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New results of spectra and pseudospectra of multivalued linear operators

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Abstract. In this paper, we give some properties and results of stability related to the S-spectra, S-pseudospectra, S-essential spectra and S-essential pseudospectra of multivalued linear operators and we show some of their characteristics.

1. Introduction

The concept of pseudospectrum was presumably introduced in the domain of linear operators by J. M. Varah [26] and has been subsequently employed by other authors, such as, H. Landau [23], L. N. Trefethen [25], D. Hinrichsen, A. J. Pritchard [19] and E. B. Davies [17] and more particularly L. N. Trefethen, who developed this idea for matrices and operators, and used this concept to study interesting problems in mathematical physics. The definition of a pseudospectrum of a closed densely linear operator T is as follows: for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon} \right\}.$$

By convention we write $\|(\lambda - T)^{-1}\| = \infty$ if $(\lambda - T)^{-1}$ which is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(T)$. This means that the pseudospectrum can be introduced as a zone of spectral instability.
In [8–10, 20, 22**?**] A. Ammar and A. Jeribi defined the notion of essential pseudospectra of a densely closed, linear operator in the Banach space by:

$$\sigma_{w,\varepsilon}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(T + K)$$

where $\mathcal{K}(X)$ is the subspace of compact operators from X into X, which gives some properties of essential pseudospectra (or Weyl pseudospectra).

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Recently, in [11, 12] A. Ammar, H. Daoud and A. Jeribi, have extended the notion of pseudospectra and essential pseudospectra on multivalued linear operators and cited some properties and results of stability of this pseudospectra.

In [13], the authors developed the notion of S-spectra, S-pseudospectra, S-essential spectra and S-essential pseudospectra for a multivalued linear operator and characterized and investigated their properties.

Linear relations (or multivalued linear operators) made their appearance in functional analysis motivated by the requirement to consider adjoins of non-densely defined linear differential operators (see J. Von. Neumann in [24] to first appearance) and additionally by the need to consider the inverses of certain operators, used, for example, in the study of some Cauchy problems associated to parabolic type equations in Banach spaces (see, example [18]).

This paper presents some characteristics of stability of S-spectra, S-pseudospectra, S-essential spectra and S-essential pseudo-spectra of multivalued linear operators and shows their specificities.

The aim of this paper is to extend and improve new results of stability related to S-spectra, S-pseudospectra, S-essential spectra and S-essential pseudospectra of multivalued linear operators. One of the central questions consists in characterizing the relationship between the norm of S-resolvent according to S-spectra, S-pseudospectra, S-essential spectra and S-essential pseudospectra of multivalued linear operators (see Theorem 3.1 and Theorem 4.3).

The paper is organized in the following way: Section 2 contains preliminary and auxiliary properties that will need to prove the main results of the other sections. Then, Section 3 presents some results about the stability of S-spectra and S-pseudospectra of linear relations. Section 4 is devoted to developing this properties and results on S-essential spectra and S-essential pseudospectra, where we apply the results obtained in Section 3 to investigate this S-essential spectra and S-essential pseudospectra.

2. Preliminary and auxiliary results

The concept of a linear relation in a linear space generalizes the one of a linear operator to that of a multivalued linear operator. A systematic treatment was given by Arens [14] and by Coddington [15]. This concept has been studied in a large number of papers, cf. [16].
Let X, Y, Z be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multivalued linear operator or linear relation T from X to Y is a mapping from a subspace $\mathcal{D}(T)$ of X, called the domain of T, into the collection of nonempty subsets of Y such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all nonzero scalars α, β and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the point of its domain to singletons, then T is said to be a single valued or simply an operator. We denote the class of linear relation from X to Y be $\mathcal{LR}(X, Y)$ and we write $\mathcal{LR}(X) = \mathcal{LR}(X, X)$. A linear relation $T \in \mathcal{LR}(X, Y)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) = \{(x, y) \in X \times Y \text{ such that } x \in \mathcal{D}(T), y \in Tx\},$$

so that we can identify T with $G(T)$. The inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) = \{(y, x) \in Y \times X \text{ such that } (x, y) \in G(T)\}.$$

For $\emptyset \neq N \subset Y$ we have

$$T^{-1}(N) = \{u \in \mathcal{D}(T) \text{ such that } N \cap Tu \neq \emptyset\}.$$

In particular, for $v \in \mathcal{R}(T)$,

$$T^{-1}v = \{u \in \mathcal{D}(T) \text{ such that } v \in Tu\}.$$

Let $T \in \mathcal{LR}(X, Y)$. The symbols $\mathcal{R}(T)$, $N(T)$ and $T(0)$ stand for the range, the null space and the multivalued part of T, which are defined by

$$\begin{array}{rcl} \mathcal{R}(T) & := & \Big\{ y : (x, y) \in G(T) \Big\}, \\ N(T) & := & \Big\{ x \in \mathcal{D}(T) : (x, 0) \in G(T) \Big\}, \text{ and} \\ T(0) & := & \Big\{ y : (0, y) \in G(T) \Big\}. \end{array}$$

T is called injective if $N(T) = 0$ and T is said to be surjective if $\mathcal{R}(T) = Y$. We denote $\alpha(T) := \dim N(T)$, $\beta(T) := \dim Y/\mathcal{R}(T)$, $\overline{\beta}(T) := \dim Y/\overline{\mathcal{R}(T)}$ and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite.
For $S, T \in \mathcal{LR}(X, Y)$, we define the relation $S + T$ by

$$G(S + T) = \{(x, y) \in X \times Y : y = s + t, \text{ where } (x, s) \in G(S) \text{ and } (x, t) \in G(T)\}.$$

For $S \in \mathcal{LR}(X, Y)$ and $R \in \mathcal{LR}(Y, Z)$ where $\mathcal{R}(S) \cap \mathcal{D}(R) \neq \emptyset$, the product RS is defined by

$$G(RS) = \{(x, z) \in X \times Z \text{ such that } (x, y) \in G(S) \text{ and } (y, z) \in G(R) \text{ for some } y \in Y\}.$$

For a given closed linear subspace E of X let Q_E^X (or simply, Q_E) denote the natural quotient map with domain X and null space E. We shall denote $Q_{\overline{T(0)}}^Y$ by Q_T, or simply Q when T is understood. We define $\|Tx\| := \|QTx\|$ $(x \in \mathcal{D}(T))$ and $\|T\| := \|QT\|$.

For U and V be nonempty subsets of a normed space, we define the distance between U and V by the formula $\text{dist}(U, V) := \inf\{\|u - v\| \text{ such that } u \in U, v \in V\}$. We shall write $\text{dist}(x, V)$, or $\text{dist}(V, x)$ for the distance between $\{x\}$ and V. The minimum modulus of T is the quantity $\gamma(T) := \sup\{\lambda \text{ such that } \|Tx\| \geq \lambda \text{dist}(x, N(T)) \text{ for } x \in \mathcal{D}(T)\}$. T is said to be continuous if $\|T\| < \infty$ and T is called open if $\gamma(T) > 0$. If $\mathcal{D}(T) = X$ and if $\|T\| < \infty$, then we shall say that T is bounded.

The relation T is called closed if its graph $G(T)$ is closed in $X \times Y$, or equivalently, if $T = \overline{T}$. T is said to be closable if $\overline{T}$ is an extension of T i.e., if

$$Tx = \overline{T}x \text{ for all } x \in \mathcal{D}(T).$$

We denote the class of all closed linear relations from X to Y by $\mathcal{CR}(X, Y)$ and we write $\mathcal{CR}(X) = \mathcal{CR}(X, X)$ and $\mathcal{KR}(X, Y)$ will denote the class of all compact linear relations from X to Y where $T \in \mathcal{LR}(X, Y)$ is called compact if $\overline{Q_T T B_X}$ is compact and B_X is the unit ball of X. Let $\tilde{X}$ denote the completion of the normed space X and let $\tilde{T}$ denote the linear relation in $\mathcal{LR}(\tilde{X}, \tilde{Y})$ whose graph is the completion of $G(T)$, we call $\tilde{T}$ the completion (or complete closure) of T.
Let $\tilde{X}$ denote the completion of the normed space X and let $\tilde{T}$ denote the linear relation in $\mathcal{LR}(\tilde{X}, \tilde{Y})$ whose graph is the completion of $G(T)$, we call $\tilde{T}$ the completion (or complete closure) of T.

Definition 2.1. *Let $T \in \mathcal{LR}(X, Y)$ where X, Y are normed space.*

(i) *T is said to be upper semi-Fredholm, if there exists a closed, finite, codimensional subspace M of X, such that the restriction $T|_M$ has a single valued continuous inverse.*

(ii) *T is said to be lower semi-Fredholm linear relation if its conjugate T' is upper semi-Fredholm linear relation.*

We denote by $F_+(X, Y)$, which we abbreviate as F_+, the set of upper semi- Fredholm linear relations and by $F_-(X, Y)$ (or F_-) the set of lower semi-Fredholm linear relations. ◊

In the case when X and Y are Banach spaces, we extend the classes of closed single-valued Fredholm type operators given earlier to include closed multivalued operators, and note that the definitions of the classes $F_+(X, Y)$ and $F_-(X, Y)$ are consistent, respectively, with

$$\Phi_+(X,Y) := \Big\{T \in \mathcal{CR}(X,Y) : \ \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\Big\}, \text{ and}$$

$$\Phi_-(X,Y) := \Big\{T \in \mathcal{CR}(X,Y) : \ \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\Big\},$$

$$\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y), \Big(\text{resp. } \Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)\Big).$$

If $X = Y$, then , $\Phi_+(X,Y)$,$\Phi_-(X,Y)$, $\Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced respectively by $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ and $\Phi(X)$. T. Several authors in [3, 4, 6, 7] has studied some properties of Fredholm relations that we need to study the concept of demicompactness.

Lemma 2.1. *Let X and Y be normed spaces and* $T \in \mathcal{LR}(X,Y)$ *then*

(*i*) [16, Proposition II.1.2] QT is single valued.
(*ii*) [16, Proposition II.1.4] $\|Tx\| = dist(y, T(0))$ for any $y \in Tx$.
(*iii*) [16, Proposition II.1.4] $\|Tx\| = dist(Tx, T(0)) = dist(Tx, 0)$ $(x \in \mathcal{D}(T))$.
(*iv*) [16, Proposition II.1.6] $\|T\| = \sup\limits_{x\in B_X}\|Tx\|$ with $B_X := \{x \in X : \|x\| \leq 1\}$.
(*v*) [16, Theorem II.2.5] $\gamma(T) = \|T^{-1}\|^{-1}$. ◊

Lemma 2.2. [16, Proposition I.2.8] *Let* $T \in \mathcal{LR}(X,Y)$ *where X and Y linear spaces. Then for* $x \in \mathcal{D}(T)$*, we have the following equivalence:*

(*i*) $$y \in Tx \Leftrightarrow Tx = y + T(0).$$
In particular,
(*ii*) $$0 \in Tx \Leftrightarrow Tx = T(0).$$ ◊

Remark 2.1. *Let X and Y be linear spaces and* $T \in \mathcal{LR}(X,Y)$ *then from Lemma 2.2,*

$$N(T) = \{x \in \mathcal{D}(T) \text{ such that } Tx = T(0)\}.$$ ◊

Lemma 2.3. [16, Corollary I.2.4] *Let T be a linear relation. Then* $T(0)$ *and* $T^{-1}(0)$ *are linear subspaces.* ◊

Lemma 2.4. [16, Corollary III.7.7] *Let* $T \in \mathcal{LR}(X,Y)$ *where X, Y are normed spaces be open and injective with dense range. Then for any relation S such that* $S(0) \subset \overline{T(0)}$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ *and* $\|S\| < \gamma(T)$,
we have $T + S$ *is open injective with dense range.* ◊

Lemma 2.5. [12, Lemma 2.5] *Let* $T, S \in \mathcal{LR}(X)$ *where X is a normed space such that T is injective, open and* $S(0) \subset T(0)$*. Then*

$$\gamma(T - S) \geq \gamma(T) - \|S\|.$$ ◊

Definition 2.2. *Let* $T \in \mathcal{LR}(X)$ *and where X is a normed space, let* $\lambda \in \mathbb{C}$,

$$R(\lambda, T) = (\lambda - T)^{-1}$$

called the resolvent of T (corresponding to λ*). The resolvent set of T is the set*

$$\rho(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is injective, open with dense range on } X\}.$$

The spectrum of T is the set $\sigma(T) := \mathbb{C}\backslash\rho(T)$. ◊

Remark 2.2. *From* [16, Definitions VI.1.1] *and* [16, Exercise VI.1.2], *we observe that if* $T \in \mathcal{CR}(X)$ *where X is a complete space, we have*

$$\rho(T) = \{\lambda \in \mathbb{C} \text{ such that } (\lambda - T)^{-1} \text{ is a bounded linear operator on } X\}.$$ ◊

Definition 2.3. *Let X be a Banach space and let $S, T \in \mathcal{LR}(X)$ such that S is continuous, T is closed with $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, in [2], T. Alvarez, A. Ammar and A. Jeribi defined the notion of $\mathcal{S}$- resolvent set of T by :*

$$\rho_S(T) = \{\lambda \in \mathbb{C} \textit{ such that } \lambda S - T \text{ is bijective}\}$$

and the $\mathcal{S}$-spectrum of T by $\sigma_S(T) = \mathbb{C} \backslash \rho_S(T)$ and the $\mathcal{S}$- essential spectra of T by

$$\sigma_{w,S}(T) = \bigcap_{K \in \mathcal{K}_T(X)} \sigma_S(T + K)$$

where

$$\mathcal{K}_T(X) = \{K \in \mathcal{KR}(X) \textit{ such that } \mathcal{D}(K) \supset \mathcal{D}(T) \textit{ and } K(0) \subset T(0)\}. \qquad \diamond$$

Proposition 2.1. [2, Theorem 3.1]. *Let $T \in \mathcal{CR}(X)$ where X is a Banach space and S is continuous, with $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then*

$$\sigma_{w,S}(T) = \mathbb{C} \backslash \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \in \Phi(X) \text{ and } i(\lambda S - T) = 0\}. \qquad \diamond$$

3. Stability of S-spectra and S-pseudospectra of linear relations

In this section, we study results of stability and properties of S-spectra and S-pseudospectra.

Definition 3.1. *In [13], A. Ammar, H. Daoud and A. Jeribi introduced the definition of S-spectrum of a linear relation in a normed space X. Let $T \in \mathcal{LR}(X)$, S a continuous linear relation such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then we the $\mathcal{S}$-resolvent set of T by*

$$\rho_S(T) := \{\lambda \in \mathbb{C} \text{ such that } (\lambda S - T) \text{ is injective, open with dense range on } X\}.$$

We denote the S- spectra set of T by:

$$\sigma_S(T) = \mathbb{C} \backslash \rho_S(T). \qquad \diamond$$

It is clear that if $T \in \mathcal{CR}(X)$ and X is complete, we will return to the $\mathcal{S}$- spectrum definition in a Banach space with closed linear relation. In this case

$$\begin{aligned} \rho_S(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda S - T \text{ is bijective}\} \textit{ (see Definition 2.3).} \\ &= \{\lambda \in \mathbb{C} \text{ such that } (\lambda S - T)^{-1} \text{ is a bounded linear operator on } X\}. \qquad \diamond \end{aligned}$$

Proposition 3.1. *Let $T, S \in \mathcal{LR}(X)$ where X is a normed space, S is continuous such that $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then for any $\lambda, \beta \in \mathbb{C}$ with $\beta \neq 0$*

$$\begin{array}{rrcl} & N(\lambda S - \beta T) & = & N(\beta^{-1}\lambda S - T), \\ & \mathcal{R}(\lambda S - \beta T) & = & \mathcal{R}(\beta^{-1}\lambda S - T) \\ \text{and} & \gamma(\lambda S - \beta T) & = & |\beta|\, \gamma(\beta^{-1}\lambda S - T). \qquad \diamond \end{array}$$

Proof. Let $x \in N(\lambda S - \beta T)$, if and only if, $x \in \mathcal{D}(\lambda S - \beta T) = \mathcal{D}(\lambda S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ and $0 \in (\lambda S - \beta T)x$, if and only if, $x \in \mathcal{D}(\beta^{-1}\lambda S - T) = \mathcal{D}(\beta^{-1}\lambda S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ and $0 \in (\beta^{-1}\lambda S - T)x$, i.e., $x \in N(\beta^{-1}\lambda S - T)$.
On the other hand, since $\mathcal{D}(\lambda S - \beta T) = \mathcal{D}(\beta^{-1}\lambda S - T) = \mathcal{D}(T)$,

$$\begin{array}{rcl} \mathcal{R}(\lambda S - \beta T) & = & (\lambda S - \beta T)(\mathcal{D}(\lambda S - \beta T)) \\ & = & (\lambda S - \beta T)(\mathcal{D}(T)) \\ & = & \beta^{-1}(\lambda S - \beta T)(\mathcal{D}(T)) \\ & = & (\beta^{-1}\lambda S - T)(\mathcal{D}(T)) \\ & = & (\beta^{-1}\lambda S - T)(\mathcal{D}(\beta^{-1}\lambda S - T)) \\ & = & \mathcal{R}(\beta^{-1}\lambda S - T). \end{array}$$

Finally, we have

$$\begin{aligned}\gamma(\lambda S-\beta T) &= \sup\{\lambda \text{ such that } \|(\lambda S-\beta T)x\| \geq \lambda \mathrm{dist}(x, N(\lambda S-\beta T)) \text{ for } x\in\mathcal{D}(T)\}\\ &= \sup\{\lambda \text{ such that } |\beta|\|(\beta^{-1}\lambda S-T)x\| \geq \lambda \mathrm{dist}(x, N(\beta^{-1}\lambda S-T))\\ &\quad \text{for } x\in\mathcal{D}(T)\}\\ &= |\beta|\,\gamma(\beta^{-1}\lambda S-T).\end{aligned}$$

Q.E.D.

Corollary 3.1. *Let T, $S\in\mathcal{LR}(X)$ where X is a normed space, S is continuous such that $S(0)\subset\overline{T(0)}$ and $\mathcal{D}(S)\supset\mathcal{D}(T)$, then for any $\beta\in\mathbb{C}$ with $\beta\neq 0$*

$$\rho_S(\beta T)=\rho_S(T)\,\beta$$

and

$$\sigma_S(\beta T)=\sigma_S(T)\,\beta.$$

◇

Proposition 3.2. *Let T, $S\in\mathcal{LR}(X)$ where X is a normed space, S is continuous such that $S(0)\subset\overline{T(0)}$ and $\mathcal{D}(S)\supset\mathcal{D}(T)$, and $\varepsilon>0$. Then for any α, $\beta\in\mathbb{C}$ with $\beta\neq 0$ we have,*

$$\rho_S(\alpha S+\beta T)=\alpha\,+\,\rho_S(T)\,\beta$$

and

$$\sigma_S(\alpha S+\beta T)=\alpha\,+\,\sigma_S(T)\,\beta.$$

◇

Proof. Let $\lambda\in\rho_S(\alpha S+\beta T)$, if, and only if, $(\lambda-\alpha)S-\beta T$ is injective, open with dense range, if, and only if, $(\lambda-\alpha)\in\rho_S(\beta T)$, if, and only if, $(\lambda-\alpha)\in\rho_S(T)\,\beta$.

In similar way, we obtain $\sigma_S(\alpha S+\beta T)=\alpha\,+\,\sigma_S(T)\,\beta$. Q.E.D.

Definition 3.2. *Let X be a Banach space, $\varepsilon>0$, let $T\in\mathcal{CR}(X)$, and $S\in\mathcal{LR}(X)$ such that S continuous, $S(0)\subset T(0)$ and $\mathcal{D}(S)\supset\mathcal{D}(T)$. In [13], A. Ammar, H. Daoud and A. Jeribi defined the $\mathcal{S}$-pseudospectra of T as follows:*

$$\sigma_{\varepsilon,S}(T) \;=\; \sigma_S(T)\cup\left\{\lambda\in\mathbb{C} \text{ such that } \|(\lambda S-T)^{-1}\|>\frac{1}{\varepsilon}\right\}.$$

We denote the $\mathcal{S}$-pseudoresolvent set of T

$$\rho_{\varepsilon,S}(T)=\mathbb{C}\backslash\sigma_{\varepsilon,S}(T)=\rho_S(T)\cap\left\{\lambda\in\mathbb{C} \text{ such that } \|(\lambda S-T)^{-1}\|\leq\frac{1}{\varepsilon}\right\}.$$

◇

Proposition 3.3. *Let X be a Banach space, $\varepsilon>0$. Let $T\in\mathcal{CR}(X)$ and $S\in\mathcal{LR}(X)$ is a continuous linear relation such that $S(0)\subset T(0)$ and $\mathcal{D}(S)\supset\mathcal{D}(T)$. Then $\|S\|=0$ implies $\rho_{\varepsilon,S}(T)=\emptyset$ or $\mathbb{C}$.* ◇

Proof. In [13, Lemma 2.3], we have $\lambda\in\rho_S(T)$, if, and only if, T is injective, open with dense range, i.e, $\lambda S-T$ is injective, open with dense range, if, and only if, T is injective, open with dense range. Thus, we discus two case:

case 1: if T is not injective, open with dense range, then $\rho_{\varepsilon,S}(T)=\emptyset$.

case 2: if T is injective, open with dense range, then

$$\rho_{\varepsilon,S}(T)=\{\lambda\in\mathbb{C} \text{ such that } \|(\lambda S-T)^{-1}\|\leq\frac{1}{\varepsilon}\}.$$

Using Lemma 2.1 (v), we have

$$\rho_{\varepsilon,S}(T)=\{\lambda\in\mathbb{C} \text{ such that } \gamma(\lambda S-T)\geq\varepsilon\}.$$

Since $S(0) \subset T(0)$, then $\lambda S(0) \subset T(0)$ and hence $T(0) - \lambda S(0) \subset T(0)$.On the other hand, it is clear that $T(0) \subset T(0) - \lambda S(0)$ and for $x \in \mathcal{D}(\lambda S - T) = \mathcal{D}(\lambda S) \cap \mathcal{D}(T) = \mathcal{D}(T)$, we have

$$\begin{aligned}\|(\lambda S - T)x\| &= \|(T - \lambda S)x\| \\ &= \text{dist}(Tx - \lambda Sx, (T - \lambda S)(0)) \\ &= \text{dist}(Tx - \lambda Sx, T(0))\end{aligned}$$

Now, since $\|S\| = 0$, then, using [13, Lemma 2.2], $\lambda Sx \subset \mathcal{R}(S) \subset \overline{S(0)} \subset \overline{T(0)}$, hence,

$$\begin{aligned}\|(\lambda S - T)x\| &= \text{dist}(Tx, T(0)) \\ &= \|Tx\|.\end{aligned}$$

Finally, it is obvious, since T is injective, open with dense range, that $\lambda S - T$ is injective, open with dense range and

$$\begin{aligned}\gamma(\lambda S - T) &= \sup \{\alpha \text{ such that } \|(\lambda S - T)x\| \geq \alpha \text{ dist}(x, N(\lambda S - T)) \text{ for } x \in \mathcal{D}(\lambda S - T)\} \\ &= \sup \{\alpha \text{ such that } \|(\lambda S - T)x\| \geq \alpha \text{ dist}(x, 0) \text{ for } x \in \mathcal{D}(\lambda S - T)\} \\ &= \sup \{\alpha \text{ such that } \|Tx\| \geq \alpha \text{ dist}(x, N(T)) \text{ for } x \in \mathcal{D}(T)\} \\ &= \gamma(T),\end{aligned}$$

and we obtain that

$$\rho_{\varepsilon,S}(T) = \{\lambda \in \mathbb{C} \text{ such that } \gamma(T) \geq \varepsilon\}.$$

Thus $\rho_{\varepsilon,S}(T) = \emptyset$ or $\mathbb{C}$. Q.E.D.

In the sequel of the paper, X will denote a Banach space, $\varepsilon > 0$ and $S, T \in \mathcal{LR}(X)$ such that S is continuous, T is closed with $S(0) \subset T(0)$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| \neq 0$ except where stated otherwise.

Proposition 3.4. *Let α, $\beta \in \mathbb{C}$ such that $\beta \neq 0$, then,*

$$\sigma_{\varepsilon,S}(\alpha S + \beta T) = \alpha + \sigma_{\frac{\varepsilon}{|\beta|},S}(T)\,\beta. \qquad \diamond$$

Proof. $\lambda \in \sigma_{\varepsilon,S}(\alpha S + \beta T)$, if, and only if, $\lambda \in \sigma_S(\alpha S + \beta T)$ or $\|((\lambda - \alpha)S - \beta T)^{-1}\| > \frac{1}{\varepsilon}$, if, and only if, $(\lambda - \alpha)S - \beta T$ is not injective, open with dense range or $\|(\beta^{-1}(\lambda - \alpha)S - T)^{-1}\| = |\beta| \|((\lambda - \alpha)S - \beta T)^{-1}\| > \frac{|\beta|}{\varepsilon}$, i.e., $(\lambda - \alpha) \in \sigma_S(\beta T)$ or $\|(\beta^{-1}(\lambda - \alpha)S - T)^{-1}\| > \frac{|\beta|}{\varepsilon}$, if, and only if, $\beta^{-1}(\lambda - \alpha) \in \sigma_S(T)$ or $\|(\beta^{-1}(\lambda - \alpha)S - T)^{-1}\| > \frac{|\beta|}{\varepsilon}$, if, and only if, $\beta^{-1}(\lambda - \alpha) \in \sigma_{\frac{\varepsilon}{|\beta|},S}(T)$. Q.E.D.

Proposition 3.5. *For $\delta > 0$, we have*

$$\sigma_{\varepsilon,S}(T) \subseteq D_\delta + \sigma_{\varepsilon,S}(T) \subseteq \sigma_{\varepsilon+\delta,S}(T),$$

where $D_\delta = \{\lambda \in \mathbb{C}$ such that $|\lambda| \leq \frac{\delta}{\|S\|}\}$. $\diamond$

Proof. Let $\lambda \in D_\delta + \sigma_{\varepsilon,S}(T)$, then there exists $\lambda_1 \in D_\delta$ and $\lambda_2 \in \sigma_{\varepsilon,S}(T)$ such that $\lambda = \lambda_1 + \lambda_2$. ewline Assume that $\lambda \notin \sigma_{\varepsilon+\delta,S}(T)$, then $\lambda_1 + \lambda_2 \in \rho_S(T)$ and $\|((\lambda_1 + \lambda_2)S - T)^{-1}\| \leq \frac{1}{\varepsilon + \delta}$. Therefore $(\lambda_1 + \lambda_2)S - T$ is injective, surjective, open and $\gamma((\lambda_1 + \lambda_2)S - T) \geq \varepsilon + \delta$. Using the fact that $\|\lambda_1 S\| = |\lambda_1| \|S\| \leq \delta < \delta + \varepsilon$ and applying Lemma 2.4, we obtain $(\lambda_1 + \lambda_2)S - T - \lambda_1 S = \lambda_2 S - T$ is injective, surjective and open, i.e., $\lambda_2 \in \rho_S(T)$. On the other hand, by Lemma 2.5,

$$\begin{aligned}\gamma(\lambda_2 S - T) &= \gamma(T - \lambda_2 S) \\ &= \gamma(T - \lambda_2 S - \lambda_1 S + \lambda_1 S) \\ &\geq \gamma(T - (\lambda_2 + \lambda_1)S) - \|\lambda_1 S\| \\ &\geq \varepsilon + \delta - |\lambda_1| \|S\| \\ &\geq \varepsilon.\end{aligned}$$

Thus

$$\|(\lambda_2 S - T)^{-1}\| \leq \frac{1}{\varepsilon}.$$

Finally, we conclude

$$\lambda_2 \notin \sigma_{\varepsilon,S}(T),$$

and this is a contradiction. Q.E.D.

Proposition 3.6. *Let $B \in \mathcal{LR}(X)$ such that $B(0) \subset T(0)$, $\mathcal{D}(B) \supset \mathcal{D}(T)$ and $\|B\| < \varepsilon$, then*

$$\sigma_{\varepsilon-\|B\|,S}(T) \subseteq \sigma_{\varepsilon,S}(T+B) \subseteq \sigma_{\varepsilon+\|B\|,S}(T). \qquad \diamond$$

Proof. It is simply to show, since $B(0) \subset T(0)$, $\mathcal{D}(B) \supset \mathcal{D}(T)$ and B is continuous, using [1, Lemma 3.5], that $T + B$ is closed

Let $\lambda \notin \sigma_{\varepsilon+\|B\|,S}(T)$, i.e., $\lambda \in \rho_S(T)$ and $\|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon + \|B\|}$, i.e., $\lambda S - T$ is injective, open, surjective and $\gamma(\lambda S - T) \geq \varepsilon + \|B\|$. We have $B(0) \subset (\lambda S - T)(0) = T(0)$, $\mathcal{D}(B) \supset \mathcal{D}(\lambda S - T) = \mathcal{D}(T)$ and $\|B\| < \varepsilon + \|B\| \leq \gamma(\lambda S - T)$, then, using Lemma 2.4, we obtain $\lambda S - T - B$ is injective, open with dense range, i.e., $\lambda \in \rho_S(T + B)$. On the other hand, by Lemma 2.5, we have

$$\begin{array}{rcl} \gamma(\lambda S - T - B) & \geq & \gamma(\lambda S - T) - \|B\| \\ & \geq & \varepsilon + \|B\| - \|B\| \\ & \geq & \varepsilon. \end{array}$$

Then

$$\|(\lambda S - (T + B))^{-1}\| \leq \frac{1}{\varepsilon}.$$

Hence

$$\lambda \notin \sigma_{\varepsilon,S}(T + B).$$

For the first inclusion, let $\lambda \notin \sigma_{\varepsilon,S}(T + B)$. Then $\lambda \in \rho_S(T + B)$ and $\|(\lambda S - T - B)^{-1}\| \leq \frac{1}{\varepsilon}$. Hence $\lambda S - T - B$ is injective, open, surjective and $\gamma(\lambda S - T - B) \geq \varepsilon > \|B\|$. Using Lemma 2.4, $\lambda S - T - B + B = \lambda S - T$ is injective, open with dense range. Then $\lambda \in \rho_S(T)$. In similar way, by Lemma 2.5, we have

$$\begin{array}{rcl} \gamma(\lambda S - T) & = & \gamma(\lambda S - T - B + B) \\ & \geq & \gamma(\lambda S - T - B) - \|B\| \\ & \geq & \varepsilon - \|B\|. \end{array}$$

Then

$$\|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon - \|B\|}.$$

Hence

$$\lambda \notin \sigma_{\varepsilon-\|B\|,S}(T).$$

Q.E.D.

We give some further results on the location of the pseudospectra. We start with the following general result. Although the result is well known, we include the proof. For a subset $\Omega \in \mathbb{C}$ we set as usual

$$\operatorname{dist}(\lambda, \Omega) = \inf\{|z - \lambda| \text{ such that } z \in \Omega\},$$

and note that if Ω is compact, then the infimum is attained for some point in Ω.

Theorem 3.1. (*i*) *Let $T, S \in \mathcal{LR}(X)$, where X is a normed space, such that S is continuous,, $\|S\| \neq 0$, $S(0) \subset \overline{T(0)}$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then $\lambda \notin \sigma_S(T)$, implies*

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\text{dist}(\lambda, \sigma_S(T))\|S\|}.$$

(*ii*) *Let $T \in \mathcal{CR}(X)$, $S \in \mathcal{LR}(X)$, where X is a Banach space, such that S is continuous, $\|S\| \neq 0$, $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then $\lambda \notin \sigma_{\varepsilon,S}(T)$, implies*

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\text{dist}(\lambda, \sigma_{\varepsilon,S}(T))\|S\| + \varepsilon}. \qquad \diamond$$

Proof. (*i*) Let $\lambda \in \rho_S(T)$, we have $\text{dist}(\lambda, \sigma_S(T)) = \inf\{|z - \lambda| \text{ such that } z \in \sigma_S(T)\}$. Then for all $\eta > 0$ there exists $z_\eta \in \sigma_S(T)$ such that

$$|\lambda - z_\eta| < \text{dist}(\lambda, \sigma_S(T)) + \eta.$$

We first show that $\overline{(\lambda S - T)(0)} = \overline{T(0)}$. In fact, since $S(0) \subset \overline{T(0)}$ then $(\lambda S - T)(0) \subset \overline{T(0)}$, so, $\overline{(\lambda S - T)(0)} \subset \overline{T(0)}$. On the other hand, since $S(0)$ is a linear subspace (Lemma 2.3), then $T(0) \subset \lambda S(0) - T(0) = (\lambda S - T)(0)$, hence $\overline{T(0)} \subset \overline{(\lambda S - T)(0)}$.

We suppose that $|\lambda - z_\eta| < \frac{\gamma(\lambda S - T)}{\|S\|}$, since $\lambda S - T$ is injective open with dense range, $(z_\eta - \lambda)S(0) = S(0) \subset \overline{(\lambda S - T)(0)} = \overline{T(0)}$, $\mathcal{D}((z_\eta - \lambda)S) = \mathcal{D}(S) \supset \mathcal{D}(\lambda S - T) = \mathcal{D}(T)$ and $\|(z_\eta - \lambda)S\| = |\lambda - z_\eta|\|S\| < \gamma(\lambda S - T)$, then, $\lambda S - T + (z_\eta - \lambda)S = z_\eta S - T$ is injective open with dense range (using Lemma 2.4). Hence $z_\eta \in \rho_S(T)$. This is a contradiction.

Therefore $|\lambda - z_\eta| \geq \frac{\gamma(\lambda S - T)}{\|S\|}$ for all $\eta > 0$. Thus

$$\begin{aligned} \frac{\gamma(\lambda S - T)}{\|S\|} &\leq |\lambda - z_\eta| \\ &< \text{dist}(\lambda, \sigma_S(T)) + \eta \quad \text{for all } \eta > 0. \end{aligned}$$

So

$$\frac{\gamma(\lambda S - T)}{\|S\|} \leq \text{dist}(\lambda, \sigma_S(T)).$$

Hence

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\text{dist}(\lambda, \sigma_S(T))\|S\|}.$$

(*ii*) Let $\lambda \in \rho_{\varepsilon,S}(T)$, since $\text{dist}(\lambda, \sigma_{S,\varepsilon}(T)) = \inf\{|z - \lambda| \text{ such that } z \in \sigma_{S,\varepsilon}(T)\}$, then for $\eta > 0$, there exists $z_\eta \in \sigma_{S,\varepsilon}(T)$ such that $|\lambda - z_\eta| < \text{dist}(\lambda, \sigma_{S,\varepsilon}(T)) + \eta$. We discuss two case:

Case 1: if $|\lambda - z_\eta| \geq \frac{\gamma(\lambda S - T)}{\|S\|}$, then,

$$\begin{aligned} \frac{\gamma(\lambda S - T)}{\|S\|} &\leq |\lambda - z_\eta| \\ &< \text{dist}(\lambda, \sigma_{\varepsilon,S}(T)) + \eta, \quad \text{for all } \eta > 0. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\gamma(\lambda S - T)}{\|S\|} &\leq \text{dist}(\lambda, \sigma_{\varepsilon,S}(T)), \\ \gamma(\lambda S - T) &\leq \text{dist}(\lambda, \sigma_{\varepsilon,S}(T))\|S\|. \end{aligned}$$

Thus

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\text{dist}(\lambda, \sigma_{\varepsilon,S}(T))\|S\|}.$$

Case 2: if $|\lambda - z_\eta| < \frac{\gamma(\lambda S-T)}{\|S\|}$, i.e., $\|(\lambda - z_\eta)S\| < \gamma(\lambda S - T)$, since $(z_\eta - \lambda)S(0) = S(0) \subset (\lambda S - T)(0) = T(0)$, $\mathcal{D}((z_\eta - \lambda)S) = \mathcal{D}(S) \supset \mathcal{D}((\lambda S - T)) = \mathcal{D}(\lambda S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ and $\lambda S - T$ is injective open with dense range (as $\lambda \in \rho_S(T)$), then, using Lemma 2.4, we have $\lambda S - T + (z_\eta - \lambda)S = z_\eta S - T$ is injective open with dense range, i.e., $z_\eta \in \rho_S(T)$. But $z_\eta \in \sigma_{\varepsilon,S}(T)$, then $\|(z_\eta S - T)^{-1}\| > \dfrac{1}{\varepsilon}$, i.e., $\gamma(z_\eta S - T) < \varepsilon$.
Now, using Lemma 2.5, we obtain

$$\begin{array}{rcl}\gamma(z_\eta S - T) & = & \gamma(\lambda S - T + (z_\eta - \lambda)S) \\ & \geq & \gamma(\lambda S - T) - |z_\eta - \lambda|\|S\|.\end{array}$$

Therefore

$$\begin{array}{rcl}\dfrac{\gamma(\lambda S - T)}{\|S\|} & \leq & \dfrac{\gamma(z_\eta S - T)}{\|S\|} + |z_\eta - \lambda| \\ & < & \dfrac{\varepsilon}{\|S\|} + \operatorname{dist}(\lambda, \sigma_{\varepsilon,S}(T)) + \eta, \qquad \forall\, \eta > 0.\end{array}$$

Thus

$$\frac{\gamma(\lambda S - T)}{\|S\|} \leq \frac{\varepsilon}{\|S\|} + \operatorname{dist}(\lambda, \sigma_{\varepsilon,S}(T)),$$

i.e.,

$$\gamma(\lambda S - T) \leq \varepsilon + \operatorname{dist}(\lambda, \sigma_{\varepsilon,S}(T))\|S\|.$$

Then

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma_{\varepsilon,S}(T))\|S\| + \varepsilon}. \qquad \text{Q.E.D.}$$

Corollary 3.2. *Let $T \in \mathcal{CR}(X)$, $S \in \mathcal{LR}(X)$, where X is a Banach space, such that S is continuous, $\|S\| \neq 0$, $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$, then*

$$\{\lambda \in \mathbb{C} \text{ such that } \operatorname{dist}(\lambda, \sigma_S(T)) < \frac{\varepsilon}{\|S\|}\} \subseteq \sigma_{\varepsilon,S}(T). \qquad \diamond$$

Proof. Let $\lambda \notin \sigma_{\varepsilon,S}(T)$, then $\lambda \notin \sigma_S(T)$, using Theorem 3.1 (i), we have

$$\frac{1}{\operatorname{dist}(\lambda, \sigma_S(T))\|S\|} \leq \|(\lambda S - T)^{-1}\| \leq \frac{1}{\varepsilon}.$$

Therefore

$$\operatorname{dist}(\lambda, \sigma_S(T))\|S\| \geq \varepsilon.$$

Hence

$$\operatorname{dist}(\lambda, \sigma_S(T)) \geq \frac{\varepsilon}{\|S\|}. \qquad \text{Q.E.D.}$$

Theorem 3.2. *Let $T \in \mathcal{CR}(X)$ where X is a Banach space and assume that V is a bounded single valued relation in $\mathcal{LR}(X)$ such that $0 \in \rho(V)$. Let $k = \|V\|\|V^{-1}\|$. Let $B = VTV^{-1}$. Then*

$$\sigma_S(T) = \sigma_{VSV^{-1}}(B) \tag{3.1}$$

and for $k \neq 0$, we have

$$\sigma_{\varepsilon/k, VSV^{-1}}(B) \subseteq \sigma_{\varepsilon,S}(T) \subseteq \sigma_{k\varepsilon, VSV^{-1}}(B) \tag{3.2}$$

and

$$\sigma_{\varepsilon/k, S}(T) \subseteq \sigma_{\varepsilon, VSV^{-1}}(B) \subseteq \sigma_{k\varepsilon, S}(T). \tag{3.3}$$

$\diamond$

Proof. We first show that B is closed. Since V is a bounded linear operator, i.e., $V(0) = \{0\}$, $\mathcal{D}(V) = X$ and V is continuous, then, by [16, Definitions II.5.1 (6)], V is closed thus V has a closed range, $\mathcal{R}(V) = X$, (as $0 \in \rho(V)$). We have $\alpha(V) = 0 < \infty$ and $\gamma(V) > 0$ (as V injective and open), by [16, Proposition II.5.17], VT is closed. Moreover, since V^{-1} is single valued and bounded, then VTV^{-1} is closed (using [16, Exercise II.5.18]). Hence B is closed. It is simply to verify since $S(0) \subset T(0)$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and S is continuous that $VSV^{-1}(0) = VS(0) \subset VT(0) = VTV^{-1}(0) = B(0)$,

$$\begin{array}{rcll}
\mathcal{D}(VSV^{-1}) & = & V\mathcal{D}(VS) & (\text{ using [16, I.1.3 (2)]}) \\
& = & V\{x \in X,\ \ Sx \cap \mathcal{D}(V) \neq \emptyset\} & (\text{ using [16, I.1.3 (1)]}) \\
& = & V\{x \in X,\ \ Sx \neq \emptyset\} & (\text{ since } \mathcal{D}(V) = X\text{ }) \\
& = & V(\mathcal{D}(S)) & \\
& \supset & V(\mathcal{D}(T)) & \\
& = & V\mathcal{D}(VT) = V\{x \in X,\ \ Tx \cap \mathcal{D}(V) \neq \emptyset\} & \\
& = & \mathcal{D}(VTV^{-1}) & (\text{ using [16, I.1.3 (2)]}) \\
& = & \mathcal{D}(B), &
\end{array}$$

and ,by [16, Proposition II.3.13],

$$\begin{array}{rcll}
\|VSV^{-1}\| & \leq & \|VS\|\|V^{-1}\| & (\text{ since } V^{-1}(0) = 0) \\
& \leq & \|V\|\|S\|\|V^{-1}\| & (\text{ since } \mathcal{D}(V) = X) \\
& \leq & k\|S\|. &
\end{array}$$

Using [1, Lemma 3.5], we prove that $\lambda S - T$ and $\lambda VSV^{-1} - B$ are closed.

On the other hand,

$$\begin{array}{rcll}
\lambda S - T & = & \lambda S - V^{-1}BV & \\
(\lambda S - T)V^{-1} & = & (\lambda S - VBV)V^{-1} & \\
(\lambda S - T)V^{-1} & = & (\lambda SV^{-1} - VBVV^{-1}) & (\text{ using [16, Proposition I.4.2 (d)]}) \\
(\lambda S - T)V^{-1} & = & (\lambda SV^{-1} - VB) & (\text{ as } V(0) = 0) \\
V(\lambda S - T)V^{-1} & = & V(\lambda SV^{-1} - VB) & \\
V(\lambda S - T)V^{-1} & = & (\lambda VSV^{-1} - VV^{-1}B) & (\text{ using [16, Proposition I.4.2 (e)]}) \\
V(\lambda S - T)V^{-1} & = & (\lambda VSV^{-1} - B) & (\text{ as } V(0) = 0) \\
(\lambda S - T) & = & V^{-1}(\lambda VSV^{-1} - B)V & (\text{ as } V \text{ is injective.})
\end{array}$$

Now, if $\lambda \in \rho_S(T)$ then the closed relation $\lambda S - T$ is injective, surjective and open. By [16, Proposition VI.5.2]) $V(\lambda S - T)V^{-1} = \lambda VSV^{-1} - B$ is also closed, bounded below (injective and open), surjective. Hence $\lambda \in \rho_{VSV^{-1}}(B)$.
Conversely, if $\lambda \in \rho_{VSV^{-1}}(B)$ then the closed relation $(\lambda VSV^{-1} - B)$ is injective, surjective and open, by [16, Proposition VI.5.2], $V^{-1}(\lambda VSV^{-1} - B)V = \lambda S - T$ is also closed, bounded below (injective and open), surjective. Hence $\lambda \in \rho_S(T)$, which implies the first result.

Now, we have $V^{-1}(\lambda VSV^{-1} - B)V = (\lambda S - T)$ and $V(\lambda S - T)V^{-1} = (\lambda VSV^{-1} - B)$. Then $V^{-1}(\lambda VSV^{-1} - B)^{-1}V = (\lambda S - T)^{-1}$ and $V(\lambda S - T)^{-1}V^{-1} = (\lambda VSV^{-1} - B)^{-1}$. Thus

$$\begin{array}{rcll}
\|(\lambda S - T)^{-1}\| & = & \|V^{-1}(\lambda VSV^{-1} - B)^{-1}V\| & \\
& \leq & \|V^{-1}(\lambda VSV^{-1} - B)^{-1}\|\|V\| & (\text{using [16, Proposition II.3.13]}, \\
& & & \text{since } V(0) = 0.) \\
& \leq & \|V^{-1}\|\|(\lambda VSV^{-1} - B)^{-1}\|\|V\| & (\text{using [16, Proposition II.3.13]}, \\
& & & \text{since } \mathcal{D}(V^{-1}) = \mathcal{R}(V) = X.) \\
& \leq & k\|(\lambda VSV^{-1} - B)^{-1}\|. &
\end{array}$$

In the same way,

$$\|(\lambda VSV^{-1} - B)^{-1}\| \leq k\|(\lambda S - T)^{-1}\|.$$

For $\lambda \in \sigma_{\varepsilon/k,VSV^{-1}}(B)$,

$$\lambda \in \sigma_{VSV^{-1}}(B) \text{ or } \|(\lambda VSV^{-1} - B)^{-1}\| > \frac{k}{\varepsilon}.$$

Then,

$$\lambda \in \sigma_S(T) \text{ or } \|(\lambda S - T)^{-1}\| \geq \frac{1}{k}\,\|(\lambda VSV^{-1} - B)^{-1}\| > \frac{1}{\varepsilon}.$$

Hence

$$\lambda \in \sigma_{\varepsilon,S}(T).$$

Therefore

$$\sigma_{\varepsilon/k,VSV^{-1}}(B) \subseteq \sigma_{\varepsilon,S}(T).$$

On the other hand, for $\lambda \in \sigma_{\varepsilon,S}(T)$,

$$\lambda \in \sigma_S(T) \text{ or } \|(\lambda S - T)^{-1}\| > \frac{1}{\varepsilon}.$$

Then,

$$\lambda \in \sigma_{VSV^{-1}}(B) \text{ or } \|(\lambda VSV^{-1} - B)^{-1}\| \geq \frac{1}{k}\,\|(\lambda S - T)^{-1}\| > \frac{1}{k\varepsilon}.$$

Hence

$$\lambda \in \sigma_{k\varepsilon,VSV^{-1}}(T).$$

Therefore

$$\sigma_{\varepsilon,S}(T) \subseteq \sigma_{k\varepsilon,VSV^{-1}}(B).$$

With similar reasoning, we prove that

$$\sigma_{\varepsilon/k,S}(T) \subseteq \sigma_{\varepsilon,VSV^{-1}}(B) \subseteq \sigma_{k\varepsilon,S}(T).$$ Q.E.D.

Corollary 3.3. *Let $T \in \mathcal{CR}(X)$ where X is a Banach space, let $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$ and assume that V is a bounded single valued relation in $\mathcal{LR}(X)$ such that $0 \in \rho(V)$. Let $k = \|V\|\|V^{-1}\|$. Let $B = VTV^{-1}$.*

If S commute with V (i.e., $SV = VS$) or S commute with V^{-1} (i.e., $SV^{-1} = V^{-1}S$) (for example $S = I$), then

$$\sigma_S(T) = \sigma_S(B) \tag{3.4}$$

and for $k \neq 0$, we have

$$\sigma_{\varepsilon/k,S}(B) \subseteq \sigma_{\varepsilon,S}(T) \subseteq \sigma_{k\varepsilon,S}(B) \tag{3.5}$$

and

$$\sigma_{\varepsilon/k,S}(T) \subseteq \sigma_{\varepsilon,S}(B) \subseteq \sigma_{k\varepsilon,S}(T). \tag{3.6}$$

◊

4. Stability of S-essential spectra and S-essential pseudospectra of linear relations

In this section, we study results of stability and properties of S-essential spectra and S-essential pseudospectra.

Definition 4.1. *In* [13, Definition 4.1], *the authors introduced the following definition: Let T be a linear relation in $\mathcal{CR}(X)$ where X is a Banach space. The S-essential pseudospectra of T is the set*

$$\sigma_{w,\varepsilon,S}(T) = \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T+K)$$

where $\mathcal{K}_T(X) := \{K \in \mathcal{KR}(X)$ such that $\mathcal{D}(K) \supset \mathcal{D}(T)$ and $K(0) \subset T(0)\}$,
and The S-essential pseudoresolvent set

$$\rho_{w,\varepsilon,S}(T) = \mathbb{C}\backslash\sigma_{w,\varepsilon,S}(T). \qquad \diamond$$

Theorem 4.1. [13, Theorem 4.2] *The following properties are equivalent:*

(i) $\lambda \notin \sigma_{w,\varepsilon,S}(T)$.

(ii) *For all continuous linear relations $B \in \mathcal{LR}(X)$ such that $\mathcal{D}(B) \supset \mathcal{D}(T)$, $B(0) \subset T(0)$ and $\|B\| < \varepsilon$, we have*

$$T + B - \lambda S \in \Phi(X) \quad \text{and} \quad i(T + B - \lambda S) = 0.$$

(iii) *For all continuous single valued relations $D \in \mathcal{LR}(X)$ such that $\mathcal{D}(D) \supset \mathcal{D}(T)$ and $\|D\| < \varepsilon$, we have*

$$T + D - \lambda S \in \Phi(X) \quad \text{and} \quad i(T + D - \lambda S) = 0. \qquad \diamond$$

Proposition 4.1. [13, Proposition 4.4] *Let $T \in \mathcal{CR}(X)$. Then*

(i) *If $0 < \varepsilon_1 < \varepsilon_2$ then $\sigma_{w,S}(T) \subset \sigma_{w,\varepsilon_1,S}(T) \subset \sigma_{w,\varepsilon_2,S}(T)$.*

(ii) *For $\varepsilon > 0$, $\sigma_{w,\varepsilon,S}(T) \subset \sigma_{\varepsilon,S}(T)$.*

(iii) $\bigcap_{\varepsilon>0} \sigma_{w,\varepsilon,S}(T) = \sigma_{w,S}(T)$. $\diamond$

Theorem 4.2. [13, Theorem 4.9]

$$\sigma_{w,\varepsilon,S}(T) = \bigcap_{P\in\mathcal{P}_T(X)} \sigma_{\varepsilon,S}(T+P). \qquad \diamond$$

Proposition 4.2. *Let $\mathcal{J}(X)$ be a subset of $LR(X)$.*
If $\mathcal{K}_T(X) \subset \mathcal{J}(X) \subset \mathcal{P}_T(X)$, then

$$\sigma_{w,\varepsilon,S}(T) = \bigcap_{J\in\mathcal{J}(X)} \sigma_{\varepsilon,S}(T+J).$$

Proof.

$$\sigma_{w,\varepsilon,S}(T) = \bigcap_{P\in\mathcal{P}_T(X)} \sigma_{\varepsilon,S}(T+P) \subset \bigcap_{J\in\mathcal{J}(X)} \sigma_{\varepsilon,S}(T+J) \subset \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T+K) = \sigma_{w,\varepsilon,S}(T).$$

Q.E.D.

Corollary 4.1. *It follows, from the definition of $\mathcal{S}$-Essential Pseudospectra, Theorem 4.2 and Corollary 4.2 that*

(i) $\sigma_{w,\varepsilon,S}(T+K) = \sigma_{w,\varepsilon,S}(T)$ *for all* $K \in \mathcal{K}_T(X)$.

(ii) $\sigma_{w,\varepsilon,S}(T+P) = \sigma_{w,\varepsilon,S}(T)$ *for all* $P \in \mathcal{P}_T(X)$.

(iii) $\sigma_{w,\varepsilon,S}(T+J) = \sigma_{w,\varepsilon,S}(T)$ *for all* $J \in \mathcal{J}(X)$ *such that* $\mathcal{K}_T(X) \subset \mathcal{J}(X) \subset \mathcal{P}_T(X)$. $\diamond$

Proposition 4.3. *Let $\alpha, \beta \in \mathbb{C}$ such that $\beta \neq 0$, then*

$$\sigma_{w,\varepsilon,S}(\alpha S + \beta T) = \alpha + \sigma_{w,\frac{\varepsilon}{|\beta|},S}(T)\,\beta. \qquad \diamond$$

Proof. Let $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$. It is simply to verify that $\{\beta K : K \in \mathcal{K}_T(X)\} = K_T(X)$. In fact, if $K \in K_T(X)$, $\beta K(0) = K(0) \subset T(0)$, $\mathcal{D}(\beta K) = \mathcal{D}(K) \supset \mathcal{D}(T)$, moreover since K is compact, $\overline{Q_{\beta K}\beta K B_X} = \overline{\beta Q_K K B_X} = \beta\overline{Q_K K B_X}$ is compact, then βK is compact hence $\beta K \in \mathcal{K}_T(X)$. Therefore $\{\beta K : K \in \mathcal{K}_T(X)\} \subset \mathcal{K}_T(X)$.
Conversely, by the same way, if $K \in \mathcal{K}_T(X)$, $\beta^{-1}K \in \mathcal{K}_T(X)$. Then $\mathcal{K}_T(X) \subset \{\beta K : K \in \mathcal{K}_T(X)\}$. Therefore

$$\begin{aligned}\sigma_{w,\varepsilon,S}(\alpha S + \beta T) &= \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon,S}(\alpha S + \beta T + K)\\ &= \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon,S}(\alpha S + \beta T + \beta K)\\ &= \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon,S}(\alpha S + \beta(T + K)).\end{aligned}$$

Using Proposition 3.4,

$$\begin{aligned}\sigma_{w,\varepsilon,S}(\alpha S + \beta T) &= \bigcap_{K \in \mathcal{K}_T(X)} (\alpha + \sigma_{\frac{\varepsilon}{|\beta|},S}(T + K)\,\beta)\\ &= \alpha + (\bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\frac{\varepsilon}{|\beta|},S}(T + K))\,\beta.\\ &= \alpha + \sigma_{w,\frac{\varepsilon}{|\beta|},S}(T)\,\beta.\end{aligned}$$

Q.E.D.

Proposition 4.4. *For $\delta > 0$, we have*

$$\sigma_{w,\varepsilon,S}(T) \subseteq D_\delta + \sigma_{w,\varepsilon,S}(T) \subseteq \sigma_{w,\varepsilon+\delta,S}(T). \qquad \diamond$$

Proof. Let $K \in \mathcal{K}_T(X)$, then K is compact hence it is continuous (by [16, Corollary V.2.3]). Using [1, Lemma 3.5], $T + K$ is closed. By Proposition 3.5,

$$D_\delta + \sigma_{\varepsilon,S}(T + K) \subseteq \sigma_{\varepsilon+\delta,S}(T + K).$$

Then

$$D_\delta + \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T + K) \subseteq \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon+\delta,S}(T + K).$$

Hence

$$D_\delta + \sigma_{w,\varepsilon,S}(T) \subseteq \sigma_{w,\varepsilon+\delta,S}(T). \qquad \text{Q.E.D.}$$

Proposition 4.5. *Let $B \in \mathcal{LR}(X)$ such that $B(0) \subset T(0)$, $\mathcal{D}(B) \supset \mathcal{D}(T)$ and $\|B\| < \varepsilon$, then we have*

$$\sigma_{w,\varepsilon-\|B\|,S}(T) \subseteq \sigma_{w,\varepsilon,S}(T + B) \subseteq \sigma_{w,\varepsilon+\|B\|,S}(T). \qquad \diamond$$

Proof. Let $K \in \mathcal{K}_T(X)$, then K is compact hence it is continuous (by [16, Corollary V.2.3]). Using [1, Lemma 3.5], $T + K$ is closed.
Moreover, since $B \in \mathcal{LR}(X)$ such that $B(0) \subset T(0) = (T + K)(0)$, $\mathcal{D}(B) \supset \mathcal{D}(T) = \mathcal{D}(T + K)$ and $\|B\| < \varepsilon$, then, from Proposition 3.6,

$$\sigma_{\varepsilon-\|B\|,S}(T + K) \subseteq \sigma_{\varepsilon,S}(T + B + K) \subseteq \sigma_{\varepsilon+\|B\|,S}(T + K).$$

Hence

$$\bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon-\|B\|,S}(T + K) \subseteq \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T + B + K) \subseteq \bigcap_{K \in \mathcal{K}_T(X)} \sigma_{\varepsilon+\|B\|,S}(T + K).$$

But, since $T(0) = (T + B)(0)$ and $\mathcal{D}(T) = \mathcal{D}(T + B)$, then $\mathcal{K}_T(X) = \mathcal{K}_{(T+B)}(X)$. Thus

$$\bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon-\|B\|,S}(T+K) \subseteq \bigcap_{K\in\mathcal{K}_{T+B}(X)} \sigma_{\varepsilon,S}(T+B+K) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon+\|B\|,S}(T+K).$$

Therefore

$$\sigma_{w,\varepsilon-\|B\|,S}(T) \subseteq \sigma_{w,\varepsilon,S}(T+B) \subseteq \sigma_{w,\varepsilon+\|B\|,S}(T). \qquad \text{Q.E.D.}$$

Theorem 4.3. *Let $T \in \mathcal{CR}(X)$, $S \in \mathcal{LR}(X)$, where X is a Banach space, such that S is continuous, $\|S\| \neq 0$, $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \overline{\mathcal{D}(T)}$,*

(i) if $\lambda \notin \sigma_{w,S}(T)$, then

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma_{w,S}(T))\|S\|}.$$

(ii) if $\lambda \notin \sigma_{w,\varepsilon,S}(T)$ and $(\lambda S - T)$ is injective and open (for example $\lambda \notin \sigma_S(T)$), then

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma_{w,\varepsilon,S}(T))\|S\| + \varepsilon}. \qquad \diamond$$

Proof. (*i*) Let $\lambda \notin \sigma_{w,S}(T)$, then, by Proposition 2.1, $\lambda S - T \in \Phi(X)$ and $i(\lambda S - T) = 0$. Since $\operatorname{dist}(\lambda, \sigma_{w,S}(T)) = \inf\{|z - \lambda|$ such that $z \in \sigma_{w,S}(T)\}$, then, for all $\eta > 0$, there exists $z_\eta \in \sigma_{w,S}(T)$ such that

$$|\lambda - z_\eta| < \operatorname{dist}(\lambda, \sigma_{w,S}(T)) + \eta.$$

If $|\lambda - z_\eta| < \frac{\gamma(\lambda S - T)}{\|S\|}$, since $(z_\eta - \lambda)S(0) = S(0) \subset (\lambda S - T)(0)$, $\mathcal{D}((z_\eta - \lambda)S) = \mathcal{D}(S) \supset \overline{\mathcal{D}((\lambda S - T))} = \overline{\mathcal{D}(T)}$ and $\|(z_\eta - \lambda)S\| = |\lambda - z_\eta|\|S\| < \gamma(\lambda S - T)$ then, by [7, Proposition 10], $\lambda S - T + (z_\eta - \lambda)S = z_\eta S - T \in \Phi(X)$ and $i(z_\eta S - T) = i(\lambda S - T + z_\eta S - \lambda S) = i(\lambda S - T) = 0$. Hence $z_\eta \in \rho_{w,S}(T)$, and this is a contradiction.

Therefore $|\lambda - z_\eta| \geq \frac{\gamma(\lambda - T)}{\|S\|}$ for all $\eta > 0$. Thus

$$\begin{aligned} \frac{\gamma(\lambda S - T)}{\|S\|} &\leq |\lambda - z_\eta| \\ &< \operatorname{dist}(\lambda, \sigma_{w,S}(T)) + \eta \quad \text{for all } \eta > 0. \end{aligned}$$

So

$$\frac{\gamma(\lambda S - T)}{\|S\|} \leq \operatorname{dist}(\lambda, \sigma_{w,S}(T)).$$

Hence

$$\|(\lambda S - T)^{-1}\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma_{w,S}(T))\|S\|}.$$

(*ii*) Since $\operatorname{dist}(\lambda, \sigma_{w,\varepsilon,S}(T)) = \inf\{|z - \lambda|$ such that $z \in \sigma_{w,\varepsilon,S}(T)\}$, then, for $\eta > 0$, there exists $z_\eta \in \sigma_{w,\varepsilon,S}(T)$ such that $|\lambda - z_\eta| < \operatorname{dist}(\lambda, \sigma_{w,\varepsilon,S}(T)) + \eta$.

Let B be a linear relation such that $B(0) \subset T(0)$, $\mathcal{D}(B) \supset \mathcal{D}(T)$ and $\|B\| < \varepsilon$. Let $\lambda \notin \sigma_{w,\varepsilon,S}(T)$, then by Theorem 4.1, $\lambda S - T - B \in \Phi(X)$ and $i(\lambda S - T - B) = 0$.

Now, if $|\lambda - z_\eta| < \frac{\gamma(\lambda S - T - B)}{\|S\|}$, since $(z_\eta - \lambda)S(0) \subset (\lambda S - T - B)(0)$, $\mathcal{D}((z_\eta - \lambda)S) \supset \overline{\mathcal{D}((\lambda S - T - B))} = \overline{\mathcal{D}(T)}$ and $\|(z_\eta - \lambda)S\| = |\lambda - z_\eta|\|S\| < \gamma(\lambda S - T - B)$, then, by [7, Proposition 10], $\lambda S - T - B + z_\eta S - \lambda S = z_\eta S - T - B \in \Phi(X)$ and $i(z_\eta S - T - B) = i(\lambda S - T - B + z_\eta S - \lambda S) = i(\lambda S - T - B) = 0$. Hence $z_\eta \in \rho_{w,\varepsilon,S}(T)$. This is a contradiction. Therefore $\frac{\gamma(\lambda S - T - B)}{\|S\|} \leq |\lambda - z_\eta|$. On the other hand, since $(\lambda S - T)$ is injective and open, then

$$\begin{aligned} \gamma(\lambda S - T - B) &\geq \gamma(\lambda S - T) - \|B\| \\ &> \gamma(\lambda S - T) - \varepsilon \end{aligned}$$

Thus

$$\begin{aligned}\frac{\gamma(\lambda S-T)}{\|S\|} &< \frac{\gamma(\lambda S-T-B)}{\|S\|}+\frac{\varepsilon}{\|S\|}\\ &< |\lambda-z_\eta|+\frac{\varepsilon}{\|S\|}\\ &< \operatorname{dist}(\lambda,\sigma_{w,\varepsilon,S}(T))+\eta+\frac{\varepsilon}{\|S\|}.\end{aligned}$$

Hence

$$\gamma(\lambda S-T)\leq \operatorname{dist}(\lambda,\sigma_{w,\varepsilon,S}(T))\|S\|+\varepsilon.$$

Then

$$\|(\lambda S-T)^{-1}\|\geq \frac{1}{\operatorname{dist}(\lambda,\sigma_{w,\varepsilon,S}(T))\|S\|+\varepsilon}.$$ Q.E.D.

Theorem 4.4. *Let $T\in\mathcal{CR}(X)$ where X is a Banach space and assume that V is a bounded single valued relation in $\mathcal{LR}(X)$ such that $0\in\rho(V)$. Let $k=\|V\|\|V^{-1}\|$, $k\neq 0$. Let $B=VTV^{-1}$. Then*

$$\sigma_{w,S}(T)=\sigma_{w,VSV^{-1}}(B),$$
$$\sigma_{w,\varepsilon/k,VSV^{-1}}(B)\subseteq\sigma_{w,\varepsilon,S}(T)\subseteq\sigma_{w,k\varepsilon,VSV^{-1}}(B)$$

and

$$\sigma_{w,\varepsilon/k,S}(T)\subseteq\sigma_{w,\varepsilon,VSV^{-1}}(B)\subseteq\sigma_{w,k\varepsilon,S}(T).$$ ◊

Proof. Let $K\in\mathcal{K}_T(X)$, then K is compact hence it is continuous (by [16, Corollary V.2.3]). Using [1, Lemma 3.5], $T+K$ is closed. Moreover

$$\begin{aligned}V(T+K)V^{-1} &= V(TV^{-1}+KV^{-1}) && (\text{ using [16, Proposition I.4.2 (d)]})\\ &= VTV^{-1}+VKV^{-1} && (\text{ using [16, Proposition I.4.2 (e)]})\\ &= B+VKV^{-1}.\end{aligned}$$

From Theorem 3.2, for $\varepsilon>0$, we have

$$\sigma_{\varepsilon/k,S}(T+K)\subseteq\sigma_{\varepsilon,VSV^{-1}}(B+VKV^{-1})\subseteq\sigma_{k\varepsilon,S}(T+K).$$

Hence

$$\bigcap_{K\in\mathcal{K}_T(X)}\sigma_{\varepsilon/k,S}(T+K)\subseteq\bigcap_{K\in\mathcal{K}_T(X)}\sigma_{\varepsilon,VSV^{-1}}(B+VKV^{-1})\subseteq\bigcap_{K\in\mathcal{K}_T(X)}\sigma_{k\varepsilon,S}(T+K).$$

But $\{VKV^{-1}:K\in\mathcal{K}_T(X)\}=\mathcal{K}_B(X)$. In fact if $K\in\mathcal{K}_T(X)$, then $K(0)\subset T(0)$, hence $VKV^{-1}(0)=VK(0)\subset VT(0)=VTV^{-1}(0)=B(0)$. Moreover, since V is bounded, $\mathcal{D}(VK)=\{x\in X: Kx\cap\mathcal{D}(V)\neq\emptyset\}=\mathcal{D}(K)$ and $\mathcal{D}(VT)=\{x\in X: Tx\cap\mathcal{D}(V)\neq\emptyset\}=\mathcal{D}(T)$. Then $\mathcal{D}(VKV^{-1})=\{x\in X: V^{-1}x\cap\mathcal{D}(VK)\neq\emptyset\}\supset\{x\in X: V^{-1}x\cap\mathcal{D}(VT)\neq\emptyset\}=\mathcal{D}(VTV^{-1})=\mathcal{D}(B)$.

On the other hand, K is compact V is continuous and $V(0)=\{0\}\subset\mathcal{D}(K)$. Then by [16, Proposition V.2.10] VK is precompact. Thus, by [16, Theorem V.2.2], $\overline{\Gamma}_0(VK)=0$. Furthermore, since V^{-1} is single valued ($0\in\rho(V)$), by [16, Proposition IV.2.15], we have $\overline{\Gamma}_0(VKV^{-1})\leq\overline{\Gamma}_0(VK)\overline{\Gamma}_0(V^{-1})=0$. Hence, by [16, Theorem V.2.2], VKV^{-1} is precompact and we have X is complete then VKV^{-1} is compact. Therefore

$$\{VKV^{-1}:K\in\mathcal{K}_T(X)\}\subset\mathcal{K}_B(X).$$

In the similar way, if $K_B\in\mathcal{K}_B(X)$, then $K_B(0)\subset S(0)$, hence $V^{-1}K_BV(0)=V^{-1}K_B(0)\subset V^{-1}B(0)=V^{-1}BV(0)=T(0)$. Moreover, since V^{-1} is bounded (as $0\in\rho(V)$), $\mathcal{D}(V^{-1}K_B)=\{x\in X: K_Bx\cap\mathcal{D}(V^{-1})\neq\emptyset\}=\mathcal{D}(K_B)$ and

$\mathcal{D}(VS) = \{x \in X : Sx \cap \cap\mathcal{D}(V^{-1})eq\emptyset\} = \mathcal{D}(B)$. Then $\mathcal{D}(V^{-1}K_BV) = \{x \in X : Vx \cap \mathcal{D}(V^{-1}K_B)eq\emptyset\} \supset \{x \in X : Vx \cap \mathcal{D}(V^{-1}B)eq\emptyset\} = \mathcal{D}(V^{-1}BV) = \mathcal{D}(T)$.

Moreover, K_B is compact, V^{-1} is continuous and $V^{-1}(0) = \{0\} \subset \mathcal{D}(K_B)$ as $0 \in \rho(V)$. Then, by [16, Proposition V.2.10], $V^{-1}K_B$ is precompact. Thus by [16, Theorem V.2.2] $\overline{\Gamma}_0(V^{-1}K_B) = 0$. Furthermore, since V is single valued, by [16, Proposition IV.2.15], $\overline{\Gamma}_0(V^{-1}K_BV) \leq \overline{\Gamma}_0(V^{-1}K_B)\overline{\Gamma}_0(V) = 0$. Hence by [16, Theorem V.2.2], $V^{-1}K_BV$ is precompact and we have X is complete then $V^{-1}K_BV$ is compact. Therefore $V^{-1}K_BV \in \mathcal{K}_T(X)$. Hence

$$\mathcal{K}_B(X) \subset \{VKV^{-1} : K \in \mathcal{K}_T(X)\}.$$

Therefore

$$\bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon/k,S}(T+K) \subseteq \bigcap_{K\in\mathcal{K}_B(X)} \sigma_{\varepsilon,VSV^{-1}}(B+K) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{k\varepsilon,S}(T+K).$$

Then, for any $\varepsilon > 0$

$$\sigma_{w,\varepsilon/k,S}(T) \subseteq \sigma_{w,\varepsilon,VSV^{-1}}(B) \subseteq \sigma_{w,k\varepsilon,S}(T).$$

In similar way, we prove that

$$\bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon/k,VSV^{-1}}(B+VKV^{-1}) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T+K) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{k\varepsilon,VSV^{-1}}(B+VKV^{-1}),$$

which implies that

$$\bigcap_{K\in\mathcal{K}_B(X)} \sigma_{\varepsilon/k,S}(B+K) \subseteq \bigcap_{K\in\mathcal{K}_T(X)} \sigma_{\varepsilon,S}(T+K) \subseteq \bigcap_{K\in\mathcal{K}_B(X)} \sigma_{k\varepsilon,VSV^{-1}}(B+K).$$

Then

$$\sigma_{w,\varepsilon/k,VSV^{-1}}(B) \subseteq \sigma_{w,\varepsilon,S}(T) \subseteq \sigma_{w,k\varepsilon,VSV^{-1}}(B).$$

Moreover, by Proposition 4.1,

$$\bigcap_{\varepsilon>0} \sigma_{w,\varepsilon/k,S}(T) \subseteq \bigcap_{\varepsilon>0} \sigma_{w,\varepsilon,VSV^{-1}}(B) \subseteq \bigcap_{\varepsilon>0} \sigma_{w,k\varepsilon,S}(T).$$

and from the proof of Theorem 3.2, we show that B is closed, then, we have,

$$\sigma_{w,S}(T) \subseteq \sigma_{w,VSV^{-1}}(B) \subseteq \sigma_{w,S}(T).$$

Which implies the first result. Q.E.D.

Corollary 4.2. *Let $T \in \mathcal{CR}(X)$ where X is a Banach space, let $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$ and $\mathcal{D}(S) \supset \mathcal{D}(T)$ and assume that V is a bounded single valued relation in $\mathcal{LR}(X)$ such that $0 \in \rho(V)$. Let $k = \|V\|\|V^{-1}\|$, $k \neq 0$. Let $B = VTV^{-1}$.*
If S commute with V (i.e., $SV = VS$) or S commute with V^{-1} (i.e., $SV^{-1} = V^{-1}S$) (for example $S = I$), then

$$\sigma_{w,S}(T) = \sigma_{w,S}(B),$$

$$\sigma_{w,\varepsilon/k,S}(B) \subseteq \sigma_{w,\varepsilon,S}(T) \subseteq \sigma_{w,k\varepsilon,S}(B)$$

and

$$\sigma_{w,\varepsilon/k,S}(T) \subseteq \sigma_{w,\varepsilon,S}(B) \subseteq \sigma_{w,k\varepsilon,S}(T).$$ ◊

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