if and only if } (\mathbb{S}_1\mathbb{S}_2, \mathbb{T}_1\mathbb{T}_2) \text{ is } m - \text{isometric} \\
&\text{and } (\mathbb{S}_1, \mathbb{T}_1) \text{ is } m_1 - \text{isometric imply } (\mathbb{S}_2, \mathbb{T}_2) \text{ is } m_2 - \text{isometric.}
\end{aligned}$$

Let $t \leq m_1$ be the least positive integer such that $(\mathbb{S}_1, \mathbb{T}_1) \in t$-isometric. (Thus, $(\mathbb{S}_1, \mathbb{T}_1)$ is strict t-isometric.) Then

$$\begin{aligned}
& \triangle^m_{\mathbb{S}_1\mathbb{S}_2,\mathbb{T}_1\mathbb{T}_2}(\mathbb{I}) = (I - \mathcal{E}_{\mathbb{S}_1\mathbb{S}_2,\mathbb{T}_1\mathbb{T}_2})^m(\mathbb{I}) \\
= \;& (\mathcal{E}_{\mathbb{S}_1,\mathbb{T}_1}\triangle_{\mathbb{S}_2,\mathbb{T}_2} + \triangle_{\mathbb{S}_1,\mathbb{T}_1})^m)(\mathbb{I}) \\
= \;& \sum_{j=0}^{m} \binom{m}{j} \mathcal{E}^{m-j}_{\mathbb{S}_1,\mathbb{T}_1}\triangle^{m-j}_{\mathbb{S}_2,\mathbb{T}_2}\triangle^{j}_{\mathbb{S}_1,\mathbb{T}_1}(\mathbb{I}) = 0.
\end{aligned}$$

Since the operators $\mathcal{E}_{\mathbb{S}_1,\mathbb{T}_1}$ and $\triangle_{\mathbb{S}_2,\mathbb{T}_2}$ commute, as also do the operators $\triangle_{\mathbb{S}_1,\mathbb{T}_1}$ and $\triangle_{\mathbb{S}_2,\mathbb{T}_2}$,

$$\begin{aligned}
& \mathcal{E}^{m-j}_{\mathbb{S}_1,\mathbb{T}_1}(\triangle^{j}_{\mathbb{S}_1,\mathbb{T}_1})(\mathbb{I}) \\
= \;& \left(\sum_{k=0}^{j}(-1)^k \binom{j}{k} \mathcal{E}^{m-j+k}_{\mathbb{S}_1,\mathbb{T}_1}\right)(\mathbb{I}) \\
= \;& \sum_{k=0}^{j}(-1)^k \binom{j}{k} \lfloor \sum_{i=1}^{d} L_{A_{1i}\otimes I}R_{B_{1i}\otimes I}\rfloor^{m-j+k}(\mathbb{I}) \\
= \;& \sum_{k=0}^{j}(-1)^k \binom{j}{k} \lfloor \sum_{i=1}^{d} A_{1i}B_{1i} \otimes I\rfloor^{m-j+k} \\
& (\lfloor \sum_{i=1}^{d} A_{1i}B_{1i} \otimes I\rfloor^{n} = \sum_{i=1}^{d}(A_{1i}\otimes I)\lfloor \sum_{i=1}^{d} A_{1i}B_{1i} \otimes I\rfloor^{n-1}(B_{1i}\otimes I), \\
& \text{all positive integers } n) \\
= \;& \sum_{k=0}^{j}(-1)^k \binom{j}{k} \lfloor \sum_{i=1}^{d} A_{1i}B_{1i}\rfloor^{m-j+k} \otimes I \\
= \;& X_j \otimes I \text{ (say).}
\end{aligned}$$

Hence

$$\begin{aligned}
&\triangle^{m-j}_{\mathbb{S}_2,\mathbb{T}_2}\left(\mathcal{E}^{m-j}_{\mathbb{S}_1,\mathbb{T}_1}(\triangle^{j}_{\mathbb{S}_1,\mathbb{T}_1}\mathbb{I})\right)\\
= &\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j\\ p\end{pmatrix}\left(I\otimes\lfloor\sum_{i=1}^{d}A_{2i}B_{2i}\rfloor\right)^p\left(X_j\otimes I\right)\\
= &\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j\\ p\end{pmatrix}\left(X_j\otimes\lfloor\sum_{i=1}^{d}A_{2i}B_{2i}\rfloor^p\right)\\
= &X_j\otimes Y_j \text{ say,}
\end{aligned}$$

and

$$\triangle^{m}_{\mathbb{S}_1\mathbb{S}_2,\mathbb{T}_1\mathbb{T}_2}(\mathbb{I})=\sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}(X_j\otimes Y_j).$$

The sequence $\{X_j\}_{j=0}^{t-1}$ being linearly independent, we must have $Y_j=0$ for all $0\le j\le t-1$. Since $j\le t-1$ implies $m-j=m_1+m_2-1-t+1\ge m_2$, we have

$$\begin{aligned}
&I\otimes\sum_{p=0}^{m_2}(-1)^p\begin{pmatrix} m_2\\ p\end{pmatrix}\lfloor\sum_{i=1}^{d}A_{2i}B_{2i}\rfloor^p=0\\
\Longleftrightarrow\ &\sum_{p=0}^{m_2}(-1)^p\begin{pmatrix} m_2\\ p\end{pmatrix}\lfloor\sum_{i=1}^{d}A_{2i}B_{2i}\rfloor^p\\
= \ &\triangle^{m_2}_{\mathbb{A}_2,\mathbb{B}_2}(I)=0.
\end{aligned}$$

This completes the proof. □

An anlogue of Theorem 3.3 holds for products of m-symmetric operators.

Theorem 3.4. *Given commuting d-tuples $\mathbb{A}_i,\mathbb{B}_i\in B(\mathcal{X})^d$ such that $\mathbb{A}_i$ is left invertible for $1\le i\le 2$, any two of the following conditions implies the third.*

(i). $(\mathbb{A}_1\otimes\mathbb{A}_2,\mathbb{B}_1\otimes\mathbb{B}_2)$ *is m-symmetric;* $m=m_1+m_2-1$.
(ii). $(\mathbb{A}_1,\mathbb{B}_1)$ *is* m_1*-symmetric.*
(iii). $(\mathbb{A}_2,\mathbb{B}_2)$ *is* m_2*-symmetric.*

Proof. That (*ii*) and (*iii*) imply (*i*), without any hypothesis on the left invertibility of $\mathbb{A}_1$ and $\mathbb{A}_2$, is well known [7]. We prove (*i*) and (*ii*) imply (*iii*); the proof of (*i*) and (*iii*) imply (*ii*) is similar and left to the reader.

Assume $t\le m_1$ is the least positive integer such that $\delta^{t-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\neq 0$. (Thus, $\mathbb{A}_1,\mathbb{B}_1$ is strict t-symmetric.) Then

$$\begin{aligned}
\delta^m_{\mathbb{A}_1\otimes\mathbb{A}_2,\mathbb{B}_1\otimes\mathbb{B}_2}(\mathbb{I}) &= (\mathbb{L}_{\mathbb{A}_1\otimes\mathbb{A}_2}-\mathbb{R}_{\mathbb{B}_1\otimes\mathbb{B}_2})^m(\mathbb{I})\\
&= (\mathbb{L}_{\mathbb{A}_1}\otimes\mathbb{L}_{\mathbb{A}_2}-\mathbb{R}_{\mathbb{B}_1}\otimes\mathbb{R}_{\mathbb{B}_2})^m(\mathbb{I})\\
&= ((\mathbb{L}_{\mathbb{A}_1}\otimes\mathbb{L}_{\mathbb{A}_2}-\mathbb{R}_{\mathbb{B}_1}\otimes\mathbb{L}_{\mathbb{A}_2})+(\mathbb{R}_{\mathbb{B}_1}\otimes\mathbb{L}_{\mathbb{A}_2}-\mathbb{R}_{\mathbb{B}_1}\otimes\mathbb{R}_{\mathbb{B}_2}))^m(\mathbb{I})\\
&= \left(\sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\left(\delta^{m-j}_{\mathbb{A}_1,\mathbb{B}_1}\otimes\mathbb{L}^{m-j}_{\mathbb{A}_2}\right)\times\left(\mathbb{R}^{j}_{\mathbb{B}_1}\otimes\delta^{j}_{\mathbb{A}_2,\mathbb{B}_2}\right)\right)(\mathbb{I})\\
&= \sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\left(\mathbb{R}^{j}_{\mathbb{B}_1}\delta^{m-j}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right)\otimes\left(\mathbb{L}^{m-j}_{\mathbb{A}_2}\delta^{j}_{\mathbb{A}_2,\mathbb{B}_2}(I)\right).
\end{aligned}$$

Since $(\mathbb{A}_1, \mathbb{B}_1)$ is strict t-symmetric and $\mathbb{L}_{\mathbb{A}_1}\delta^{t-1}_{\mathbb{A}_1,\mathbb{B}_1}(I) = \mathbb{R}_{\mathbb{B}_1}\delta^{t-1}_{\mathbb{A}_1,\mathbb{B}_1}(I) \neq 0$, the argument of the proof of Lemma 3.1 implies the linear independence of the sequence $\{\mathbb{R}^j_{\mathbb{B}_1}\delta^{m-j}_{\mathbb{A}_1,\mathbb{B}_1}(I)\}^{t-1}_{m-j=0}$. Hence $\mathbb{L}^{m-j}_{\mathbb{A}_2}\delta^j_{\mathbb{A}_2,\mathbb{B}_2}(I) = 0$ for all $m - j \leq t - 1$, equivalently, $j \geq m - t + 1 \geq m_1 + m_2 - 1 - m_1 + 1 = m_2$. But then, since $\mathbb{L}_{\mathbb{A}_2}$ is left invertible, $\delta^j_{\mathbb{A}_2,\mathbb{B}_2}(I) = 0$ for all $j \geq m_2$. □

Strictness in conditions $(i) - (iii)$ of Theorem 3.3 requires more: thus whereas $(\mathbb{A}_1 \otimes \mathbb{A}_2, \mathbb{B}_1 \otimes \mathbb{B}_2)$ is strictly m isometric implies $(\mathbb{A}_i, \mathbb{B}_i)$ is strictly m_i-isometric for both $i = 1$ and $i = 2$, $(\mathbb{A}_i, \mathbb{B}_i)$ is strictly m_i-isometric for both $i = 1$ and $i = 2$ does not in general imply $(\mathbb{A}_1 \otimes \mathbb{A}_2, \mathbb{B}_1 \otimes \mathbb{B}_2)$ is strictly m isometric.

Theorem 3.5. *Given commuting d-tuples* $\mathbb{A}_i, \mathbb{B}_i \in B(\mathcal{X})^d$, $1 \leq i \leq 2$, *such that* $[\mathbb{A}_1, \mathbb{A}_2] = [\mathbb{B}_1, \mathbb{B}_2] = 0$, *if* $(\mathbb{A}_i, \mathbb{B}_i)$ *is* m_i*-isometric and* $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ *is m isometric,* $m = m_1 + m_2 - 1$, *then:*

(i) $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ *is strictl m-isometric if and only if*

$$\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}\left(\triangle^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}(I)\right) = \triangle^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}\left(\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right) \neq 0; \tag{3}$$

(ii) $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ *is strict m-isometric implies* $(\mathbb{A}_i, \mathbb{B}_i)$ *is strict* m_i*-isometric for* $1 \leq i \leq 2$*;*
(iii) $(\mathbb{A}_i, \mathbb{B}_i)$ *is strict* m_i*-isometric for* $1 \leq i \leq 2$ *does not imply* $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ *is strict m-isometric*

Proof. The hypothesis $[\mathbb{A}_1, \mathbb{A}_2] = [\mathbb{B}_1, \mathbb{B}_2] = 0$ implies

$$[\triangle_{\mathbb{A}_1,\mathbb{B}_1}, \triangle_{\mathbb{A}_2,\mathbb{B}_2}] = 0 = [\mathcal{E}^{m_2-1}_{\mathbb{A}_1,\mathbb{B}_1}, \triangle_{\mathbb{A}_2,\mathbb{B}_2}].$$

Since

$$\begin{aligned} & \mathcal{E}^{m_2-1}_{\mathbb{A}_1,\mathbb{B}_1}\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}\left(\triangle^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}(I)\right) \\ = \; & \triangle^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}\left(\mathcal{E}^{m_2-1}_{\mathbb{A}_1,\mathbb{B}_1}\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right) \\ = \; & \triangle^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}\left(\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right), \end{aligned}$$

whenever $\triangle^{m_1}_{\mathbb{A}_1,\mathbb{B}_1}(I) = 0$ (equivalently, $\mathcal{E}_{\mathbb{A}_1,\mathbb{B}_1}\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I) = \triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)$), if $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is m-isometric and either of $(\mathbb{A}_i, \mathbb{B}_i)$, $1 \leq i \leq 2$, is not strict m_i-isometric then (3) is contradicted. Hence (i) implies (ii).

To prove (i), assume $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is strict m-isometric . Then $\triangle^m_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I) = 0$ and $\triangle^{m-1}_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I) \neq 0$. We have

$$\begin{aligned} 0 \;\neq\; & \triangle^{m-1}_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I) \\ = \; & \sum_{j=0}^{m-1}\binom{m-1}{j}\triangle^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}\left(\mathcal{E}^{m-1-j}_{\mathbb{A}_1,\mathbb{B}_1}\triangle^j_{\mathbb{A}_1,\mathbb{B}_1}(I)\right) \\ & \text{(see the proof ofTheorem 3.3)} \\ = \; & \sum_{j=0}^{m_1-1}\binom{m-1}{j}\triangle^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}\left(\mathcal{E}^{m-1-j}_{\mathbb{A}_1,\mathbb{B}_1}\triangle^j_{\mathbb{A}_1,\mathbb{B}_1}(I)\right) \\ & \text{(since } \triangle^{m_1}_{\mathbb{A}_1,\mathbb{B}_1}(I) = 0 \text{ for } j \geq m_1\text{)} \\ = \; & \binom{m-1}{m_1-1}\triangle^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}\left(\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right), \end{aligned}$$

since $\triangle^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}(I) = 0$ for all $m - 1 - j \geq m_2$, equivalently, $m_1 - 2 \geq j$, and $\mathcal{E}^{m-1-(m_1-1)}_{\mathbb{A}_1,\mathbb{B}_1}\left(\triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right) = \triangle^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)$.

To complete the proof, we give an example proving (*iii*). Let $I_2 = I \oplus I$, and let $A, B \in B(\mathcal{X})$ be such that (A, B) is strict m-isometric. Define operators $\mathbb{A}_i, \mathbb{B}_i \in B(\mathcal{X} \oplus \mathcal{X})^d$, $1 \le i \le 2$, by

$$\begin{aligned}
\mathbb{A}_1 &= (A_{11}, \cdots, A_{1d}) = \frac{1}{\sqrt{d}}(A \oplus I, \cdots, A \oplus I),\\
\mathbb{A}_2 &= (A_{21}, \cdots, A_{2d}) = \frac{1}{\sqrt{d}}(I \oplus A, \cdots, I \oplus A),\\
\mathbb{B}_1 &= (B_{11}, \cdots, B_{1d}) = \frac{1}{\sqrt{d}}(B \oplus I, \cdots, B \oplus I),\\
\mathbb{B}_2 &= (B_{21}, \cdots, B_{2d}) = \frac{1}{\sqrt{d}}(I \oplus B, \cdots, I \oplus B).
\end{aligned}$$

Then

$$\begin{aligned}
\triangle^m_{\mathbb{A}_1,\mathbb{B}_1}(I_2) &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} (\mathbb{L}_{\mathbb{A}_1} * \mathbb{R}_{\mathbb{B}_1})^{m-j}(I_2)\\
&= \sum_{j=0}^{m}(-1)^j \binom{m}{j} \lfloor \sum_{i=1}^{d} \frac{1}{d}(AB \oplus I)\rfloor^{m-j}\\
&\quad \left(\lfloor \sum_{i=1}^{d}(AB \oplus I)\rfloor^t = \sum_{i=1}^{d}(A \oplus I)\lfloor \sum_{i=1}^{d}(AB \oplus I)\rfloor^{t-1}(B \oplus I) \text{ for integers } t \ge 1 \right)\\
&= \sum_{j=0}^{m}(-1)^j \binom{m}{j} \lfloor (AB \oplus I)\rfloor^{m-j}\\
&= \sum_{j=0}^{m}(-1)^j \binom{m}{j} (A^{m-j}B^{m-j} \oplus I)\\
&= 0,
\end{aligned}$$

and

$$\triangle^{m-1}_{\mathbb{A}_1,\mathbb{B}_1}(I_2) \neq 0.$$

Similarly, $\triangle^m_{\mathbb{A}_2,\mathbb{B}_2}(I_2) = 0$ and $\triangle^{m-1}_{\mathbb{A}_2,\mathbb{B}_2}(I_2) \neq 0$.

Consider now $\triangle^m_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I_2)$. Since $\mathbb{A}_i$ and $\mathbb{B}_i$ are commuting d-tuples such that $[\mathbb{A}_1, \mathbb{A}_2] = [\mathbb{B}_1, \mathbb{B}_2] = 0$, $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is $(2m-1)$-isometric. Again, since

$$\mathbb{A}_1\mathbb{A}_2 = \frac{1}{d}(A \oplus A, \cdots, A \oplus A) \in B(\mathcal{X} \oplus \mathcal{X})^{d^2}$$

and

$$\mathbb{B}_1\mathbb{B}_2 = \frac{1}{d}(B \oplus B, \cdots, B \oplus B) \in B(\mathcal{X} \oplus \mathcal{X})^{d^2},$$

$$\begin{aligned}
&\triangle^m_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I_2)\\
= &\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\mathbb{L}_{\mathbb{A}_1\mathbb{A}_2} * \mathbb{R}_{\mathbb{B}_1\mathbb{B}_2})^{m-j}(I_2)\\
= &\sum_{j=0}^{m}(-1)^j\binom{m}{j}\lfloor\sum_{i=1}^{d^2}\frac{1}{d^2}(AB\oplus AB)\rfloor^{m-j}\\
&(\lfloor\sum_{i=1}^{d^2}(AB\oplus AB)\rfloor^t = \sum_{i=1}^{d^2}(A\oplus A)\lfloor\sum_{i=1}^{d^2}(AB\oplus AB)\rfloor^{t-1}(B\oplus B)\\
&\text{for all integers } t\geq 1)\\
= &\sum_{j=0}^{m}(-1)^j\binom{m}{j}\lfloor(AB\oplus AB)\rfloor^{m-j}\\
= &\sum_{j=0}^{m}(-1)^j\binom{m}{j}(A^{m-j}B^{m-j}\oplus A^{m-j}B^{m-j})\\
= &\ 0,
\end{aligned}$$

i.e., $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is m-isometric. Thus $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is not strict $(2m-1)$-isometric for all $m \geq 2$. □

Remark 3.6. Choosing $A, B \in B(\mathcal{X})$, and $\mathbb{A}_i, \mathbb{B}_i \in B(\mathcal{X}\oplus\mathcal{X})^d$, to be the operators of the example proving part (*iii*), define operators $\mathbb{S}_i$ and $\mathbb{T}_i \in B((\mathcal{X}\oplus\mathcal{X})\overline{\otimes}(\mathcal{X}\oplus\mathcal{X}))^d$ by $\mathbb{S}_i = \mathbb{A}_i \otimes I_2$ and $\mathbb{T}_i = \mathbb{B}_i \otimes I_2$; $1 \leq i \leq 2$ and $I_2 = I \oplus I$. Then $(\mathbb{S}_i, \mathbb{T}_i)$ and $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$, $1 \leq i \leq 2$, are all strict m-isometric. Recall from Remark 2.6 of [8] that (A_i, B_i), $1 \leq i \leq 2$, strict m_i-isometric and $[A_1, A_2] = [B_1, B_2] = 0$ does not in general imply (A_1A_2, B_1B_2) is strict $(m_1 + m_2 - 1)$-isometric.

Just as for m-isometric operators, strictness for m-symmetric operators requires more.

Theorem 3.7. *Given commuting d-tuples $\mathbb{A}_i, \mathbb{B}_i \in B(\mathcal{X})^d$ such that $[\mathbb{A}_1, \mathbb{A}_2] = [\mathbb{B}_1, \mathbb{B}_2] = 0$, and $\mathbb{A}_i$ is left invertible for $1 \leq i \leq 2$, if $(\mathbb{A}_i, \mathbb{B}_i)$ is m_i-symmetric, $1 \leq i \leq 2$, and $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is m-symmetric, $m = m_1 + m_2 - 1$, then:*

(i) $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is strict m-symmetric if and only if

$$\mathbb{L}^{m_2-1}_{\mathbb{A}_1} \times \delta^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}\left(\mathbb{R}^{m_1-1}_{\mathbb{A}_2} \times \delta^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}(I)\right) \tag{4}$$

$$= \mathbb{R}^{m_1-1}_{\mathbb{A}_2} \times \delta^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}\left(\mathbb{L}^{m_2-1}_{\mathbb{A}_1} \times \delta^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}(I)\right) \neq 0; \tag{5}$$

(ii) $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is strict m-symmetric implies $(\mathbb{A}_i, \mathbb{B}_i)$ is strict m_i-symmetric for $1 \leq i \leq 2$;
(iii) $(\mathbb{A}_i, \mathbb{B}_i)$ is strict m_i-symmetric for $1 \leq i \leq 2$ does not imply $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is strict m-symmetric.

Proof. (*i*) and (*ii*). Evidently, if either of $(\mathbb{A}_i, \mathbb{B}_i)$ is not strict m_i-symmetric, then (4) and (5) equal 0 and (*i*) is violated. Hence (*i*) implies (*ii*). We prove (*i*), and then modify the example in the proof of Theorem 3.5 to prove (*iii*).

$(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is strict m-symmetric if and only if $\delta^m_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I) = 0$ and

$$\begin{aligned}
0 \neq &\ \delta^{m-1}_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I) = (\mathbb{L}_{\mathbb{A}_1\mathbb{A}_2} - \mathbb{R}_{\mathbb{B}_1\mathbb{B}_2})^{m-1}(I)\\
= &\sum_{j=0}^{m-1}\binom{m-1}{j}\left(L^{m-1-j}_{\mathbb{A}_1} \times \delta^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}\right)\left(\mathbb{R}^j_{\mathbb{A}_2} \times \delta^j_{\mathbb{A}_1,\mathbb{B}_1}\right)(I)\\
= &\sum_{j=0}^{m-1}\binom{m-1}{j}\left(\mathbb{R}^j_{\mathbb{A}_2} \times \delta^j_{\mathbb{A}_1,\mathbb{B}_1}\right)\left(L^{m-1-j}_{\mathbb{A}_1} \times \delta^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}\right)(I)
\end{aligned}$$

by the commutativity hypotheses on $\mathbb{A}_i, \mathbb{B}_i$ and the commutativity of the left and the right multiplication operators. Since

$$\mathbb{R}^j_{\mathbb{A}_2} \times \delta^j_{\mathbb{A}_1,\mathbb{B}_1}(I) = (\sum_{i=1}^{d} R_{A_{2i}})^j (\delta^j_{\mathbb{A}_1,\mathbb{B}_1}(I)) = 0$$

for all $j \geq m_1$,

$$0 \neq \sum_{j=0}^{m_1-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \left(\mathbb{R}^j_{\mathbb{A}_2} \times \delta^j_{\mathbb{A}_1,\mathbb{B}_1}\right)\left(L^{m-1-j}_{\mathbb{A}_1} \times \delta^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}\right)(I).$$

But then, since $\delta^{m-1-j}_{\mathbb{A}_2,\mathbb{B}_2}(I) = 0$ for $m-1-j = m_1+m_2-2-j \geq m_2$,

$$0 \neq \begin{pmatrix} m-1 \\ m_1-1 \end{pmatrix} \left(\mathbb{R}^{m_1-1}_{\mathbb{A}_2} \times \delta^{m_1-1}_{\mathbb{A}_1,\mathbb{B}_1}\right)\left(L^{m_2-1}_{\mathbb{A}_1} \times \delta^{m_2-1}_{\mathbb{A}_2,\mathbb{B}_2}\right)(I).$$

(*iii*). Define operators $\mathbb{A}_i$ and $\mathbb{B}_i$, $1 \leq i \leq 2$, as in the proof of Theorem 3.5(iii) but with $\sqrt{d}$ replaced by d. Choose the operators A, B this time to be such that A is left invertible and (A, B) is strict m-symmetric. Let, as before, $I_2 = I \oplus I$. Then, since

$$\begin{aligned}
\delta^m_{\mathbb{A}_1,\mathbb{B}_1}(I_2) &= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}\left(\sum_{i=1}^{d} L_{A_{1i}}\right)^{m-j}\left(\sum_{i=1}^{d} R_{B_{1i}}\right)^j\right)(I_2) \\
&= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}\left(L^{m-j}_{A\oplus I} R^j_{B\oplus I}\right)\right)(I_2) \\
&= \left(\sum_{j=0}^{m_1-1}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}\left(L^{m-j}_{A} R^j_{B} \oplus I\right)\right)(I_2) \\
&= \delta^m_{A,B}(I) \oplus 0 = 0,
\end{aligned}$$

i.e., $(\mathbb{A}_1, \mathbb{B}_1)$ is m-symmetric. Similarly, $(\mathbb{A}_2, \mathbb{B}_2)$ is m-symmetric. This, in view of the fact that $[\mathbb{A}_1, \mathbb{A}_2] = [\mathbb{B}_1, \mathbb{B}_2] = 0$, implies $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is $(2m-1)$-symmetric. However,

$$\begin{aligned}
\delta^m_{\mathbb{A}_1\mathbb{A}_2,\mathbb{B}_1\mathbb{B}_2}(I_2) &= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}\left(\frac{1}{d^2}\sum_{i=1}^{d^2} L_{A\oplus A}\right)^{m-j}\left(\frac{1}{d^2}\sum_{i=1}^{d^2} R_{\mathbb{B}\oplus\mathbb{B}}\right)^j\right)(I_2) \\
&= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}\left(L^{m-j}_{A} R^j_{B} \oplus L^{m-j}_{A} R^j_{B}\right)\right)(I_2) \\
&= 0.
\end{aligned}$$

Hence $(\mathbb{A}_1\mathbb{A}_2, \mathbb{B}_1\mathbb{B}_2)$ is not strict $(2m-1)$-symmetric for all $m > 1$. □

4. Results: the inverse problem

By definition, $(S_1 \otimes S_2, T_1 \otimes T_2)$ is m-isometric if and only if

$$\begin{aligned}
\triangle^m_{S_1\otimes S_2,T_1\otimes T_2}(I \otimes I) &= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}(L_{S_1\otimes S_2} R_{T_1\otimes T_2})^j\right)(I \otimes I) \\
&= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \lfloor S_1T_1 \rfloor^j \otimes \lfloor S_2T_2 \rfloor^j \\
&= 0
\end{aligned}$$

and it is strict m-isometric if and only if it is m-isometric and

$$\triangle^{m-1}_{S_1\otimes S_2,T_1\otimes T_2}(I\otimes I)=\sum_{j=0}^{m-1}(-1)^j\begin{pmatrix} m-1\\ j\end{pmatrix}\lfloor S_1T_1\rfloor^j\otimes\lfloor S_2T_2\rfloor^j\neq 0.$$

(Recall that, given operators $S_i,T_i\in B(\mathcal{X})$, $1\leq i\leq d$, $\lfloor\sum_{i=1}^d S_iT_i\rfloor^t=S_1\lfloor\sum_{i=1}^d S_iT_i\rfloor^{t-1}T_1+\cdots+S_d\lfloor\sum_{i=1}^d S_iT_i\rfloor^{t-1}T_d$.) Paul and Gu [14, Theorem 1.1] prove that "if $(S_1\otimes S_2,T_1\otimes T_2)$ is m-isometric, then there exist integers $m_i>0$, and a non-zero scalar c, such that $m=m_1+m_2-1$, $(S_1,\frac{1}{c}T_1)$ is m_1-isometric and (S_2,cT_2) is m_2-isometric". Translating this into the terminology above, one has the following.

Proposition 4.1. *Given operators $S_i,T_i\in B(\mathcal{X})$, $1\leq i\leq 2$, if*

$$\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\lfloor S_1T_1\rfloor^j\otimes\lfloor S_2T_2\rfloor^j=0$$

then there exist integers $m_i>0$, and a non-zero scalar c, such that $m=m_1+m_2-1$,

$$\sum_{j=0}^{m_1}(-1)^j\begin{pmatrix} m_1\\ j\end{pmatrix}\lfloor S_1(\frac{1}{c}T_1)\rfloor^j=0=\sum_{j=0}^{m_2}(-1)^j\begin{pmatrix} m_2\\ j\end{pmatrix}\lfloor S_2(cT_2)\rfloor^j.$$

The following theorem is an analogue of [14, Theorem 1.1] for commuting d-tuples of operators. Our proof, which depends on an application of Proposition 4.1, is achieved by reducing the problem to that for single linear operators.

Theorem 4.2. *If $\mathbb{S}_i,\mathbb{T}_i\in B(\mathcal{X})^d$, $1\leq i\leq 2$, are commuting d-tuples such that $(\mathbb{S}_1\otimes\mathbb{S}_2,\mathbb{T}_1\otimes\mathbb{T}_2)$ is strict m-isometric, then there exist integers $m_i>0$, and a non-zero scalar c, such that $m=m_1+m_2-1$, $(\mathbb{S}_1,\frac{1}{c}\mathbb{T}_1)$ is strict m_1-isometric and $(\mathbb{S}_2,c\mathbb{T}_2)$ is strict m_2-isometric.*

Proof. If $(\mathbb{S}_1\otimes\mathbb{S}_2,\mathbb{T}_1\otimes\mathbb{T}_2)$ is strict m-isometric, then, since

$$\mathbb{S}_1\otimes\mathbb{S}_2=(S_{11}\otimes S_{21},\cdots,S_{11}\otimes S_{2d},S_{12}\otimes S_{21},\cdots,S_{12}\otimes S_{2d},\cdots,S_{1d}\otimes S_{21},\cdots,S_{1d}\otimes S_{2d})$$

and

$$\mathbb{T}_1\otimes\mathbb{T}_2=(T_{11}\otimes T_{21},\cdots,T_{11}\otimes T_{2d},T_{12}\otimes T_{21},\cdots,T_{12}\otimes T_{2d},\cdots,T_{1d}\otimes T_{21},\cdots,T_{1d}\otimes T_{2d}),$$

$$\begin{aligned}0&=\triangle^m_{\mathbb{S}_1\otimes\mathbb{S}_2,\mathbb{T}_1\otimes\mathbb{T}_2}(\mathbb{I})\\&=\left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}(\mathbb{L}_{\mathbb{S}_1\otimes\mathbb{S}_2}*\mathbb{R}_{\mathbb{T}_1\otimes\mathbb{T}_2})^j\right)(\mathbb{I}\\&=\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m\\ j\end{pmatrix}\lfloor\sum_{i,k=1}^{d}(S_{1i}\otimes S_{2k})(T_{1i}\otimes T_{2k})\rfloor^j.\end{aligned}$$

This, since

$$\begin{aligned}&\lfloor\sum_{i,k=1}^{d}(S_{1i}\otimes S_{2k})(T_{1i}\otimes T_{2k})\rfloor=\lfloor\sum_{i,k=1}^{d}S_{1i}T_{1i}\otimes S_{2i}T_{2k}\rfloor\\=&\lfloor\sum_{i=1}^{d}S_{1i}T_{1i}\rfloor\otimes\lfloor\sum_{k=1}^{d}S_{2k}T_{2k}\rfloor,\end{aligned}$$

implies

$$\begin{aligned} 0 &= \triangle^m_{\mathbb{S}_1\otimes\mathbb{S},\mathbb{T}_1\otimes\mathbb{T}_2}(\mathbb{I}) \\ &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\lfloor\sum_{i=1}^{d}S_{1i}T_{1i}\rfloor^j\otimes\lfloor\sum_{k=1}^{d}S_{2k}T_{2k}\rfloor^j \end{aligned}$$

and

$$\begin{aligned} 0 &\neq \triangle^{m-1}_{\mathbb{S}_1\otimes\mathbb{S}_2,\mathbb{T}_1\otimes\mathbb{T}_2}(\mathbb{I}) \\ &= \sum_{j=0}^{m-1}(-1)^j\begin{pmatrix} m-1 \\ j \end{pmatrix}\lfloor\sum_{i=1}^{d}S_{1i}T_{1i}\rfloor^j\otimes\lfloor\sum_{k=1}^{d}S_{2k}T_{2k}\rfloor^j. \end{aligned}$$

Applying Proposition 4.1 we have the existence of a non-zero scalar c and positive integers m_i, $m = m_1+m_2-1$, such that

$$\sum_{j=0}^{m_1}(-1)^j\begin{pmatrix} m_1 \\ j \end{pmatrix}\lfloor\sum_{i=1}^{d}S_{1i}(\frac{1}{c}T_{1i})\rfloor^j = 0 = \sum_{j=0}^{m_2}(-1)^j\begin{pmatrix} m_2 \\ j \end{pmatrix}\lfloor\sum_{i=1}^{d}S_{2i}(cT_{2i})\rfloor^j.$$

The strictness of the m-isometric property of the tensor products pair $(S_1\otimes S_2, T_1\otimes T_2)$ implies

$$\sum_{j=0}^{m_1-1}(-1)^j\begin{pmatrix} m_1-1 \\ j \end{pmatrix}\lfloor\sum_{i=1}^{d}S_{1i}(\frac{1}{c}T_{1i})\rfloor^j \neq 0 \neq \sum_{j=0}^{m_2-1}(-1)^j\begin{pmatrix} m_2-1 \\ j \end{pmatrix}\lfloor\sum_{i=1}^{d}S_{2i}(cT_{2i})\rfloor^j,$$

i.e., $(\mathbb{S}_1, \frac{1}{c}\mathbb{T}_1)$ is strict m_1-isometric and $(\mathbb{S}_2, c\mathbb{T}_2)$ is strict m_2-isometric. □

Observe from the proof above that the strictness property of the m-isometric operator pair $(\mathbb{S}_1\otimes\mathbb{S}_2, \mathbb{T}_1\otimes\mathbb{T}_2)$ plays no role in the determination of the scalar c or positive integers m_i such $(\mathbb{S}_1, \frac{1}{c}\mathbb{T}_1)$ is m_1-isometric and $(\mathbb{S}_2, c\mathbb{T}_2)$ is m_2-isometric: strictness plays a role only in the determining of the strictness of the m_1-isometric property of $(\mathbb{S}_1, \frac{1}{c}\mathbb{T}_1)$ and the strictness of the m_2-isometric property of $(\mathbb{S}_2, c\mathbb{T}_2)$. The reverse implication, i.e., the implication that $(\mathbb{S}, \frac{1}{c}\mathbb{T})$ is strict m_1-isometric and $(\mathbb{S}_2, c\mathbb{T}_2)$ is strict m_2-isometric, fails: this follows from Theorem 3.5 (see also [8]).

If an operator $T \in B(\mathcal{H})$ is m-symmetric, then $\sigma_a(T)$, the approximate point spectrum of T, is a subset of $\mathbb{R}$. Hence T is left invertible if and only if it is invertible. Consider operators $S_i, T_i \in B(\mathcal{X})$, $1 \leq i \leq 2$, such that S_i is invertible and $\delta^m_{S_1\otimes S_2,T_1\otimes T_2}(I\otimes I) = 0$. Then

$$\begin{aligned} &\delta^m_{S_1\otimes S_2,T_1\otimes T_2}(I\otimes I) = \left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}L^{m-j}_{S_1\otimes S_2}R^j_{T_1\otimes T_2}\right)(I\otimes I) = 0 \\ \Longleftrightarrow\ &\left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}L^{-j}_{S_1\otimes S_2}R^j_{T_1\otimes T_2}\right)(I\otimes I) = 0 \\ \Longleftrightarrow\ &\triangle^m_{S_1^{-1}\otimes S_2^{-1},T_1\otimes T_2}(I\otimes I) = 0. \end{aligned}$$

Assuming, further, $(S_1\otimes S_2, T_1\otimes T_2)$ to be strict m-symmetric, it follows that

$$(S_1\otimes S_2, T_1\otimes T_2) \text{ is strict } m-\text{symmetric} \Longleftrightarrow (S_1^{-1}\otimes S_2^{-1}, T_1\otimes T_2) \text{ is strict } m-\text{isometric}.$$

Hence there exists a non-zero scalar c and positive integers m_i, $m = m_1 + m_2 - 1$, such that

$$(S_1^{-1}, \frac{1}{c}T_1) \text{ is strict } m_1-\text{isometric and } (S_2^{-1}, cT_2) \text{ is strict } m_2-\text{isometric}.$$

Since

$$\begin{aligned}\triangle^{m_i}_{S_i^{-1},\alpha T_i}(I)=0 &\iff \left(\sum_{j=0}^{m_i}(-1)^j\begin{pmatrix} m_i \\ j\end{pmatrix}L^{-j}_{S_i}(\alpha R_{T_i})^j\right)(I)=0\\ &\iff \left(\sum_{j=0}^{m_i}(-1)^j\begin{pmatrix} m_i \\ j\end{pmatrix}L^{m_i-j}_{S_i}(\alpha R_{T_i})^j\right)(I)=0\\ &\iff \delta^{m_i}_{S_i,\alpha T_i}(I)=0\end{aligned}$$

and, similarly,

$$\triangle^{m_i-1}_{S_i^{-1},\alpha T_i}(I)\neq 0 \iff \delta^{m_i-1}_{S_i,\alpha T_i}(I)\neq 0,$$

we have the following Banach space analogue of [14, Theorem 1.2].

Proposition 4.3. *If $S_1, S_2 \in B(\mathcal{X})$ are invertible and $(S_1\otimes S_2, T_1\otimes T_2)$ is m-symmetric, for some operators $T_1, T_2 \in B(\mathcal{X})$, then there exists a non-zero scalar c and positive integers m_i, $m = m_1+m_2-1$, such that $(S_1, \frac{1}{c}T_1)$ is m_1-symmetric and (S_2, cT_2) is m_2-symmetric.*

Looking upon $\delta^m_{S_1\otimes S_2,T_1\otimes T_2}(I\otimes I)$ as the sum

$$\begin{aligned}\delta^m_{S_1\otimes S_2,T_1\otimes T_2}(I\otimes I) &= \left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}L^{m-j}_{S_1\otimes S_2}R^{j}_{T_1\otimes T_2}\right)(I\otimes I)\\ &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}S_1^{m-j}T_1^j\otimes S_2^{m-j}T_2^j\\ &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}\left(\lfloor S_1\rfloor^{m_i-j}\times\lfloor T_1\rfloor^j\right)\otimes\left(\lfloor S_2\rfloor^{m-j}\times\lfloor T_2\rfloor^j\right),\end{aligned}$$

Proposition 4.3 says the following.

Proposition 4.4. *If $S_1, S_2 \in B(\mathcal{X})$ are invertible operators such that*

$$\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j\end{pmatrix}\left(\lfloor S_1\rfloor^{m-j}\times\lfloor T_1\rfloor^j\right)\otimes\left(\lfloor S_2\rfloor^{m-j}\times\lfloor T_2\rfloor^j\right)=0,$$

and

$$\sum_{j=0}^{m-1}(-1)^j\begin{pmatrix} m-1 \\ j\end{pmatrix}\left(\lfloor S_1\rfloor^{m-1-j}\times\lfloor T_1\rfloor^j\right)\otimes\left(\lfloor S_2\rfloor^{m-1-j}\times\lfloor T_2\rfloor^j\right)\neq 0,$$

then there exists a non-zero scalar c and positive integers m_i $(1\le i\le 2)$, $m=m_1+m_2-1$, such that

$$\begin{aligned}&\sum_{j=0}^{m_1}(-1)^j\begin{pmatrix} m_1 \\ j\end{pmatrix}\left(\lfloor S_1\rfloor^{m_1-j}\times\lfloor \frac{1}{c}T_1\rfloor^j\right)\\ =\ & 0\\ =\ & \sum_{j=0}^{m_2}(-1)^j\begin{pmatrix} m_2 \\ j\end{pmatrix}\left(\lfloor S_2\rfloor^{m_2-j}\times\lfloor cT_2\rfloor^j\right)\end{aligned}$$

and

$$\begin{aligned} & \sum_{j=0}^{m_1-1}(-1)^j \binom{m_1-1}{j}\left(\lfloor S_1\rfloor^{m_1-1-j}\times\lfloor\frac{1}{c}T_1\rfloor^j\right) \\ \neq \quad & 0 \\ \neq \quad & \sum_{j=0}^{m_2-1}(-1)^j \binom{m_2-1}{j}\left(\lfloor S_2\rfloor^{m_2-1-j}\times\lfloor cT_2\rfloor^j\right). \end{aligned}$$

Corresponding to Theorem 4.2, we have the following result for tensor products of commuting d-tuples satisfying a strict m-symmetric property.

Theorem 4.5. *If $\mathbb{S}_i, \mathbb{T}_i \in B(\mathcal{X})^d$, $1 \leq i \leq 2$, are commuting d-tuples, $\mathbb{S}_1$ and $\mathbb{S}_2$ are invertible, and $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$ is strict m symmetric, then there exists a non-zero scalar c and positive integers m_i, $m = m_1 + m_2 - 1$, such that $(\mathbb{S}_1, \frac{1}{c}\mathbb{T}_1)$ is strict m_1-symmetric and $(\mathbb{S}_2, c\mathbb{T}_2)$ is strict m_2-symmetric.*

Proof. If $(\mathbb{S}_1 \otimes \mathbb{S}_2, \mathbb{T}_1 \otimes \mathbb{T}_2)$ is strict m-symmetric, then

$$\begin{aligned} & \delta^m_{\mathbb{S}_1\otimes\mathbb{S}_2,\mathbb{T}_1\otimes\mathbb{T}_2}(I\otimes I) \\ = \quad & \left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathbb{L}^{m-j}_{\mathbb{S}_1\otimes\mathbb{S}_2}\times\mathbb{R}^j_{\mathbb{T}_1\otimes\mathbb{T}_2}\right)(I\otimes I) \\ = \quad & \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\sum_{i,k=1}^{d}S_{1i}\otimes S_{2k}\right)^{m-j}\left(\sum_{i,k=1}^{d}T_{1i}\otimes T_{2k}\right)^j \\ = \quad & \sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\lfloor\sum_{i=1}^{d}S_{1i}\rfloor^{m-j}\lfloor\sum_{i=1}^{d}T_{1i}\rfloor^j\right)\otimes\left(\lfloor\sum_{i=1}^{d}S_{2i}\rfloor^{m-j}\lfloor\sum_{i=1}^{d}T_{2i}\rfloor^j\right) \\ = \quad & 0, \end{aligned}$$

and

$$\begin{aligned} & \delta^{m-1}_{\mathbb{S}_1\otimes\mathbb{S}_2,\mathbb{T}_1\otimes\mathbb{T}_2}(\mathbb{I}\otimes\mathbb{I}) \\ = \quad & \sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\left(\lfloor\sum_{i=1}^{d}S_{1i}\rfloor^{m-1-j}\lfloor\sum_{i=1}^{d}T_{1i}\rfloor^j\right)\otimes\left(\lfloor\sum_{i=1}^{d}S_{2i}\rfloor^{m-1-j}\lfloor\sum_{i=1}^{d}T_{2i}\rfloor^j\right) \\ \neq \quad & 0. \end{aligned}$$

The operators $\mathbb{S}_1$ and $\mathbb{S}_2$ being invertible, $\sum_{i=1}^d S_{1i}$ and $\sum_{i=1}^d S_{2i}$ are invertible, Proposition 4.4 applies and we conclude the existence of a non-zero scalar c and positive integers m_i, $m = m_1 + m_2 - 1$, such that

$$\sum_{j=0}^{m_1}(-1)^j\binom{m_1}{j}\left(\lfloor\sum_{i=1}^{d}S_{1i}\rfloor^{m_1-j}\lfloor\sum_{i=1}^{d}\frac{1}{c}T_{1i}\rfloor^j\right) = \delta^{m_1}_{\mathbb{S}_1,\frac{1}{c}\mathbb{T}_1}(\mathbb{I}) = 0,$$

$$\sum_{j=0}^{m_2}(-1)^j\binom{m_2}{j}\left(\lfloor\sum_{i=1}^{d}S_{2i}\rfloor^{m_2-j}\lfloor\sum_{i=1}^{d}cT_{2i}\rfloor^j\right) = \delta^{m_2}_{\mathbb{S}_2,c\mathbb{T}_2}(\mathbb{I}) = 0$$

and

$$\delta^{m_1-1}_{\mathbb{S}_1,\frac{1}{c}\mathbb{T}_1}(\mathbb{I}) \neq 0,\ \delta^{m_2-1}_{\mathbb{S}_2,c\mathbb{T}_2}(\mathbb{I}) \neq 0.$$

This completes the proof. □

Remark 4.6. Paul and Gu [14, Theorem 5.2] state that "if the operators S_i are left invertible and the operators T_i are right invertible, $1 \le i \le 2$, then $(S_1 \otimes S_2, T_1 \otimes T_2)$ is m-symmetric if and only if there exist a non-zero scalar c and positive integers m_i, $m = m_1 + m_2 - 1$, such that $(S_1, \frac{1}{c}T_1)$ is strict m_1-symmetric and (S_2, cT_2) is strict m_2-symmetric". The hypothesis S_i are left invertible and T_i are right invertible is a bit of an overkill, as we show below. As seen in the proof of Theorem 4.5, the invertibility of S_1 and S_2 - a fact gurantateed by the left invertibility of S_i and the right invertibility of T_i - is sufficient. If S_i, $1 \le i \le 2$, is left invertible, then there exist operators E_i such that $(E_1 \otimes E_2)(S_1 \otimes S_2) = (I \otimes I)$ $(= \mathbb{I})$ and

$$\begin{aligned} 0 &= \delta^m_{S_1\otimes S_2, T_1\otimes T_2}(\mathbb{I}) \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j}(S_1\otimes S_2)^{m-j}(T_1\otimes T_2)^j \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j}(E_1\otimes E_2)^{j}(T_1\otimes T_2)^j \\ &= \triangle^m_{E_1\otimes E_2, T_1\otimes T_2}(\mathbb{I}). \end{aligned}$$

For conveneience, set $E_1 \otimes E_2 = A$ and $T_1 \otimes T_2 = B$. Then $\triangle^m_{A,B}(\mathbb{I}) = 0$. It is easily seen, use induction, that $(a-1)^m = a^m - \sum_{j=0}^{m-1}\binom{m}{j}(a-1)^j$; hence

$$(L_A R_B - \mathbb{I})^m = (L_A R_B)^m - \sum_{j=0}^{m-1}\binom{m}{j}(L_A R_B - \mathbb{I})^j$$

and upon letting $(L_A R_B - \mathbb{I}) = \nabla_{A,B}$ that

$$\begin{aligned} &\nabla^m_{A,B}(\mathbb{I}) = 0 \iff (L_A R_B)^m(\mathbb{I}) - \sum_{j=0}^{m-1}\binom{m}{j}\nabla^j_{A,B}(\mathbb{I}) = 0 \\ \Longrightarrow\ & (L_A R_B)^{m+1}(\mathbb{I}) = \sum_{j=0}^{m-1}\binom{m}{j}(L_A R_B)\nabla^j_{A,B}(\mathbb{I}) \\ =\ & \sum_{j=0}^{m-1}\binom{m}{j}\nabla^{j+1}_{A,B}(\mathbb{I}) + \sum_{j=0}^{m-1}\binom{m}{j}\nabla^j_{A,B}(\mathbb{I}) \\ =\ & \binom{m}{m-1}\nabla^m_{A,B}(\mathbb{I}) + \sum_{j=0}^{m-1}\binom{m+1}{j}\nabla^j_{A,B}(\mathbb{I}) \\ =\ & \sum_{j=0}^{m-1}\binom{m+1}{j}\nabla^j_{A,B}(\mathbb{I}). \end{aligned}$$

An induction argument now proves

$$\begin{aligned} & (L_A R_B)^n(\mathbb{I}) = \sum_{j=0}^{m-1}\binom{n}{j}\nabla^j_{A,B}(\mathbb{I}) \\ =\ & \binom{n}{m-1}\nabla^{m-1}_{A,B}(\mathbb{I}) + \sum_{j=0}^{m-2}\binom{n}{j}\nabla^j_{A,B}(\mathbb{I}) \end{aligned}$$

for all integers $n \ge m$. Observe that $\triangle^m_{A,B}(\mathbb{I}) = 0$ implies A is right invertible and B is left invertible; since already $B = T_1 \otimes T_2$ is right invertible, B is invertible, and then

$$\begin{aligned} \delta^m_{A,B}(\mathbb{I}) = 0 &\iff \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} L_A^{m-j} R_B^{j}(\mathbb{I}) = 0 \\ &\iff \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} L_A^{m-j} R_B^{-m+j}(\mathbb{I}) = 0 \\ &\iff \triangle^m_{A,B^{-1}}(\mathbb{I}) = 0 \Longrightarrow A \text{ is right invertible} \Longrightarrow A \text{ is invertible.} \end{aligned}$$

The invertibility of A and B implies that of $L_A R_B$. We have

$$\frac{1}{\begin{pmatrix} n \\ m-1 \end{pmatrix}} \left(\mathbb{I} - \sum_{j=0}^{m-2} \begin{pmatrix} n \\ j \end{pmatrix} (L_A R_B)^{-n} \nabla^j_{A,B}(\mathbb{I}) \right) = \nabla^{m-1}_{A,B}(\mathbb{I}).$$

Since $\begin{pmatrix} n \\ m-1 \end{pmatrix}$ is of the order of n^{m-1} and $\begin{pmatrix} n \\ m-2 \end{pmatrix}$ is of the order of n^{m-2}, letting $n \longrightarrow \infty$ this implies

$$\nabla^{m-1}_{A,B}(\mathbb{I}) = 0 \iff \triangle^{m-1}_{A,B}(\mathbb{I}) = 0.$$

Repeating the argument, we eventually have

$$\triangle_{A,B}(\mathbb{I}) = 0 \iff (S_1^{-1} \otimes S_2^{-1})(T_1 \otimes T_2) = \mathbb{I} \iff S_1 \otimes S_2 = T_1 \otimes T_2;$$

hence there exists a scalar c such that $S_1 = cT_1$ and $S_2 = \frac{1}{c}T_2$. In particular, if S_i, T_i are Hilbert space operators such that $S_i = T_i^*$, then $T_1 \otimes T_2$ is self-adjoint.

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