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Lower characteristic, weakly demicompact and semi-Fredholm linear operators

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Abstract. In this paper, we show that a lower characteristic linear operator T acting on a Banach space, can be characterized by some measures of weak noncompactness and the weakly demicompact.

1. Introduction

Let X and Y be two Banach spaces. By $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y and by $\mathcal{K}(X,Y)$ the subspace of all compact operators of $\mathcal{L}(X,Y)$. If $T \in \mathcal{L}(X,Y)$ then $\alpha(T)$ denotes the dimension of the kernel $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in Y. The classes of upper semi-Fredholm from X into Y are defined respectively by

$$\Phi_+(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ closed in } Y\},$$

and

$$\Phi_-(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } R(T) \text{ closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \bigcap \Phi_-(X,Y)$ is the set of Fredholm operators from X into Y. If $X = Y$, the sets $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, and $\Phi_-(X)$, respectively. The index of an operator $T \in \Phi(X)$ is $i(T) := \alpha(T) - \beta(T)$.

The operator T is said to be a Dunford-Pettis (for short property DP operator) if it maps weakly compact sets into compact sets. In particular, if T is a DP operator, then $x_n \rightharpoonup 0$ implies $\lim \|Tx_n\| = 0$ (see [5]). Given an operator $T \in \mathcal{L}(X)$, we denote by

$$R^\infty(T) = \bigcap_{n=0}^{\infty} R(T^n).$$

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We start this section by recalling some notations, results and definitions of weak noncompactness measure [4]. If $x \in X$ and $r > 0$, then $B(x,r)$ will denote the closed ball of X with a center at x and a radius r. We denote by B_X the closed unit ball in X and

$$S_X = \{x \in X \ : \ \|x\| = 1\}.$$

Throughout this section, X denotes a Banach space. For any $r > 0$, B_r denotes the closed ball in X centered at 0_X with radius r, and B_X denotes the closed ball in X centered at 0_X with radius 1. Ω_X is the collection of all nonempty bounded subsets of X, and $\mathcal{K}^w$ is the subset of Ω_X consisting of all weakly compact subsets of X. Recall that the notion of the measure of weak noncompactness was introduced by De Blasi [3]; it is the map $\omega : \Omega_X \longrightarrow [0,+\infty)$ defined in the following way:

$$\omega(\mathcal{M}) = \inf\Big\{r > 0 : \text{ there exists } \mathrm{K} \in \mathcal{K}^w \text{ such that } \mathcal{M} \subset \mathrm{K} + B_r\Big\},$$

for all $\mathcal{M} \in \Omega_X$. For more convenience, let us recall some basic properties of $\omega(.)$ needed below (see, for example, [2, 3]) (see also [1], where an axiomatic approach to the notion of a measure of weak noncompactness is presented).

Lemma 1.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two elements of Ω_X. Then, the following conditions are satisfied:*

(1) $\mathcal{M}_1 \subset \mathcal{M}_2$ *implies* $\omega(\mathcal{M}_1) \leq \omega(\mathcal{M}_2)$.

(2) $\omega(\mathcal{M}_1) = 0$ *if, and only if,* $\overline{\mathcal{M}_1}^w \in \mathcal{K}^w$, *where* $\overline{\mathcal{M}_1}^w$ *is the weak closure of the subset* $\mathcal{M}_1$.

(3) $\omega(\overline{\mathcal{M}_1}^w) = \omega(\mathcal{M}_1)$.

(4) $\omega(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)\}$.

(5) $\omega(\lambda\mathcal{M}_1) = |\lambda|\omega(\mathcal{M}_1)$ *for all* $\lambda \in \mathbb{R}$.

(6) $\omega(co(\mathcal{M}_1)) = \omega(\mathcal{M}_1)$, *where* $co(\mathcal{M}_1)$ *is the convex hull of* $\mathcal{M}_1$.

(7) $\omega(\mathcal{M}_1 + \mathcal{M}_2) \leq \omega(\mathcal{M}_1) + \omega(\mathcal{M}_2)$.

(8) *if* $(\mathcal{M}_n)_{n\geq 1}$ *is a decreasing sequence of nonempty, bounded, and weakly closed subsets of X with* $\lim\limits_{n\to\infty} \omega(\mathcal{M}_n) = 0$, *then* $\mathcal{M}_\infty := \cap_{n=1}^{\infty}\mathcal{M}_n$ *is nonempty and* $\omega(\mathcal{M}_\infty) = 0$ *i.e.,* $\mathcal{M}_\infty$ *is relatively weakly compact.*

Remark 1.1. $\omega(B_X) \in \{0,1\}$. *Indeed, it is obvious that* $\omega(B_X) \leq 1$. *Let* $r > 0$ *be given such that there is a weakly compact K of X satisfying* $B_X \subset K + rB_X$. *Hence,* $\omega(B_X) \leq r\omega(B_X)$. *If* $\omega(B_X) \neq 0$, *then* $r \geq 1$. *Thus,* $\omega(B_X) \geq 1$.

Let X be a Banach space and $\mathcal{M}_X$ consisting of all relatively weakly compact sets. Let us recall an expressive example of measure of weak noncompactness of operators (see [6–9]).

Definition 1.1. *Let X and Y be two Banach spaces and ω be the De Blasi measure of weak noncompactness in the space Y. We define the function*

$$\begin{aligned} \Theta_\omega : \mathcal{L}(X,Y) &\longrightarrow [0,\infty[\\ T &\longrightarrow \Theta_\omega(T) = \omega(T(B_X)). \end{aligned}$$

Θ_ω *is a measure of weak noncompactness of operators associated with* ω. ◇

Proposition 1.1. *[9] Let X and Y be two complex Banach spaces and* $T, S \in \mathcal{L}(X,Y)$. *Then,*

(*i*) $\omega(T(D)) \leq \Theta_\omega(T)\omega(D)$, *for every* $D \in \mathcal{M}_X$.
(*ii*) $\Theta_\omega(T) \leq \|T\|$.
(*iii*) $\Theta_\omega(ST) \leq \Theta_\omega(S)\Theta_\omega(T)$.
(*iv*) $\Theta_\omega(\lambda T) = \||\lambda|\Theta_\omega(T)$ *for all* $\lambda \in \mathbb{C}$.

(*v*) *If* $X = Y$, *then* $\Theta_\omega(T^n) \le (\Theta_\omega(T))^n$ *for every* $n \in \mathbb{N}$.
(*vi*) *Let* $C \ge 0$ *such that for every* $x \in X$, $\|Tx\| \le C\|x\|$. *Then,*

$$\omega(T(D)) \le C\omega(D),$$

for every $D \in \mathcal{M}_X$.
(*vii*) *Let* $C \ge 0$ *such that for every* $x \in X$, $\|x\| \le C\|Tx\|$. *Then,*

$$\omega(D) \le C\omega(T(D)),$$

for every $D \in \mathcal{M}_X$. ◊

Definition 1.2. *Let* X *be a Banach space and let* $T : X \longrightarrow X$, *be a bounded linear operator. The operator* T *is said to be weakly demicompact (or weakly relative demicompact), if for every bounded sequence* $(x_n)_n \in X$ *such that* $x_n - Tx_n \rightharpoonup x \in X$, *then there exists a weakly convergent subsequence of* $(x_n)_n$. ◊

Definition 1.3. *For* $T \in \mathcal{L}(X, Y)$, *we define the "lower" characteristic*

$$[T]_a = \sup\{k \,:\, k > 0,\ \omega(T(M)) \ge k\omega(M) \text{ for all bounded } M \subset X\} \tag{1.1}$$

as elements of $[0, \infty]$. ◊

Note that in finite dimensional spaces we have $[T]_a = \infty$. In infinite dimensional spaces, where this characteristic is of more use, we get

$$[T]_a = \inf_{0<\omega(M)<\infty} \frac{\omega(T(M))}{\omega(M)}.$$

Sets with $\omega(M) = 0$ can be left out here, since the continuity of T assures that also $\omega(T(M)) = 0$. This can be seen by considering $\omega(T(M)) \le \omega(T(\overline{M}))$.

The paper is organized in the following way. In Section 2, we present the main results of this paper. We prove that T is weakly demicompact if, and only if, $[I - T]_a > 0$, where $[\cdot]_a$ is the "lower" characteristic. In Section 3, we present some results concerning the weak demicompactness, semi-Fredholm and lower characteristic.

2. Main results

Theorem 2.1. Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^\infty(T)$ is relatively compact. Then, T is weakly demicompact if, and only if, $[I - T]_a > 0$. ◊

Proof. Assume that T is weakly demicompact. We first show that $N(I - T)$ is finite dimensional. Let

$$\begin{aligned} M &= \{x \in X \text{ such that } (I - T)x = 0 \text{ and } \|x\| = 1\} \\ &= S_X \bigcap N(I - T), \end{aligned}$$

and $(x_n)_n$ be a bounded sequence of M. Observe that

$$N(I - T) \subset R^\infty(T).$$

Then,

$$M = S_X \bigcap N(I - T) \subset S_X \bigcap R^\infty(T).$$

Since M is closed and $\overline{S_X \bigcap R^\infty(T)}$ is compact, then M is compact. Consequently, $N(I - T)$ is of finite dimension. Now, in order to show that $R(I - T)$ is closed it suffices to show that $I - T$ takes closed bounded sets of X into a closed set of X. For this purpose, let $(x_n)_n$ be a sequence of a closed bounded set D such that $y_n = (I-T)x_n \to y \in X$. Since T is weakly demicompact, there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ which converges weakly to $x \in X$. As D is closed, it follows that $x \in D$. Taking into account that $I-T$ is a bounded operator, we deduce that $y = (I-T)x$ and consequently, $(I-T)(D)$ is closed. Thus, we conclude that $I-T$ is upper semi-Fredholm. Since $\dim N(I-T) < \infty$, we may find a closed subspace X_0 of X with $X = X_0 \oplus N(I-T)$. The projection $P : X \longrightarrow X_0$ satisfies $[P]_a = 1$, since $I-P$ is compact. Consider the canonical isomorphism $\widetilde{L} : X_0 \longrightarrow R(I-T)$. Since $I-T = \widetilde{L}P$ and $[\widetilde{L}]_a > 0$, we conclude that also

$$[I-T]_a \geq [\widetilde{L}]_a[P]_a > 0.$$

Inversely, suppose that $[I-T]_a > 0$ and fix $k \in (0, [I-T]_a)$. Since the set $M = N(I-T) \bigcap B_X$ is mapped into $(I-T)(M) = \{0\}$, we get

$$\omega(M) \leq \frac{1}{k}\omega((I-T)(M)) = 0,$$

which show that $\overline{M}$ is compact, and hence $N(I-T)$ is finite dimensional. We prove now that the range $R(I-T)$ of $I-T$ is closed. Since $\dim N(I-T) < \infty$, there exists a closed subspace $X_0 \subset X$ such that $X = X_0 \oplus N(I-T)$. Let $(y_n)_n$ be a sequence in $R(I-T)$ converging to some $y \in Y$, and choose $(x_n)_n$ in X with $(I-T)x_n = y_n$. Now, we distinguish two cases. First, suppose that $(x_n)_n$ is bounded. With $k > 0$ as before we get then

$$\omega(\{x_1, x_2, \cdots, x_n, \cdots\}) \leq \frac{1}{k}\omega(\{y_1, y_2, \cdots, y_n, \cdots\}) = 0,$$

and hence $x_{n_k} \to x$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and suitable $x \in X$. By continuity we see that $(I-T)x = y$, and so $y \in R(I-T)$. On the other hand, suppose that $\|x_n\| \to \infty$. Set $e_n = \frac{x_n}{\|x_n\|}$ and $E = \{e_1, e_2, \cdots, e_n, \cdots\}$. Then, clearly $E \subset \{x \in X \ : \ \|x\| = 1\}$ and

$$(I-T)e_n = \frac{(I-T)x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to 0 \text{ as } n \to \infty.$$

Hence, $\omega((I-T)(E)) = 0$. On the other hand, $\omega((I-T)(E)) \geq k\omega(E)$, by (1.1), and thus $\omega(E) = 0$. Whithout loss of generality we may assume that the sequence $(e_n)_n$ converge to some element $e \in \{x \in X_0 \ : \ \|x\| = 1\}$. So, $(I-T)e = 0$, contradicting the fact that $X_0 \bigcap N(I-T) = \{0\}$. So, $I-T \in \Phi_+(X)$. Thus, there exist $T_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that

$$T_l(I-T) = I-K.$$

Let $(x_n)_n$ be a bounded sequence of X such that Then,

$$(I-T)x_n \rightharpoonup y \in X.$$

$$T_l(I-T)x_n = (I-K)x_n \rightharpoonup T_l y.$$

As K is compact, then $(Kx_n)_n$ has a convergent and then a weakly convergent subsequence. We deduce that $(x_n)_n$ admits a weakly convergent subsequence. So, weakly demicompact and the proof is achieved. Q.E.D.

Corollary 2.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^\infty(T)$ is relatively compact. Then, T is weakly demicompact if, and only if, $I-T \in \Phi_+(X,Y)$.* ◊

Proof. A consequence direct of the proof of Theorem 2.1. Q.E.D.

Corollary 2.2. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then, $T \in \Phi_+(X,Y)$ if, and only if, $[T]_a > 0$.* ◊

Proof. A consequence direct of the proof of Theorem 2.1. Q.E.D.

3. Weak demicompactness, semi-Fredholm and lower characteristic

Lemma 3.1. *[11] An operator A is in $\Phi_+(X)$ with $i(A) \leq 0$, if and only if, there exists two operators A_0 and K such that A_0 is in $\Phi_+(X)$ and one to one, and K is a finite rank operator such that $A = A_0 + K$.* ◊

Lemma 3.2. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Let $(x_n)_n$ be a bounded sequence of X such that $(I-T)x_n \rightharpoonup x \in X$. Assume that $S_X \bigcap R^\infty(T)$ is relatively compact. If $[T]_a > 0$ and $i(I-T) > 0$, then $(x_n)_n$ admits a weakly convergent subsequence.* ◊

Proof. If $[T]_a > 0$, then by using Corollary 2.2, we have $I - T \in \Phi_+(X)$. Since $i(I - T) > 0$, then $I - T \in \Phi(X)$. By using [10, Theorem 7.2], there exists $A \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that

$$A(I - T) = I + K.$$

Let $(x_n)_n$ be a bounded sequence of X such that

$$(I - T)x_n \rightharpoonup x \in X.$$

Then,

$$A(I - T)x_n \rightharpoonup Ax.$$

Hence, $(x_n + Kx_n)_n$ converges weakly to Ax. Since K is compact, then $(Kx_n)_n$ has a convergent, and then a weakly convergent, subsequence. It follows that $(x_n)_n$ admits a weakly convergent subsequence. Q.E.D.

Lemma 3.3. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Let $(x_n)_n$ be a bounded sequence of X such that $(I-T)x_n \rightharpoonup x \in X$. Assume that $S_X \bigcap R^\infty(T)$ is relatively compact. If $[T]_a > 0$ and $i(I-T) \leq 0$, then $(x_n)_n$ admits a weakly convergent subsequence.* ◊

Proof. If $[T]_a > 0$, then by using Corollary 2.2, we have $I - T \in \Phi_+(X)$. Since $i(I - T) \leq 0$, then, in view of Lemma 3.1, there exists a bounded below operator A_0 and $K \in \mathcal{K}(X)$ such that

$$I - T = A_0 + K.$$

Let $(x_n)_n$ be a bounded sequence in X such that $(I - T)x_n \rightharpoonup x \in X$. Then, $((A_0 + K)x_n)_n$ converges weakly on X. Since K is compact, then $(Kx_n)_n$ has a convergent subsequence $(Kx_{\varphi(n)})_n$. Consequently, $(A_0x_{\varphi(n)})_n$ is a weakly convergent sequence. Since A_0 is bounded below, then there exist a positive constant C such that

$$||x|| \leq C||A_0x||$$

for all $x \in X$. Let $D = \{x_n \,;\, n \in \mathcal{N}\}$, then by using Proposition 1.1, we get

$$\omega(D) \leq C\omega(A_0(D)).$$

Thus, $\omega(D) = 0$ and therefore, $(x_{\varphi(n)})_n$ has a weakly convergent subsequence. Q.E.D.

Lemma 3.4. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(T)$ is a DP operator and $f(1) = 1$ and $S_X \bigcap R^\infty(T)$ is relatively compact. If $[I - T]_a > 0$, then $\alpha(I - T - K) < \infty$ for any compact operator $K \in \mathcal{K}(X)$.* ◊

Proof. Let $K \in \mathcal{K}(X)$ and take a sequence $(x_n)_n \in B_X \bigcap N(I - T - K)$. Then, for every $n \in \mathbb{N}$, we have

$$(I - T - K)x_n = 0.$$

Since K is compact, there exists a subsequence of $(x_n)_n$, still denoted $(x_n)_n$, such that

$$Kx_n \to x \in X.$$

Hence,

$$(I-T)x_n \to x \in X.$$

Therefore,

$$(I-T)x_n \rightharpoonup x \in X.$$

Taking into account the fact that $[I-T]_a > 0$ and Theorem 2.1, we infer that T is weakly demicompact, we deduce that $(x_n)_n$ has a subsequence $(x_{\varphi(n)})_n$ such that

$$x_{\varphi(n)} \rightharpoonup a \in X.$$

It comes that

$$(I-T)a = x.$$

Furthermore, it is easy to see that

$$a \in B_X \bigcap N(I-T-K)$$

and

$$f(T+K)a = a \quad \text{and} \quad f(T+K)x_n = x_n \ \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$, then from (3.2)

$$f(T+K) - f(T) = \sum_{n=1}^{\infty} a_n[(T+K)^n - T^n] \in \mathcal{K}(X). \tag{3.2}$$

This implies that $f(T+K)$ is a DP operator and

$$f(T+K)x_n \rightharpoonup f(T+K)a.$$

By using (3.1), we infer that

$$x_n \to a.$$

We conclude that $B_X \bigcap N(I-T-K)$ is a compact set. Q.E.D.

Proposition 3.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that $S_X \bigcap R^\infty(T)$ is relatively compact and there exists an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(T)$ is a DP operator and $f(1) = 1$. Then, T is weakly demicompact, if and only if, $[T]_a > 0$.* ◊

Proof. If $[T]_a > 0$, then by using Corollary 2.2, we have $I - T \in \Phi_+(X)$. Let $(x_n)_n$ be a bounded sequence of X such that

$$(I-T)x_n \rightharpoonup x \in X.$$

There are two cases.
First case: If $i(I-T) > 0$, then by using Lemma 3.2, we have $(x_n)_n$ admits a weakly convergent subsequence.
Second case: If $i(I-T) \leq 0$, then by using Lemma 3.3, we have $(x_n)_n$ admits a weakly convergent subsequence.
So, in the two cases T is weakly demicompact.
Inversely, suppose that T is weakly demicompact. By using Lemma 3.4, we have $\alpha(I-T-K) < \infty$ for any compact operator $K \in \mathcal{K}(X)$. The result follows from [11, Theorem 1] and Corollary 2.2. Q.E.D.

Lemma 3.5. *Let X be a Banach space, $T \in \mathcal{L}(X)$. If $[I-T]_a > 0$ and $i(I-T) > 0$, then for all bounded set D of X, we have*

$$\omega(D) \leq C\omega((I-T)(D)),$$

where $C > 0$. ◊

Proof. If $[I-T]_a > 0$ and $i(I-T) > 0$, then by using [10, Theorem 7.2], there exists a bounded operator A and a compact operator K such that

$$A(I-T) = I + K.$$

Let D be a bounded set of X. Then,

$$\begin{aligned}\omega(D) &\leq \omega((A(I-T)(D))\\ &\leq \|A\|\omega((I-T)(D)).\end{aligned}$$

This completes the proof. Q.E.D.

Lemma 3.6. *Let X be a Banach space, $T \in \mathcal{L}(X)$. If $[I-T]_a > 0$ and $i(I-T) \leq 0$, then for all bounded set D of X, we have*

$$\omega(D) \leq C\omega((I-T)(D)),$$

where $C > 0$. ◊

Proof. If $[I-T]_a > 0$ and $i(I-T) \leq 0$, then by using Lemma 3.1, there exists a compact operator K and a bounded below operator A_0 such that

$$I - T = K + A_0.$$

Since A_0 is bounded below, there exists a positive constant C such that

$$\|x\| \leq C\|A_0 x\|,$$

for all $x \in X$. Hence, by applying Proposition 1.1, we get

$$\omega(D) \leq C\omega((I-T)(D)),$$

for any bounded set $D \subset X$. Q.E.D.

Lemma 3.7. *Assume that there exists a positive constant C such that for every bounded set D of X, we have*

$$\omega(D) \leq C\omega((I-T)(D)).$$

Let $(x_n)_n$ be a bounded sequence of X such that $(I-T)x_n \rightharpoonup x \in X$. Then, $(x_n)_n$ has a weakly convergent subsequence. ◊

Proof. Choose $D = \{x_n \ : \ n \in \mathbb{N}\}$. It is clear that D is a bounded set of X such that $\omega((I-T)(D)) = 0$. Hence, $\omega(D) = 0$. Hence, $(x_n)_n$ has a weakly convergent subsequence. Q.E.D.

Theorem 3.1. *Let X be a Banach space, $T \in \mathcal{L}(X)$. Assume that $S_X \bigcap R^{\infty}(T)$ is relatively compact and there exists a complex polynomial P satisfying $P(1) = 1$ such that $P(T+K)$ is DP for all $K \in \mathcal{K}(X)$. Then, $[T]_a > 0$ if and only if, there exists a positive constant C such that for all bounded sets $D \subset X$,*

$$\omega(D) \leq C\omega((SI-T)(D)).$$ ◊

Proof. If $[T]_a > 0$, then by using Corollary 2.2, we have $I - T \in \Phi_+(X)$. If $i(I-T) > 0$, then by using Lemma 3.5, we have

$$\omega(D) \leq \|A\|\omega((I-T)(D)),$$

for all $D \subset X$. If $i(I-T) \leq 0$, then by using Lemma 3.6, we have

$$\omega(D) \leq C\omega((I-T)(D)),$$

for all $D \subset X$. Now, choose $C' = \max(\|A\|, C)$, then for any bounded subset D of X, we have

$$\omega(D) \leq C'\omega((I-T)(D)).$$

Inversely, by using Lemma 3.7, we have T is weakly demicompact. The result follows from Theorem 2.1. Q.E.D.

Proposition 3.2. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^\infty(T)$ is relatively compact and assume that there exists an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that $f(T)$ is a DP operator and $f(1) = 1$. Then, $[I - T]_a > 0$ if and only if for all $D \in \mathcal{M}_X$ such that $\omega(T(D)) = 0$, we have $\omega(D) = 0$.* ◊

Proof. Suppose that $[I - T]_a > 0$. Then, by using Proposition 3.1, shows that $I - T$ is weakly demicompact. Theorem 3.1 ensures the existence of a positive constant C such that for all bounded sets $D \subset X$), we have

$$\omega(D) \leq C\omega(T(D)).$$

Hence, if $\omega(T(D)) = 0$, then $\omega(D) = 0$ for all $D \in \mathcal{M}_X$. Conversely, assume that $\omega(T(D)) = 0$, then $\omega(D) = 0$ whenever $D \in \mathcal{M}_X$. According to Proposition 3.1, to prove that $[T]_a > 0$. By using Corollary 2.2, we have $T \in \Phi_+(X)$, it suffices to prove that $I-T$ is weakly demicompact. For this purpose, let $(x_n)_n \in X$ be a bounded sequence such that $Tx_n \rightharpoonup x$, for some $x \in X$. Put $D = \{x_n,\ n \in \mathbb{N}\}$. Hence, $\omega(T(D)) = 0$ and so $\omega(D) = 0$. Accordingly, $I - T$ is weakly demicompact. This ends the proof. Q.E.D.

Lemma 3.8. *[10] Let X be a Banach space and $K \in \mathcal{K}(X)$. Then, $I \pm K \in \Phi(X)$ and $i(I \pm K) = 0$.* ◊

Proposition 3.3. *Let X be a Banach space and P be a bounded projection on X. Suppose that $S_X \bigcap R^\infty(P)$ is relatively compact. Assume that P is a DP operator. Then, the following assertions are equivalents*

(i) $[I - P]_a > 0$.
(ii) $P \in \mathcal{K}(X)$.
(iii) $I \pm P \in \Phi(X)$ *and* $i(I \pm P) = 0$. ◊

Proof. *(i)* $\Longrightarrow$ *(ii)* Let P be a bounded DP projection on X. If $[I - P]_a > 0$, then by using Theorem 2.1, P is weakly demicompact. Hence, by using Corollary 2.1, we deduce that $I - P \in \Phi_+(X)$. Consequently, $R(P) = N(I-P)$ is finite dimensional. This proves that P is a finite rank operator which implies that $P \in \mathcal{K}(X)$.
(ii) $\Longrightarrow$ *(iii)* Suppose that $P \in \mathcal{K}(X)$, then $\pm P \in \mathcal{K}(X)$. By using Lemma 3.8, we get the desired result.
(iii) $\Longrightarrow$ *(i)* Since $I-P \in \Phi(X)$, then $R(I-P) = N(I-P)$ is finite dimensional. Hence, P is a finite rank operator. this proves that P is a DP operator. In view of Corollary 2.1, we deduce that P is weakly demicompact. The result follows from Theorem 2.1. Q.E.D.

Proposition 3.4. *Let X be a Banach space, $\mathbb{D}$ be the unit disk of the complex plane and $T \in \mathcal{L}(X)$. Suppose that $S_X \bigcap R^\infty(T)$ is relatively compact. Assume that $\Theta_\omega(T^m) < 1$ for some positive integer m. Then, $[I - \lambda T]_a > 0$ for all $\lambda \in \mathbb{D}$.* ◊

Proof. Let $\lambda \in \mathbb{D}$ and $(x_n)_n$ be a bounded sequence such that

$$x_n - \lambda T x_n \rightharpoonup x \in X.$$

Obviously, there exists a bounded operator $S \in \mathcal{L}(X)$ such that

$$I - \lambda^m T^m = S(I - \lambda T).$$

Hence, using the properties of the De Blasi measure of weak noncompactness, we get

$$\begin{aligned} \omega(\{x_n\}) &= \omega(\{\lambda^m T^m x_n\}) + \omega(\{x_n - \lambda^m T^m x_n\}) \\ &= |\lambda|^m \Theta_\omega(T^m) w(\{x_n\}) \\ &= \Theta_\omega(T^m)\omega(\{x_n\}). \end{aligned}$$

Thus, $\omega(\{x_n\}) = 0$. This shows that λT is weakly demicompact. The result follows from Theorem 2.1. Q.E.D.

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