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New stability for some matrix operators involving γ-relatively bounded inputs

Aymen Ammar[a], Nawrez Lazrag[a]

[a] *Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Soukra Road Km 3.5, B. P. 1171, 3000, Sfax, Tunisia*

Abstract. In this research article, we use the concept of the γ-relative boundedness, which represents as a generalization of the notion of relative boundedness, in order to give a necessary and sufficient conditions on the inputs of a block operator matrix to become self-adjoint.

1. Introduction

Let X and Y be two Banach spaces over the same fields $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let T be an operator acting from X into Y. We denote by $\mathcal{D}(T) \subset X$ its domain, by $N(T) \subset X$ its null space and by $R(T) \subset Y$ its range. Furthermore, recall that an operator T is said to be closed if, from simultaneous convergence of sequences $x_n \longrightarrow x$, with $x_n \in \mathcal{D}(T)$, and $Tx_n \longrightarrow y$, it follows that $x \in \mathcal{D}(T)$ and $Tx = y$. T is said to be bounded if, $\mathcal{D}(T) = X$ and

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : 0 \neq x \in X\right\} < \infty.$$

We denote by $\mathcal{L}(X, Y)$ (respectively, $C(X, Y)$) the set of all bounded (respectively, densely defined closed) linear operators from X into Y, and we denote by $\mathcal{K}(X, Y)$ the subspace of compact operators from X into Y. The nullity of T, $\alpha(T)$, is defined as the dimension of $N(T)$ and the deficiency of T, $\beta(T)$, is defined as the codimension of $R(T)$ in Y. If S and T are two linear operators such that $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Tx = Sx$ for all $x \in \mathcal{D}(T)$, then S an extension of T. A linear operator T is called to be closable if, it admits a closed extension. In this case, the smallest of such extension is called the closure of T and is denoted by $\overline{T}$. For a closed linear operator T, the set of semi-Fredholm operators and Fredholm operators are defined respectively as follows

$$\Phi_+(X, Y) := \{T \in C(X, Y) : \ \alpha(T) < \infty \text{ and } \ R(T) \text{ is closed in } Y\}$$

and

$$\Phi(X, Y) := \{T \in C(X, Y) : \ \alpha(T) < \infty, \ \beta(T) < \infty \text{ and } \ R(T) \text{ is closed in } Y\}.$$

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Email addresses: ammar_aymen84@yahoo.fr, aymen.ammar@fss.usf.tn (Aymen Ammar), lazragnawrez@gmail.com (Nawrez Lazrag)

The number $i(T) = \alpha(T) - \beta(T)$ is called the index of $T \in \Phi(X, Y)$.

Let H and K be two Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. The notion of adjoint operator of a densely defined linear operator T acting between H and K is determined as an operator T^* from K into H having domain

$$\mathcal{D}(T^*) = \Big\{ k \in K : \langle Th, k \rangle = \langle h, k^* \rangle, \text{ for some } k^* \in H, \text{ for all } h \in \mathcal{D}(T) \Big\},$$

and acting by

$$T^* k = k^*, \quad \text{for all } k \in \mathcal{D}(T^*).$$

Here the uniqueness of k^* in H is guaranteed by density of $\mathcal{D}(T)$. Nevertheless, it is a non-trivial task to determine T^*, that is, to describe $\mathcal{D}(T^*)$ explicitly and to specify the action of T^* on elements of $\mathcal{D}(T^*)$. Clearly, T and its adjoint T^* fulfil the adjoining identity

$$\langle Th, k \rangle = \langle h, T^* k \rangle, \quad \text{for all } h \in \mathcal{D}(T),\ k \in \mathcal{D}(T^*). \tag{1}$$

Let $T : \mathcal{D}(T) \subset H \longrightarrow K$ and $S : \mathcal{D}(S) \subset K \longrightarrow H$. The linear operators T and S are said to be adjoint to each other, in symbols $S \wedge T$, if

$$\langle Sk, h \rangle = \langle k, Th \rangle, \quad \text{for all } h \in \mathcal{D}(T) \text{ and } k \in \mathcal{D}(S).$$

A closed operator $T : \mathcal{D}(T) \subset H \longrightarrow K$ is said to be Hermitian, if T is densely defined and $T \subset T^*$, that is, if $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $Th = T^* h$, for all $h \in \mathcal{D}(T)$. Moreover, T is said to be self-adjoint if in addition $T = T^*$.

Remark 1.1. (*i*) *$T : \mathcal{D}(T) \subseteq H \longrightarrow H$ is Hermitian if, and only if, $\langle Th, h \rangle \in \mathbb{R}$, for all $h \in \mathcal{D}(T)$. Indeed, suppose that T is an Hermitian operator, then $Th = T^* h$, for all $h \in \mathcal{D}(T)$. This implies that*

$$\overline{\langle Th, h \rangle} = \langle h, Th \rangle = \langle T^* h, h \rangle = \langle Th, h \rangle.$$

Conversely, suppose that $(h, Th) \in G(T)$. Since $\langle Th, h \rangle \in \mathbb{R}$, then

$$\langle Th, h \rangle = \overline{\langle Th, h \rangle} = \langle h, Th \rangle.$$

This implies from (1) *that $h \in \mathcal{D}(T^*)$ and $Th = T^* h$. Hence, $(h, Th) \in G(T^*)$. As a result, $G(T) \subset G(T^*)$, as desired.*

(*ii*) *If T is self-adjoint, then $\sigma(T)$ is a subset of the real axis* (*see [2],* [6, Section 12.11]). ◊

Consider an operator which is defined in Banach or Hilbert space $X \times Y$ by the block operator matrix

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here $A : X \longrightarrow X$, $B : Y \longrightarrow X$, $C : X \longrightarrow Y$ and $D : Y \longrightarrow Y$ are assumed to be densely defined closable linear operators. In general, the operators occurring in $\mathcal{L}$ are unbounded and $\mathcal{L}$ does not need to be closed or to be a self-adjoint operator, even if its entries are closed or self-adjoint. In the present paper, inspired by the notion of relative boundedness with respect to an axiomatic measure of noncompactness γ, we shall study the stability of closedness, as well as the stability of self-adjointness, for the operator matrix $\mathcal{L}$ in the γ-diagonally dominant case. This case is characterized by the following condition: the operators C and B are γ-relatively bounded with respect to the operators A and D, respectively. This implies that $R(A)$, $R(B)$, $R(C)$ and $R(D)$ are finite dimensional, $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$ and there exist constants $a_C, a_B, b_C, b_B \geq 0$ such that

$$\gamma(C(\mathfrak{D}_1)) \leq a_C\, \gamma(A(\mathfrak{D}_1)) + b_C\, \gamma(\mathfrak{D}_1),$$
$$\gamma(B(\mathfrak{D}_2)) \leq a_B\, \gamma(D(\mathfrak{D}_2)) + b_B\, \gamma(\mathfrak{D}_2),$$

where $\gamma(\cdot)$ denotes an axiomatic measure of noncompactness, $\mathfrak{D}_1$ is a bounded subset of $\mathcal{D}(A)$ and $\mathfrak{D}_2$ is a bounded subset of $\mathcal{D}(D)$.

The rest of this paper is organized as follows. In section 2, some notations, basic concept and fundamental results about measure of noncompactness are recalled. In section 3, stability of closedness of the operator matrix $\mathcal{L}$ are studied under γ-relatively bounded perturbations about the off-diagonal operators B and C, with some additional conditions. Moreover, in the special case that the space is a Hilbert space, invariance of self-adjointness of $\mathcal{L}$ is discussed.

2. Preliminary and auxiliary results

In this section, we shall recall some basic concepts, and give some fundamental results about measure of noncompactness. The theory of measures of noncompactness has many applications in topology, functional analysis and operator theory [3]. Let X be a Banach space, we denote by M_X the family of all nonempty and bounded subsets of X. Denote by N_X the subfamily consisting of all relatively compact sets. Moreover, we write $\overline{\mathfrak{D}}, conv(\mathfrak{D})$ to denote the closure and the convex hull of a set $\mathfrak{D} \subset X$, respectively.

Definition 2.1. *A function $\gamma : M_X \longrightarrow [0, +\infty[$ is said to be a measure of noncompactness in the space X if, it satisfies the following conditions:*
(i) *The family $N(\gamma) := \{P \in M_X : \gamma(P) = 0\}$ is nonempty and $N(\gamma) \subset N_X$.*
(ii) *If $\mathfrak{D}_1 \subset \mathfrak{D}_2$, then $\gamma(\mathfrak{D}_1) \leq \gamma(\mathfrak{D}_2)$, for $\mathfrak{D}_1, \mathfrak{D}_2 \in M_X$.*
(iii) *$\gamma(\mathfrak{D}) = \gamma(\overline{\mathfrak{D}})$.*
(iv) *$\gamma(\overline{conv\ (\mathfrak{D})}) = \gamma(\mathfrak{D})$.*
(v) *$\gamma(\lambda\mathfrak{D}_1 + (1-\lambda)\mathfrak{D}_2) \leq \lambda\gamma(\mathfrak{D}_1) + (1-\lambda)\gamma(\mathfrak{D}_2)$, for all $\lambda \in [0, 1]$.*
(vi) *If $(\mathfrak{D}_n)$ is a sequence of closed sets from M_X such that $\mathfrak{D}_{n+1} \subset \mathfrak{D}_n$, for $n = 1, 2, \cdots$ and $\lim\limits_{n\to+\infty} \gamma(\mathfrak{D}_n) = 0$, then $\mathfrak{D}_\infty = \bigcap\limits_{n=1}^{\infty} \mathfrak{D}_n \neq \emptyset$ and $\gamma(\mathfrak{D}_\infty) = 0$.* ♢

Definition 2.2. (i) *A measure of noncompactness γ is said to be sublinear if for all $\mathfrak{D}, \mathfrak{D}_1, \mathfrak{D}_2 \in M_X$, it satisfies the following two conditions:*
(i_1) *$\gamma(\lambda\mathfrak{D}) = |\lambda|\gamma(\mathfrak{D})$, for all $\lambda \in \mathbb{R}$.*
(i_2) *$\gamma(\mathfrak{D}_1 + \mathfrak{D}_2) \leq \gamma(\mathfrak{D}_1) + \gamma(\mathfrak{D}_2)$.*
(ii) *A measure of noncompactness γ is said to have maximum property if it satisfies*

$$\max\{\gamma(\mathfrak{D}_1); \gamma(\mathfrak{D}_2)\} = \gamma(\mathfrak{D}_1 \cup \mathfrak{D}_2).$$

(iii) *A measure of noncompactness is said to be regular if, it is sublinear, $N(\gamma) \subset N_X$ and has the maximum property.* ♢

Remark 2.3. (i) *For $\mathfrak{D} \in M_X$, the most important example of measure of noncompactness γ is Kuratowskii measure of noncompactness which defined by*

$$\gamma(\mathfrak{D}) = \inf\{\epsilon > 0 : \mathfrak{D} \textit{ can be covered by a finite number of sets of diameter } \leq \epsilon\}.$$

(ii) *The Kuratowskii's measure of noncompactness of $\mathfrak{D}$ is regular.* ♢

In [3], the authors construct the measures of noncompactness in cartesian product of a given finite collection of Banach spaces. More precisely, we have the following.

Proposition 2.4. [3, Theorem 3.3.2] *Let $X_1, \cdots, X_n$ be a finite collection of Banach space, let $\gamma_1, \cdots, \gamma_n$ be measures of noncompactness in $X_1, \cdots, X_n$, respectively. Assume the function $F : ([0, +\infty[)^n \longrightarrow [0, +\infty[$ is convex and $F(x_1, \cdots, x_n) = 0$ if, and only if, $x_i = 0$ for $i = 1, \cdots, n$. Then,*

$$\gamma(x) = F(\gamma_1(\pi_1(x)), \cdots, \gamma_n(\pi_n(x)))$$

defines a measure of noncompactness in $X_1 \times \cdots \times X_n$. Here, π_i denotes the natural projection of into X_i, for $i = 1, \cdots, n$. ♢

According to the previous proposition, if γ is a measure of noncompactness in a Banach space X, then for all $\mathfrak{D} \in M_{X^n}$, the quantity

$$\gamma(\mathfrak{D}) := \max\{\gamma(\pi_1(\mathfrak{D})), \cdots, \gamma(\pi_n(\mathfrak{D}))\}$$

defines a measure of noncompactness in X^n.

Throughout this paper, we are working on two Banach spaces X and Y with their respective Kuratowskii measures of noncompactness γ_X and γ_Y. When no confusion axses we use just γ instead of γ_X or γ_Y.

Definition 2.5. *For $T \in \mathcal{L}(X)$, we define its Kuratowskii's measure by*

$$\gamma(T) = \sup\left\{\frac{\gamma(T(\mathfrak{D}))}{\gamma(\mathfrak{D})} : \mathfrak{D} \in M_X, \gamma(\mathfrak{D}) > 0\right\}. \qquad \diamond$$

In the next proposition, we recall some properties of the Kuratowskii's measure of a bounded linear operator.

Proposition 2.6. [3] *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then, we have the following properties for the Kuratowskii measure $\gamma(T)$.*
(i) $\gamma(T) = 0$ if, and only if, $T \in \mathcal{K}(X)$.
(ii) If $S \in \mathcal{L}(X)$, then $\gamma(ST) \leq \gamma(S)\gamma(T)$.
(iii) $\gamma(T + S) \leq \gamma(T) + \gamma(S)$, for every $S \in \mathcal{L}(X)$.
(iv) If $\mathfrak{D}$ is a bounded subset of X, then $\gamma(T(\mathfrak{D})) \leq \gamma(T)\gamma(\mathfrak{D})$.
(v) $\gamma(T) \leq \|T\|$. ◊

Lemma 2.7. *Let T be a bounded linear operator on X such that $\gamma(T) < \frac{1}{2}$. Then, $I - T$ is a boundedly invertible operator.* ◊

Proof. Let P and Q be two complex polynomials such that $P(z) = z$ and $Q(z) = z - 1$. Therefore, the fact that $\gamma(T) < \frac{1}{2}$ implies from [1, Theorem 3.1] that

$$Q(T) = I - T \in \Phi(X). \tag{2}$$

Now, let $\varepsilon \in [0, 1]$. Then, we infer from Definition 2.2 (i_1) that

$$\begin{aligned} \gamma(\varepsilon T) &\leq \varepsilon\gamma(T) \\ &\leq \gamma(T) \\ &< \frac{1}{2}. \end{aligned}$$

This implies that $I + \varepsilon T \in \Phi(X)$. The use of [6, Theorem 7.25] and the connectedness of $[0, 1]$, allows us to conclude that

$$i(I - \varepsilon T) = i(I - T) = i(I) = 0. \tag{3}$$

Hence, by referring to (2), (3) and [7, Lemma 3], we deduce that $Q(T) = T_0 + F$, where $T_0 \in \Phi_+(X)$ with $\alpha(T_0) = 0$ and F is a finite rank operator on X. In view of [5, Theorem 11] implies that

$$\alpha(T_0 + F) \leq \alpha(T_0) = 0.$$

Thus, $\alpha(Q(T)) = 0$, which implies from (3) that $\beta(Q(T))) = 0$. As a result, $I - T$ is a boundedly invertible operator. □

Definition 2.8. *Let X and Y be two Banach spaces. Let S and T be two linear operators from X into Y such that $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $R(T)$, $R(S)$ are finite dimensional. The operator S is called γ-relatively bounded with respect to T (or T-γ-bounded) if, there exist constants a_S and b_S such that*

$$\gamma(S(\mathfrak{D})) \leq a_S \gamma(T(\mathfrak{D})) + b_S \gamma(\mathfrak{D}), \tag{4}$$

where $\mathfrak{D}$ is a bounded subset of $\mathcal{D}(T)$. The infimum of the constants a_S which satisfy (4) for some $b_S \geq 0$ is called the T-γ-bound of S. ◇

Remark 2.9. *Let $\mathfrak{D}$ be a bounded subset of $\mathcal{D}(T)$. The boundedness of the sets $T(\mathfrak{D})$ and $S(\mathfrak{D})$ is not valid for the case of unbounded linear operators. Accordingly, we added the condition that $R(T)$ and $R(S)$ are finite dimensional to guarantee the existence of $\gamma(T(\mathfrak{D}))$ and thus of $\gamma(S(\mathfrak{D}))$.* ◇

Lemma 2.10. *Let T, S be two closed operators acting from X into Y such that $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $R(T)$, $R(S)$ are finite dimensional. If S is T-γ-bounded with T-γ-bound < 1, then $T + S$ is closed.* ◇

Proof. Let us assume that $(x_n) \subset \mathcal{D}(T)$ such that $x_n \longrightarrow x$ and $(T + S)x_n \longrightarrow y$ as $n \to \infty$. Our purpose is to show that $x \in \mathcal{D}(T)$ and $(T + S)x = y$. Since S is T-γ-bounded with T-γ-bound < 1, then there exist constants $b_S \geq 0$ and $0 \leq a_S < 1$ such that

$$\gamma(S\{x_n\}) \leq a_S \gamma(T\{x_n\}) + b_S \gamma(\{x_n\}). \tag{5}$$

In addition, we have

$$\begin{aligned} \gamma(T\{x_n\}) &= \gamma((T + S - S)\{x_n\}) \\ &\leq \gamma((T + S)\{x_n\}) + \gamma(S\{x_n\}). \end{aligned}$$

This implies that

$$\gamma((T + S)\{x_n\}) \geq \gamma(T\{x_n\}) - \gamma(S\{x_n\}).$$

Thus, we get from (5) that

$$\gamma((T + S)\{x_n\}) \geq (1 - a_S)\gamma(T\{x_n\}) - b_S \gamma(\{x_n\}). \tag{6}$$

The fact that $x_n \longrightarrow x$ and $(T + S)x_n \to y$ as $n \to \infty$ implies that $\{x_n\}$ and $\{(S + T)x_n\}$ are relatively compact. This yields from Proposition 2.6 Proposition 2.6 (i) that $\gamma(\{x_n\}) = \gamma(\{(S + T)x_n\}) = 0$. Therefore, by using (6), we obtain $\gamma(T\{x_n\}) = 0$. Again from Proposition 2.6 (i), there 2.6 (i), there exists a subsequence (x_{n_k}) such that $Tx_{n_k} \longrightarrow z$. By the closedness of T, we deduce that $x \in \mathcal{D}(T)$ and $Tx = z$. Since $Sx_{n_k} \to y - z$, then we conclude from the closedness of S that $x \in \mathcal{D}(T + S)$ and $(T + S)x = y$. □

Remark 2.11. *Let T and S be two closable operators with $R(T)$ and $R(S)$ are finite dimensional. If S is T-γ-bounded, then $\overline{S}$ is the closed extension of the operator S such that $\mathcal{D}(\overline{T}) \cup \mathcal{D}(S) \subsetneq \mathcal{D}(\overline{S})$. Indeed, let $(x_n) \subset \mathcal{D}(T)$ such that $x_n \longrightarrow x \in \mathcal{D}(\overline{T})$ and $Tx_n \longrightarrow y$. This implies that $\{x_n\}$ and $\{Tx_n\}$ are relatively compact, so $\gamma(\{x_n\}) = \gamma(\{Tx_n\}) = 0$. Then, by using (4), we infer that $\gamma(S\{x_n\}) = 0$. Hence, there exists a subsequence (x_{n_k}) such that $Sx_{n_k} \longrightarrow \alpha \in Y$. Since S is closable, then $\alpha = \overline{S}x$. By setting $Sx = \alpha$, we conclude that $\mathcal{D}(\overline{T}) \cup \mathcal{D}(S) \subset \mathcal{D}(\overline{S})$.* ◇

The example below show that the converse is false.

Example 2.12. *Let $S = O_{|\mathcal{D}(T)}$, where O stands for the zero operator and T is a closed operator with $\overline{\mathcal{D}(T)} \subsetneq X$. On the one hand, the fact that $\mathfrak{D}$ is a bounded subset of $\mathcal{D}(T)$ and O is a bounded operator implies that $S(\mathfrak{D})$ and $O(\mathfrak{D})$ are bounded sets. Hence, $\gamma(S(\mathfrak{D})) = \gamma(O(\mathfrak{D}))$. Since the zero operator is compact, then by referring to Proposition 2.6 (i), we infer that*

$$\gamma(S(\mathfrak{D})) = 0, \quad \mathfrak{D} \subset \mathcal{D}(T).$$

Hence, we deduce that

$$\gamma(S(\mathfrak{D})) \leq \gamma(T(\mathfrak{D})), \quad \mathfrak{D} \subset \mathcal{D}(T). \tag{7}$$

By using (7)*, we infer that* S *is* T*-*γ*-relatively bounded. On the other hand, we have* $\mathcal{D}(\overline{S}) = X$*, but*

$$\mathcal{D}(\overline{T}) \cup \mathcal{D}(S) = \mathcal{D}(\overline{T}) \cup \mathcal{D}(T) = \mathcal{D}(\overline{T}) \neq X.$$

As a result, $\mathcal{D}(\overline{T}) \cup \mathcal{D}(S) \subsetneq X$*, as desired.* ◇

Lemma 2.13. *Let assume that* T *and* S *are closable such that* $\mathcal{D}(T) \subset \mathcal{D}(S)$ *and* $R(T)$*,* $R(S)$ *are finite dimensional. If* S *is* T*-*γ*-bounded with* T*-*γ*-bound* δ*, then*

$$\gamma(\overline{S}(\mathfrak{D})) \leq a_S\, \gamma(\overline{T}(\mathfrak{D})) + b_S\, \gamma(\mathfrak{D}),$$

where $a_S, b_S \geq 0$ *and* $\mathfrak{D} \subset \mathcal{D}(T)$*.* ◇

Proof. Let us assume that S is T-γ-bounded with T-γ-bound δ. This implies that there exist two constants $a_S, b_S \geq 0$ such that

$$\gamma(S(\mathfrak{D})) \leq a_S\, \gamma(T(\mathfrak{D})) + b_S\, \gamma(\mathfrak{D}), \tag{8}$$

where $\mathfrak{D}$ is a bounded subset of $\mathcal{D}(T) \subset \mathcal{D}(S)$. It follows immediately from Remark 2.11 that $\mathcal{D}(\overline{T}) \subset \mathcal{D}(\overline{S})$. Since $\mathcal{D}(T) \subset \mathcal{D}(S)$, then $\overline{S}(\mathcal{D}(T)) \subset \overline{S}(\mathcal{D}(S))$. By using the fact that T and S are closable, we have $T = \overline{T}|_{\mathcal{D}(T)}$ and $S = \overline{S}|_{\mathcal{D}(S)}$. This implies that $R(T) = R(\overline{T}|_{\mathcal{D}(T)})$ and $R(S) = R(\overline{S}|_{\mathcal{D}(S)})$. Hence, $R(\overline{T}|_{\mathcal{D}(T)})$ and $R(\overline{S}|_{\mathcal{D}(T)})$ are finite dimensional. Thus, we infer from (8) that

$$\gamma(\overline{S}(\mathfrak{D})) \leq a_S\, \gamma(\overline{T}(\mathfrak{D})) + b_S\, \gamma(\mathfrak{D}),$$

where $\mathfrak{D} \subset \mathcal{D}(T) \subset \mathcal{D}(\overline{T})$. □

The following result is useful in the main result of this paper.

Theorem 2.14. *Let* X *be a Hilbert space. Suppose that* T *is self-adjoint operator and* A *is Hermitian operator with* $\mathcal{D}(T) \subset \mathcal{D}(A)$*. If* A *is* γ*-*T*-bounded with* γ*-*T*-bound less than* $\frac{1}{4}$*, then* $T + A$ *is also self-adjoint.* ◇

Proof. Obviously, $\mathcal{D}(T + A) = \mathcal{D}(T)$ and $T + A$ is closed and Hermitian since $\mathcal{D}(T) \subset \mathcal{D}(A)$. Due to [4, Theorem 3.16], it is sufficient to prove that

$$R((T + A) - \lambda I) = R((T + A) - \overline{\lambda} I) = X, \quad \text{for all } \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

On the one hand, the fact that T is self-adjoint implies from Remark 1.1 (*ii*) that $\mathbb{C}\backslash\mathbb{R} \subset \rho(T)$. Then, for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$, we have $T \pm \lambda I$ is invertible and $(T \pm \lambda I)^{-1} \in \mathcal{L}(X)$.
On the other hand, the operator $(T + A) \pm \lambda I$ can be written as follows

$$(T + A) \pm \lambda I = \left[I + A(T \pm \lambda I)^{-1}\right](T \pm \lambda I), \quad \text{for every } \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{9}$$

By virtue of representation (9), it is sufficient to show that the operator $I + A(T \pm it)^{-1}$ is invertible provided that $|t| > t_0$ and t_0 is sufficiently large.
Since T is closed, then for all $y \in \mathcal{D}(T)$, we have

$$\begin{aligned} \|(T \pm it)y\|^2 &= \langle (T \pm it)y, (T \pm it)y \rangle \\ &= \|Ty\|^2 + \|ity\|^2 + 2Re\langle Ty, ity \rangle. \end{aligned}$$

The fact that T is self-adjoint implies from Remark 1.1 (i) that

$$\|(T \pm it)y\|^2 = \|Ty\|^2 + |t|^2\|y\|^2, \text{ for all } y \in \mathcal{D}(T). \tag{10}$$

Hence, (10) gives the inequality

$$|t|\|y\| \leq \|(T \pm it)y\|, \text{ for all } y \in \mathcal{D}(T).$$

As the operator $T \pm it$ has a bounded inverse, thus by setting $x = (T \pm it)y$, we deduce that $(T \pm it)^{-1}x = y$. Hence,

$$\|(T \pm it)^{-1}x\| \leq \frac{1}{|t|}\|x\|, \text{ for all } x \in X. \tag{11}$$

Since $\mathcal{D}(T) \subset \mathcal{D}(A)$ and A is γ-T-bounded, then there exist $a_A, b_A \geq 0$ such that

$$\begin{aligned}\gamma(A(T \pm it)^{-1}(\mathfrak{D}_1)) &\leq a_A\, \gamma(T(T \pm it)^{-1}(\mathfrak{D}_1)) + b_A \gamma((T \pm it)^{-1}(\mathfrak{D}_1)) \\ &\leq a_A \gamma(\mathfrak{D}_1) + (a_A|t| + b_A)\, \gamma((T \pm it)^{-1}(\mathfrak{D}_1)),\end{aligned}$$

where $(T \pm it)^{-1}(\mathfrak{D}_1)$ is a bounded subset of $\mathcal{D}(T)$. Since $(T \pm it)^{-1} \in \mathcal{L}(X)$, then by using ($iv$) and ($v$) of Proposition 2.6, we deduce that

$$\gamma(A(T \pm it)^{-1}(\mathfrak{D}_1)) \leq a_A \gamma(\mathfrak{D}_1) + (a_A|t| + b_A)\, \|(T \pm it)^{-1}\| \gamma(\mathfrak{D}_1),$$

where $\gamma(\mathfrak{D}_1) > 0$. It follows immediately from (11) that

$$\gamma(A(T \pm it)^{-1}(\mathfrak{D}_1)) \leq \left(2a_A + \frac{b_A}{|t|}\right) \gamma(\mathfrak{D}_1).$$

This implies that $A(T \pm it)^{-1}$ is a bounded operator. Hence, by referring to Proposition 2.6 (iv), we get

$$\gamma(A(T \pm it)^{-1}) \leq 2a_A + \frac{b_A}{|t|}. \tag{12}$$

On can take number a_A close to the γ-T-bound δ_A, so that the inequality $a_A < \frac{1}{4}$ is preserved. Then,

$$\gamma(A(T \pm it)^{-1}) < \frac{1}{2}.$$

Consequently, by referring to Lemma 2.7, we have $I + A(T \pm it)^{-1}$ is a boundedly invertible operator, and thus $R(T + A \pm it) = X$. Therefore, $T + A$ is self-adjoint in X by [4, Theorem 3.16]. □

section The main results The goal of this section is to investigate the stability of closedness, as well as the stability of self-adjointness of the operator matrix $\mathcal{L}$ in the γ-diagonally dominant case.

Let X and Y be two Banach spaces. In the product of Banach spaces $X \times Y$, we consider an unbounded block operator matrix

$$\mathcal{L} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{13}$$

where $A : \mathcal{D}(A) \subset X \longrightarrow X$, $B : \mathcal{D}(B) \subset Y \longrightarrow X$, $C : \mathcal{D}(C) \subset X \longrightarrow Y$ and $D : \mathcal{D}(D) \subset Y \longrightarrow Y$ are densely defined closed linear operators such that $\mathcal{D}(A) \subset \mathcal{D}(C)$, $\mathcal{D}(D) \subset \mathcal{D}(B)$, $R(A), R(B), R(C)$ and $R(D)$ are finite dimensional. The domain of $\mathcal{L}$ is defined by

$$\mathcal{D}(\mathcal{L}) = \mathcal{D}(A) \times \mathcal{D}(D) \subset X \times Y.$$

Definition 2.15. *Let $\gamma(\cdot)$ be a measure of noncompactness.*
(i) *The block operator matrix $\mathcal{L}$ is called γ-diagonally dominant, if C is A-γ-bounded and B is D-γ-bounded.*
(ii) *The block operator matrix $\mathcal{L}$ is called γ-diagonally dominant with bound δ, if C is A-γ-bounded with bound δ_C, B is D-γ-bounded with bound δ_B, and $\delta = \max\{\delta_C; \delta_B\}$.* ◊

Theorem 2.16. *Let C and B be γ-relatively bounded with respect to A and D with A-γ-bound and D-γ-bound δ_C and δ_B, respectively. If $\delta_C + \delta_B < 1$, then the linear operator $\mathcal{L}$ is closed.* ◊

Proof. Consider the operators

$$\mathcal{T} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix},$$

with $\mathcal{D}(\mathcal{T}) = \mathcal{D}(A) \times \mathcal{D}(D)$ and $\mathcal{D}(\mathcal{S}) = \mathcal{D}(C) \times Y$. First, we have to prove that $\mathcal{T}$ is closed. Let a sequence $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathcal{D}(\mathcal{T})$ converge to $\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y$, and let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} Ax_n + By_n \\ Dy_n \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} \text{ as } n \longrightarrow \infty.$$

The fact that D is closed implies that $y \in \mathcal{D}(D)$ and $Dy = v$. Since $(y_n) \subset \mathcal{D}(D) \subset \mathcal{D}(B)$, then there exist $a_B, b_B \geq 0$ such that

$$\gamma(B\{y_n\}) \leq a_B\, \gamma(D\{y_n\}) + b_B\gamma(\{y_n\}). \tag{14}$$

Now, by using the fact that $y_n \longrightarrow y$ and $Dy_n \to Dy$, as $n \to \infty$, we infer that $\{y_n\}$ and $\{Dy_n\}$ are relatively compact. Hence, $\gamma(\{y_n\}) = \gamma(\{y_n\}) = \gamma(\{Dy_n\}) = 0$. Then, by using (14), we deduce that $\gamma(B\{y_n\}) = 0$. Thus, there exists a subsequence (y_{n_k}) such that $By_{n_k} \to z$. Since B is closed, we conclude that $y \in \mathcal{D}(B)$ and

$$By_{n_k} \to By \;\; \longrightarrow By \text{ as } n \longrightarrow \infty.$$

Therefore, $Ax_{n_k} \longrightarrow u - By$ as $n \to \infty$. The fact that A is closed implies that $x \in \mathcal{D}(A)$ and $Ax = u - By$. This is equivalent to say that $\mathcal{T}$ is closed.

Finally, we have to show that the linear operator $\mathcal{S}$ is $\mathcal{T}$-γ-bounded, and let us find its $\mathcal{T}$-γ-bound. First of all, based on the hypotheses $R(A)$ and $R(B)$ are finite dimensional, we conclude that $\dim(R(A) + R(B)) < \infty$. Hence, the fact that

$$R(\mathcal{T}) \subset (R(A) + R(B)) \times R(D)$$

implies that $\dim(R(\mathcal{T})) < \infty$. In addition, since $\dim(R(C)) < \infty$, then $\dim(R(\mathcal{S}))$ is finite dimensional. Now, for $\mathfrak{D} \subset \mathcal{D}(\mathcal{T}) \subset \mathcal{D}(\mathcal{S})$, we get

$$\gamma\left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] = \gamma(C(\pi_1(\mathfrak{D}))).$$

Here, π_i, $i = 1, 2$ denote the natural projection on X and Y, respectively. According to the assumptions, there exist constants $a_C, a_B, b_C, b_B \geq 0$ such that

$$\begin{cases} \gamma(C(\pi_1(\mathfrak{D}))) \leq a_C\, \gamma(A(\pi_1(\mathfrak{D}))) + b_C\gamma(\pi_1(\mathfrak{D})), \\ \\ \gamma(B(\pi_2(\mathfrak{D}))) \leq a_B\, \gamma(D(\pi_2(\mathfrak{D}))) + b_B\gamma(\pi_2(\mathfrak{D})). \end{cases}$$

Since $\gamma(\pi_1(\mathfrak{D})) \leq \gamma(\pi_1)\gamma(\mathfrak{D})$ and $\gamma(\pi_2(\mathfrak{D})) \leq \gamma(\pi_2)\gamma(\mathfrak{D})$, then we have

$$\gamma(C(\pi_1(\mathfrak{D}))) \leq a_C\, \gamma(A(\pi_1(\mathfrak{D}))) + b_C\gamma(\pi_1)\gamma(\mathfrak{D}). \tag{15}$$

$$\gamma(B(\pi_2(\mathfrak{D}))) \leq a_B\, \gamma(D(\pi_2(\mathfrak{D}))) + b_B\gamma(\pi_2)\gamma(\mathfrak{D}). \tag{16}$$

Let the condition $\delta_B + \delta_C < 1$ be fulfilled. Assume that $\delta_B < 1$; later, we will say about the changes in the proof in the case $\delta_C < 1$. Obviously, we can chose numbers a_C and a_B in the estimates (15) and (16) to be close to the numbers δ_C and δ_B. So that the inequalities $a_B < 1$ and $a_C + a_B < 1$ are preserved. Now, we can write

$$\gamma\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] = \gamma\left[\begin{pmatrix} A & B-B \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right].$$

This implies that

$$\begin{aligned}\gamma\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] &\leq \gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + \gamma\left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] \\ &\leq \gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + \gamma(B(\pi_2(\mathfrak{D}))).\end{aligned}$$

Hence, by using (16), we infer that

$$\begin{aligned}&\gamma\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] \\ &\leq \gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + a_B\,\gamma(D(\pi_2(\mathfrak{D}))) + b_B\gamma(\pi_2)\gamma(\mathfrak{D}) \\ &\leq \gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + a_B \max\left\{\gamma(A(\pi_1(\mathfrak{D})));\gamma(D(\pi_2(\mathfrak{D})))\right\} + b_B\gamma(\pi_2)\gamma(\mathfrak{D}) \\ &\leq \gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + a_B\gamma\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + b_B\gamma(\pi_2)\gamma(\mathfrak{D}).\end{aligned}$$

By using the fact that $a_B < 1$, we conclude that

$$\gamma\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] \leq \frac{1}{1-a_B}\,\gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + \frac{b_B\gamma(\pi_2)}{1-a_B}\,\gamma(\mathfrak{D}). \tag{17}$$

Hence, by virtue of (15) and (17), we deduce that

$$\begin{aligned}\gamma(C(\pi_1(\mathfrak{D}))) &\leq a_C\,\gamma(A(\pi_1(\mathfrak{D}))) + b_C\gamma(\pi_1)\gamma(\mathfrak{D}) \\ &\leq a_C \max\left\{\gamma(A(\pi_1(\mathfrak{D})));\gamma(D(\pi_2(\mathfrak{D})))\right\} + b_C\gamma(\pi_1)\gamma(\mathfrak{D}) \\ &\leq a_C\,\gamma\left[\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + b_C\gamma(\pi_1)\gamma(\mathfrak{D}) \\ &\leq \frac{a_C}{1-a_B}\,\gamma\left[\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + N_a\gamma(\mathfrak{D}),\end{aligned}$$

where $N_a = \dfrac{a_C b_B \gamma(\pi_2)}{1-a_B} + b_C\gamma(\pi_1)$. As a result,

$$\gamma\left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] \leq \frac{a_C}{1-a_B}\,\gamma\left[\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + N_a\gamma(\mathfrak{D}).$$

Thus, the $\mathcal{T}$-γ-bound of the operator $\mathcal{S}$ is equal to $\dfrac{a_C}{1-a_B} < 1$. Since $\mathcal{T}$ and $\mathcal{S}$ are closed, then by using Lemma 2.10, we conclude that the operator $\mathcal{L} = \mathcal{T} + \mathcal{S}$ is also closed. The proof of theorem is carried out analogously if the inequality $\delta_B < 1$ holds instead of $\delta_C < 1$. In this case, the operator $\mathcal{L}$ has to be presented in the form

$$\mathcal{L} = \mathcal{T}_1 + \mathcal{S}_1, \text{ where } \mathcal{T}_1 = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \text{ and } \mathcal{S}_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

The proof of closedness of $\mathcal{T}_1$ and $\mathcal{T}_1$-γ-boundedness of the operator $\mathcal{S}_1$ with $\mathcal{T}_1$-γ-bound < 1 is carried out as before. □

Now, let us prove theorem on stability of self-adjointness. In order to prove it, we start our investigation with the following Lemma. Let X and Y be two Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$, then the product space $X \times Y$ is still a Hilbert space with the following induced inner product, still denoted by $\langle\cdot,\cdot\rangle$ without any confusion:

$$\Big\langle (x,y),(z,t) \Big\rangle = \langle x,z\rangle + \langle y,t\rangle, \quad \text{for all } (x,y),(z,t) \in X \times Y.$$

Lemma 2.17. *Let X and Y be Hilbert spaces. Assume that $A \in \mathcal{C}(X), B \in \mathcal{C}(Y,X), C \in \mathcal{C}(X,Y)$ and $D \in \mathcal{C}(Y)$. If A, D are self-adjoint and $C \wedge B$, then $\mathcal{L}$ is a Hermitian operator.* ◊

Proof. Due to Remark 1.1 (i), it is sufficient to show that

$$\left\langle \begin{pmatrix} Ax+By \\ Cx+Dy \end{pmatrix}; \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \subset \mathbb{R}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{L}).$$

By using the fact that A and D are self-adjoint and $C \wedge B$, we infer that

$$\begin{aligned}
\left\langle \begin{pmatrix} Ax+By \\ Cx+Dy \end{pmatrix}; \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle Ax+By, x\rangle + \langle Cx+Dy, y\rangle \\
&= \langle Ax,x\rangle + \langle By,x\rangle + \langle Cx,y\rangle + \langle Dy,y\rangle \\
&= \langle Ax,x\rangle + \overline{\langle Cx,y\rangle} + \langle Cx,y\rangle + \langle Dy,y\rangle \\
&\subset \mathbb{R}.
\end{aligned}$$

□

Theorem 2.18. *Assume that*

(i) *A and D be self-adjoint operators in the Hilbert spaces X and Y, respectively.*

(ii) *$C \wedge B$ for which C and B be γ-relatively bounded with respect to A and D with A-γ-bound and D-γ-bound δ_C and δ_B, respectively.*

(iii) $\delta_C + \delta_B < \frac{1}{4}$.

Then, the operator matrix $\mathcal{L}$ is self-adjoint. ◊

Proof. By virtue of Theorem 2.16, the operator $\mathcal{L}$ is closed. Hence, we will decompose it as the following form $\mathcal{L} = \mathcal{T} + \mathcal{S}$, where

$$\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } \mathcal{S} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

with $\mathcal{D}(\mathcal{T}) = \mathcal{D}(A) \times \mathcal{D}(D)$ and $\mathcal{D}(\mathcal{S}) = \mathcal{D}(C) \times \mathcal{D}(B)$. Obviously, $\mathcal{D}(\mathcal{T}) \subset \mathcal{D}(\mathcal{S})$. Hence, due to Theorem 2.14, it is sufficient to prove that $\mathcal{T}$ is self-adjoint, $\mathcal{S}$ is Hermitian and $\mathcal{S}$ is $\gamma - \mathcal{T}-$ bounded with γ-T-bound less than $\frac{1}{4}$.

First, we propose to prove that $\mathcal{T}$ is self-adjoint. Since A and D are self-adjoint, then it follows from [4, Theorem 3.16] that $R(A - \lambda I) = R(A - \overline{\lambda} I) = X$ and $R(D - \lambda I) = R(D - \overline{\lambda} I) = Y$, for some $\lambda \in \mathbb{C}$. This implies that

$$\begin{aligned}
R(\mathcal{T} - \lambda I) &= R(\mathcal{T} - \overline{\lambda} I) \\
&= R(A - \overline{\lambda} I) \times R(D - \overline{\lambda} I) \\
&= X \times Y, \quad \text{for some } \lambda \in \mathbb{C}.
\end{aligned}$$

Thus, again by [4, Theorem 3.16], we conclude that $\mathcal{T}$ is self-adjoint. Second, we propose to show that $\mathcal{S}$ is Hermitian. The fact that $C \wedge B$ implies that

$$\begin{aligned}\left\langle \begin{pmatrix} By \\ Cx \end{pmatrix}; \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle By, x\rangle + \langle Cx, y\rangle \\ &= \overline{\langle Cx, y\rangle} + \langle Cx, y\rangle \\ &\subset \mathbb{R}.\end{aligned}$$

Thus, by referring to Remark 1.1 (i), we conclude that $\mathcal{S}$ is Hermitian.

Finally, let us show that the operator $\mathcal{S}$ is $\gamma - \mathcal{T}$-bounded, and let us find its $\gamma - \mathcal{T}$-bound. For $\mathfrak{D} \subset \mathcal{D}(\mathcal{T}) \subset \mathcal{D}(\mathcal{S})$, we get

$$\gamma\left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] = \max\left\{\gamma(B(\pi_2(\mathfrak{D}))), \gamma(C(\pi_1(\mathfrak{D})))\right\}.$$

Here, π_i, $i = 1, 2$ denote the natural projection on X and Y, respectively. According to the assumptions, there exist constants $a_C, a_B, b_C, b_B \geq 0$ such that

$$\begin{cases} \gamma(C(\pi_1(\mathfrak{D}))) \leq a_C\, \gamma(A(\pi_1(\mathfrak{D}))) + b_C\gamma(\pi_1(\mathfrak{D})), \\ \\ \gamma(B(\pi_2(\mathfrak{D}))) \leq a_B\, \gamma(D(\pi_2(\mathfrak{D}))) + b_B\gamma(\pi_2(\mathfrak{D})). \end{cases}$$

Let the condition $\delta_B + \delta_C < \frac{1}{4}$ be fulfilled. Obviously, we can chose numbers a_C and a_B to be close to the numbers δ_C and δ_B. So that the inequalities $a_C + a_B < \frac{1}{4}$ are preserved. In addition, by using the above estimates we can conclude that

$$\begin{aligned}&\gamma\left[\mathcal{S}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] \\ &\leq \max\left\{a_B\, \gamma(D(\pi_2(\mathfrak{D}))) + b_B\gamma(\pi_2(\mathfrak{D})); a_C\, \gamma(A(\pi_1(\mathfrak{D}))) + b_C\gamma(\pi_1(\mathfrak{D}))\right\} \\ &\leq \max\left\{a_B; a_C\right\}\max\left\{\gamma(A(\pi_1(\mathfrak{D}))); \gamma(D(\pi_2(\mathfrak{D})))\right\} \\ &\quad + \max\left\{b_B; b_C\right\}\max\left\{\gamma(\pi_1(\mathfrak{D})); \gamma(\pi_2(\mathfrak{D}))\right\} \\ &\leq \max\left\{a_B; a_C\right\}\gamma\left[\mathcal{T}\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right] + \max\left\{b_B; b_C\right\}\gamma\left[\begin{pmatrix} \pi_1(\mathfrak{D}) \\ \pi_2(\mathfrak{D}) \end{pmatrix}\right].\end{aligned}$$

Thus, the $\gamma - \mathcal{T}-$ bound of the operator $\mathcal{S}$ is equal to

$$\max\{a_B; a_C\} \leq a_B + a_C < \frac{1}{4}.$$

As a result, $\mathcal{T} + \mathcal{S}$ is self-adjoint, as desired. □

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