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Metrically generalized ρ-almost periodic sequences and applications

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Abstract. In this paper, we consider various classes of metrically generalized ρ-almost periodic sequences. We present several structural results about the introduced classes of generalized ρ-almost periodic sequences and provide certain applications of our results to the abstract Volterra difference equations.

1. Introduction and preliminaries

Suppose that $(X, \|\cdot\|)$ is a complex Banach space. An X-valued sequence $(x_k)_{k\in\mathbb{Z}}$ $[(x_k)_{k\in\mathbb{N}}]$ is said to be (Bohr) almost periodic if and only if, for every $\epsilon > 0$, there exists a natural number $K_0(\epsilon)$ such that among any $K_0(\epsilon)$ consecutive integers in $\mathbb{Z}$ $[\mathbb{N}]$, there exists at least one integer $\tau \in \mathbb{Z}$ $[\tau \in \mathbb{N}]$ such that

$$\left\|x_{k+\tau} - x_k\right\| \leq \epsilon, \quad k \in \mathbb{Z} \;\; [k \in \mathbb{N}].$$

It is well known that the range of any almost periodic X-valued sequence is relatively compact in X. The equivalent notion of Bochner almost periodicity of X-valued sequences is considered in the important research monograph [20] by A. M. Samoilenko and N. A. Perestyuk. We know that a sequence $(x_k)_{k\in\mathbb{Z}}$ in X is almost periodic if and only if there exists an almost periodic function $f : \mathbb{R} \to X$ such that $x_k = f(k)$ for all $k \in \mathbb{Z}$. Furthermore, for every almost periodic sequence $(x_k)_{k\in\mathbb{N}}$ in X, there exists a unique almost periodic sequence $(\tilde{x_k})_{k\in\mathbb{Z}}$ in X such that $\tilde{x_k} = x_k$ for all $k \in \mathbb{N}$. The class of almost periodic sequences appears in the qualitative analysis of solutions for various classes of impulsive Volterra integro-differential equations, Volterra integro-difference equations and ordinary differential equations; cf. also the doctoral dissertation [21] by M. Veselý for some recent results obtained in this direction. For more details about almost periodic functions and their applications, we refer the reader to the research monographs [7, 8, 10, 11, 13–15, 19, 22].

The class of Stepanov almost periodic sequences, introduced by J. Andres and D. Pennequin in [3], reduces to the class of almost periodic sequences, which is not the case for the corresponding classes of functions. This

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is no longer true for the class of equi-Weyl almost periodic sequences, which provides a proper extension of the class of almost periodic sequences; cf. A. Iwanik [12]. The class of Besicovitch almost periodic sequences has been introduced by A. Bellow, V. Losert [5] and further analyzed by V. Bergelson et al. in [6]. In our joint research article with W.-S. Du and D. Velinov [9], we have recently introduced and analyzed the classes of (equi-)Weyl-p-almost periodic sequences, Doss-p-almost periodic sequences and Besicovitch-p-almost periodic sequences with a general exponent $p \geq 1$, providing also several new applications to the abstract impulsive Volterra integro-differential inclusions. After that, we have extended these classes of generalized almost periodic sequences in our joint research study [16] with B. Chaouchi, W.-S. Du and D. Velinov.

The main aim of this research study is to continue the investigation raised in [16]. We reconsider and slightly generalize various classes of generalized ρ-almost periodic sequences examined there by using the concept of metrical generalizations of almost periodicity ([15]). We analyze here the Stepanov, Weyl, Besicovitch and Doss classes of metrically generalized ρ-almost periodic sequences, providing also certain applications to the abstract Volterra difference equations.

The paper is organized as follows. Section 2 investigates the metrically generalized ρ-almost periodic sequences. In Definition 2.1, we introduce the notion of Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho, l)$-almost periodic sequence, (equi-)Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic sequence and Doss-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic sequence of the form $F : \Lambda \times X \to Y$, where $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces and $\emptyset \neq \Lambda \subseteq \mathbb{Z}^n$ has certain properties. The class of Besicovitch-$(\mathcal{B}, \mathbb{F}, \mathcal{P})$-almost periodic sequences is introduced in Definition 2.6. We reexamine and slightly generalize several results from [16]; the main structural results of Section 2 are Theorem 2.2, Proposition 2.4 and Proposition 2.5. In Section 3, we present some illustrative applications to the abstract Volterra difference equations.

2. Metrically generalized ρ-almost periodic sequences

In this section, we analyze Stepanov, Weyl, Besicovitch and Doss classes of metrically ρ-almost periodic type sequences of the form $F : \Lambda \times X \to Y$, where $\emptyset \neq \Lambda \subseteq \mathbb{Z}^n$. We assume henceforth that $\Lambda = \Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$, where for each $j \in \mathbb{N}_n$ there exists an integer $a \in \mathbb{Z}$ such that $\Lambda_j = \mathbb{Z}$, $\Lambda_j = \{..., a-2, a-1, a\}$ or $\Lambda_j = \{a, a+1, a+2, ...\}$. Define $\Lambda'' := \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} + \Lambda \subseteq \Lambda\}$. For every integer $l \in \mathbb{N}$, we define P_l to be the set consisting of all closed subrectangles of Λ which contains exactly $(l+1)^n$ points with all integer coordinates. In the sequel, we will assume that condition [16, (FV)] automatically holds as well as that for each $l \in \mathbb{N}$ and $J \in P_l$ we have that $(P_{l,J}, d_{l,J})$ is a pseudometric space, where $P_{l,J} \subseteq Y^J$ is closed under the addition and subtraction of functions and $0 \in P_{l,J}$. Define $\|f\|_{l,J} := d_{l,J}(f, 0)$ for all $f \in P_{l,J}$. Henceforth, $\mathcal{B}$ denotes a non-empty collection of subsets of X such that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach space of all bounded linear operators from X into Y; $L(X, X) \equiv L(X)$ and I denotes the identity operator on Y.

The following notion generalizes the notion introduced recently in [16, Definition 3, Definition 6, Definition 7]:

Definition 2.1. Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $\mathbb{F} : \mathbb{N} \to [0, \infty)$, $\Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. Then we say that $F(\cdot; \cdot)$ is:

(i) Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho, l)$-almost periodic for some $l \in \mathbb{N}$ if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that, for every $J \in P_l$ and for every $j \in J$, $x \in B$, there exists $z_{j,x} \in \rho(F(j; x))$ such that

$$\sup_{x \in B} \mathbb{F}(l) \big\| F(\cdot + \tau; x) - z_{\cdot,x} \big\|_{l,J} < \epsilon; \tag{2.1}$$

(ii) equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist $l \in \mathbb{N}$ and $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that, for every $J \in P_l$ and for every $j \in J$, $x \in B$, there exists $z_{j,x} \in \rho(F(j; x))$ such that (2.1) holds;

(iii) Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that there exists an integer $l_\tau \in \mathbb{N}$ such that, for every $l \geq l_\tau$, $J \in P_l$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2.1) holds;

(iv) Doss-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic if and only if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that there exists an increasing sequence (l_k) of positive integers such that, for every $k \in \mathbb{N}$, $J \in P_{l_k}$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2.1) holds with the number l replaced by the number l_k therein.

In the sequel, we omit the term "$\mathcal{B}$" for the functions of the form $F : \Lambda \to Y$, the term "Λ'" if $\Lambda' = \Lambda''$ and the term "ρ" if $\rho = \mathrm{I}$. We will not consider here the uniformly recurrent analogues of the notion introduced above; cf. also [16, Definition 9, Definition 10]. In Definition 2.1, the natural choice is

$$\|f\|_{l,J} \equiv \left[\sum_{j\in J} \big\|f(j)\big\|^p \nu^p(j)\right]^{1/p}, \quad f \in P_{l,J}, \tag{2.2}$$

for some $p \in [1, \infty)$ but we can also consider the notion with

$$\|f\|_{l,J} \equiv \sum_{j\in J} \big\|f(j)\big\|^p \nu^p(j), \quad f \in P_{l,J}, \tag{2.3}$$

where $p \in (0, 1)$; here, $\nu : \mathbb{Z}^n \to [0, \infty)$ is an arbitrary weight sequence (in [16], we have always assumed that $\nu(\cdot) \equiv 1$ and $p \geq 1$). The similar pseudometrics will be used for the Besicovitch-$(\mathcal{B}, \mathbb{F}, \mathcal{P})$-almost periodic sequences introduced in Definition 2.6 below.

The interested reader may try to analyze the uniform convergence (or the convergence in the pseudometric of space $P_{l,J}$) of metrically generalized ρ-almost periodic sequences. We continue by reexaming two examples from [16]:

Example. (i) The sequence $(x_k)_{k\in\mathbb{N}}$, given by $x_k := 1$ if there exists $j \in \mathbb{N}$ such that $k = j^3$, and $x_k := 0$, otherwise, is equi-Weyl-$(l^{-\sigma}, p)$-almost periodic for any $\sigma > 1/2$ and $p > 0$; cf. (2.2)-(2.3) with $\nu(\cdot) \equiv 1$ and [16, Remark 2].

(ii) Let $l_0 \in \mathbb{N}$, let $(x_k)_{k\in\mathbb{N}}$ be a real sequence defined by $x_k := 0$ for $k = 1, 2, ..., l_0$; $x_{l_0+2k} := 1$ ($k \in \mathbb{N}_0$) and $x_{l_0+2k+1} := -1$ ($k \in \mathbb{N}_0$). Then $(x_k)_{k\in\mathbb{N}}$ is equi-Weyl-$(l^{-\sigma}, \mathcal{P}, -\mathrm{I})$-almost periodic for any $\sigma > 0$, where $P_{j,l}$ is given by (2.2)-(2.3) with $\nu(\cdot)$ being an arbitrary non-negative function; cf. also [16, Example 1(ii)].

In [16, Proposition 3], we have shown that the notion of Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho, l)$-almost periodicity is not satisfactory enough because it is in a close connection with the notion of metrical Bohr-$(\mathcal{B}, \Lambda', \rho)$-almost periodiity, where the pseudometric $\|\cdot\|_{l,J}$ is given by the formula (2.2). A similar statement holds for the corresponding classes of sequences with the exponents $p \in (0, 1)$, when the pseudometric $\|\cdot\|_{l,J}$ is given by the formula (2.3). Furthermore, the statement of [16, Proposition 4] remains true with the general exponents $p > 0$ but an equi-Weyl-$(\mathbb{F}, \mathcal{P})$-almost periodic sequence $F : \mathbb{Z}^n \to Y$ need not be bounded if the pseudometric is given by the formula (2.2) or (2.3) and there is no constant $c > 0$ such that $\nu(\cdot) \geq c$, which can be approved by a great number of very simple counterexamples.

If $F : \Lambda \times X \to Y$ is Stepanov-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho, l)$-almost periodic for some $l \in \mathbb{N}$, then it is clear that $F(\cdot;\cdot)$ is equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic. Furthermore, it is clear that every equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic sequence is Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic as well as that every Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic sequence is Doss-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic. All these inclusions are strict, as easily approved.

In [16, Theorem 4, Theorem 5], we have considered the extensions of (equi-)Weyl-p-almost periodic type sequences, Doss-p-almost periodic type sequences and Besicovitch-p-almost periodic type sequences, where $p \geq 1$; let us first notice that this statement holds for all exponents $p > 0$. Without going into full details, we want also to note that the argumentation contained in the proof of the above-mentioned Theorem 4 shows that the possible extensions can be considered even if the pseudometric on $P_{l,J}$ is given by (2.2) or (2.3); for example, we have the following result (see [15, Definition 4.3.6, Definition 6.2.11] for the corresponding notion):

Theorem 2.2. *Suppose that $F : \mathbb{Z} \times X \to Y$ is a given sequence, $p > 0$, $\Lambda' \subseteq \mathbb{Z}$ and $\rho = T \in L(Y)$. If $F(\cdot;\cdot)$ is (equi-)Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic [Doss-$(\mathcal{B}, \Lambda', \mathbb{F}, \mathcal{P}, \rho)$-almost periodic], where $P_{l,J}$ is given by (2.2) for $p \geq 1$ and (2.3) for $0 < p < 1$, with the function $\nu : \mathbb{Z} \to [0,\infty)$ such that there exists a finite real constant $c > 0$ such that $\nu(k) \leq c\nu(k+1)$, $k \in \mathbb{Z}$. Define $\tilde{\nu}(t) := \nu(k)$, if $t \in [k, k+1)$ for some $k \in \mathbb{Z}$, $P := L^{\infty}(\mathbb{R} : \mathbb{C})$, $P_{t,l} := L^p_{\tilde{\nu}}([t, t+l] : Y)$ for all $t \in \mathbb{R}$, $l > 0$, and $P_t := L^p_{\tilde{\nu}}([-t, t] : Y)$ for all $t \in \mathbb{R}$. Then there exists a continuous function $\tilde{F} : \mathbb{R} \times X \to Y$ such that $\tilde{F} \in (e-)W^{(x,\mathbb{F},T,\mathcal{P}_{t,l},\mathcal{P})}_{[0,1],\Lambda',\mathcal{B}}(\mathbb{R} \times X : Y)$ [$\tilde{F}(\cdot;\cdot)$ is Doss-$(\mathcal{P}, x, \mathbb{F}, \mathcal{B}, \Lambda', T)$-almost periodic] and $\tilde{F}(t;x) = F(t;x)$ for all $t \in \mathbb{Z}$ and $x \in X$.*

Remark 2.3. We can also extend the weight sequence $\nu(\cdot)$ in the following way: $\tilde{\nu}(t) := \nu(k+1)$, if $t \in (k, k+1]$ for some $k \in \mathbb{Z}$; then a similar result holds true if we assume that there exists a finite real constant $c > 0$ such that $\nu(k) \geq c\nu(k+1)$, $k \in \mathbb{Z}$.

The statement of [16, Proposition 6] can be metrically generalized in the following way (the proof is almost the same and therefore omitted; observe only that the inequality between the means implies that the inequality stated on [16, p. 13, l. 13] holds in the reverse sense for $p \in (0,1)$ so that a similar extension cannot be formulated for these values of exponent p):

Proposition 2.4. *Suppose that $F : \Lambda \to Y$ satisfies that $R(F) \subseteq K$ for some compact convex subset K of Y, $1 \leq p < +\infty$, $\Lambda' = \Lambda''$ and $\rho = \mathrm{I}$. Suppose, further, that $\mathbb{F}(l) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$. If $F(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}, \mathcal{P})$-almost periodic, where for each $l > 0$ and $J \in P_l$ we have that the pseudometric on $P_{l,J}$ is given by (2.2) and there exists a bounded sequence $\varphi : \Lambda'' \to [0,\infty)$ such that $\nu(x+y) \leq \nu(x)\varphi(y)$ for all $x \in \Lambda$ and $y \in \Lambda''$. Then for each $\epsilon > 0$ there exist a Bohr ν-almost periodic function $H : \Lambda \to Y$ [that is, for every $\epsilon > 0$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda''$, there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda''$ such that $\|[H(\mathbf{t}+\tau) - H(\mathbf{t})] \cdot \nu(\mathbf{t})\|_Y \leq \epsilon$ for all $\mathbf{t} \in \Lambda$] with values in K and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have*

$$l^{-n/p}\left[\sum_{j\in J} \|F(j) - H(j)\|^p \nu^p(j)\right]^{1/p} \leq \epsilon.$$

We will include the main details of the proof of the following extension of [16, Proposition 7]:

Proposition 2.5. *Suppose that $F : \Lambda \to Y$ is a given sequence, $p > 0$, $\Lambda' = \Lambda''$ and $\rho = T \in L(Y)$. If for each $\epsilon > 0$ there exist a Bohr (T,ν)-almost periodic sequence $H : \Lambda \to Y$ [that is, for every $\epsilon > 0$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda''$, there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda''$ such that $\|[H(\mathbf{t}+\tau) - TH(\mathbf{t})] \cdot \nu(\mathbf{t})\|_Y \leq \epsilon$ for all $\mathbf{t} \in \Lambda$] and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have*

$$\mathbb{F}(l)\left[\sum_{j\in J} \big\|F(j;x) - H(j;x)\big\|^p \nu^p(j)\right]^{1/p} \leq \epsilon, \ \text{if } p \geq 1, \quad \textit{resp.,} \tag{2.4}$$

$$\mathbb{F}(l)\sum_{j\in J} \big\|F(j;x) - H(j;x)\big\|^p \nu^p(j) \leq \epsilon, \ \text{if } p \in (0,1).$$

Suppose, further, that there exists a bounded sequence $\varphi : \Lambda \to [0,\infty)$ such that $\nu(x+y) \leq \nu(x)\varphi(y)$ for all $x \in \Lambda$, $y \in \Lambda''$ and

$$\mathbb{F}(l)\left[\sum_{j\in J} \nu^p(j)\right]^{1/p} \leq \epsilon, \ \text{if } p \geq 1, \quad \textit{resp.,} \tag{2.5}$$

$$\mathbb{F}(l)\sum_{j\in J}\nu^p(j)\leq\epsilon,\ \textit{if } p\in(0,1).$$

Then $F(\cdot)$ is equi-Weyl-$(\mathbb{F},\mathcal{P},T)$-almost periodic, where the pseudometric on $P_{l,J}$ is given by (2.2) for $p\geq 1$, resp. (2.3) for $p\in(0,1)$.

Proof. Let $L>0$ and $\tau\in\ B(\mathbf{t}_0,l)\cap\Lambda''$ be as in the formulation of proposition. Furthermore, let $\epsilon>0$ be given and let an integer $l>0$ satisfy the prescribed assumption. Fix $J\in P_l$. Using the decomposition

$$\begin{aligned}&\|F(\mathbf{t}+\tau)-TF(\mathbf{t})\|_Y\\&\leq\|F(\mathbf{t}+\tau)-H(\mathbf{t}+\tau)\|_Y+\|H(\mathbf{t}+\tau)-TH(\mathbf{t})\|_Y\\&+\|T\|\cdot\|H(\mathbf{t})-F(\mathbf{t})\|_Y,\ \mathbf{t}\in\Lambda,\ \tau\in\Lambda''\end{aligned}$$

and our assumptions on the sequence $\nu(\cdot)$, we get the existence of a finite real constant $c_p>0$ such that

$$\begin{aligned}&\Bigg[\sum_{j\in J}\big\|F(j+\tau)-F(j)\big\|^p\nu^p(j)\Bigg]^{1/p}\leq c_p\big(1+\|\varphi\|_\infty\big)\\&\times\Bigg\{\Bigg[\sum_{j\in J}\big\|F(j+\tau)-H(j+\tau)\big\|^p\nu^p(j+\tau)\Bigg]^{1/p}+\Bigg[\sum_{j\in J}\big\|H(j+\tau)-TH(j)\big\|^p\nu^p(j)\Bigg]^{1/p}\\&+\|T\|\cdot\Bigg[\sum_{j\in J}\big\|F(j)-H(j)\big\|^p\nu^p(j)\Bigg]^{1/p}\Bigg\},\ \text{if } p\geq 1.\end{aligned}$$

Keeping in mind the estimate (2.5), this simply implies the required; similarly we can consider the case $p\in(0,1)$. □

Suppose now that, for every $l\in\mathbb{N}$, (P_l,d_l) is a pseudometric space, where $P_l\subseteq Y^{[-l,l]^n\cap\Lambda}$ is closed under the addition and subtraction of functions and $0\in P_l$. Define $\|f\|_l:=d_l(f,0)$ for all $f\in P_l$. For the sequel, we need to introduce the following metrical generalization of the notion from [16, Definition 8]:

Definition 2.6. Suppose that $F:\Lambda\times X\rightarrow Y$ is a given sequence and $\mathbb{F}:(0,\infty)\rightarrow[0,\infty)$. Then we say that $F(\cdot;\cdot)$ is Besicovitch-$(\mathcal{B},\mathbb{F},\mathcal{P})$-almost periodic if and only if, for every $\epsilon>0$ and $B\in\mathcal{B}$, there exists a trigonometric polynomial $P(\cdot;\cdot)$ such that

$$\sup_{x\in B}\limsup_{l\rightarrow+\infty}\mathbb{F}(l)\big\|F(\cdot;x)-P(\cdot;x)\big\|_l<\epsilon.$$

We wil not consider here the completeness of the space of Besicovitch-$(\mathcal{B},\mathbb{F},\mathcal{P})$-almost periodic sequences; cf. [16, Theorem 6] for more details in this direction. See also [16, Corollary 3].

3. Applications to the abstract Volterra difference equations

In this section, we will consider certain applications of the metrically generalized ρ-almost periodic sequences to the abstract Volterra difference equations. We will split the exposition into three separate parts.

1. On the abstract first-order difference equation

$$u(k+1)=Au(k)+f(k),\quad k\in\mathbb{Z}.\tag{3.1}$$

In [4, Section 3], D. Araya, R. Castro and C. Lizama have analyzed the almost automorphic solutions of the first-order linear difference equation (3.1), where $A\in L(X)$ and $(f_k)_{k\in\mathbb{Z}}$ is an almost automorphic sequence. Some results about the existence and uniqueness of polynomially bounded, metrically Weyl almost automorphic solutions of (3.1) have recently been established in our joint research study [1] with S. Abbas.

For simplicity, we will first assume here that $A = \lambda \mathrm{I}$, where $\lambda \in \mathbb{C}$ and $|\lambda| \neq 1$. Keeping in mind [4, Theorem 3.1], we know that the almost automorphy of sequence $(f_k)_{k\in\mathbb{Z}}$ implies the existence of a unique almost automorphic solution $u(\cdot)$ of (3.1), which is given by

$$u(k) = \sum_{m=-\infty}^{k} \lambda^{k-m} f(m-1), \quad k \in \mathbb{Z}, \tag{3.2}$$

if $|\lambda| < 1$, and

$$u(k) = -\sum_{m=k}^{\infty} \lambda^{k-m-1} f(m), \quad k \in \mathbb{Z}, \tag{3.3}$$

if $|\lambda| > 1$. The uniqueness of polynomially bounded solutions of (3.2) can be trivially proved and the following extension of [16, Theorem 7] can be stated (keeping in mind the prescribed assumption on the weight $\nu(\cdot)$ and the estimate $\nu(j) \leq \nu(j-v-1)\psi(v+1)$ for all $j \in \mathbb{Z}$, $v \in \mathbb{N}_0$, the proof is almost the same as the proof of the above-mentioned result and therefore omitted):

Theorem 3.1. *Suppose that $\mathbb{F} : (0,\infty) \to (0,\infty)$, $1 \leq p < +\infty$, $\rho = T \in L(X)$ and $f(\cdot)$ is equi-Weyl-$(\mathbb{F},\mathcal{P},T)$-almost periodic [polynomially bounded Weyl-$(\mathbb{F},\mathcal{P},T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F},\mathcal{P},T)$-almost periodic], where for each $l > 0$ and $J \in P_l$ the pseudometric on $P_{l,J}$ is given by (2.2) with the sequence $\nu : \mathbb{Z} \to [0,\infty)$ satisfying that there exist a sequence $\psi : \mathbb{Z} \to (0,\infty)$ and a number $\sigma > 0$ such that $\nu(x+y) \leq \nu(x)\psi(y)$ for all $x,\ y \in \mathbb{Z}$ and $\sum_{v=0}^{+\infty}[\psi(v+1)]^p/(1+v^\sigma)^p < +\infty$. Then a unique (equi-)Weyl-$(\mathbb{F},\mathcal{P},T)$-almost periodic [polynomially bounded Weyl-$(\mathbb{F},\mathcal{P},T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F},\mathcal{P},T)$-almost periodic] solution of* (3.1) *is given by* (3.2) *if* $|\lambda| < 1$, *and* (3.3) *if* $|\lambda| > 1$.

The interested reader may try to extend the statements of [16, Theorem 8, Theorem 9] in this direction. For the sake of completeness, we will include the most relevant details of the proof of the following slight extension of [16, Theorem 10], which has been slightly incorrectly formulated since the estimate (3.5) with $\nu(\cdot) \equiv 1$ has been mistakenly ignored:

Theorem 3.2. *Suppose that $\mathbb{F} : (0,\infty) \to (0,\infty)$, $1 \leq p < +\infty$, $\mathbb{F}_1 : (0,\infty) \to (0,\infty)$ and $f(\cdot)$ is polynomially bounded Besicovitch-$(\mathbb{F},\mathcal{P})$-almost periodic, where for each $l > 0$ the pseudometric on P_l is given by*

$$\|f\|_l \equiv \left[\sum_{j\in[-l,l]\cap\mathbb{Z}} \|f(j)\|^p \nu^p(j)\right]^{1/p}, \quad f \in P_l, \tag{3.4}$$

with the sequence $\nu : \mathbb{Z} \to [0,\infty)$ satisfying that there exist a sequence $\psi : \mathbb{Z} \to (0,\infty)$ and a number $\sigma > 0$ such that $\nu(x+y) \leq \nu(x)\psi(y)$ for all $x,\ y \in \mathbb{Z}$ and $\sum_{v=0}^{+\infty}[\psi(v+1)]^p/(1+v^\sigma)^p < +\infty$. Suppose, further, that for each $\epsilon > 0$ there exist $l_0 > 0$, $c > 0$ and $k > 0$ such that, for every $l \geq l_0$ and $v \in \mathbb{N}_0$, we have

$$\frac{\mathbb{F}_1(l)}{\mathbb{F}(l+v)} \leq c(1+v)^k.$$

If

$$\limsup_{l\to+\infty} \mathbb{F}_1(l)\left[\sum_{j\in[-l,l]\cap\mathbb{Z}} \nu^p(j)\right]^{1/p} < +\infty, \tag{3.5}$$

then a unique polynomially bounded Besicovitch-$(\mathbb{F}_1,\mathcal{P})$-almost periodic solution of (3.1) *is given by* (3.2) *if* $|\lambda| < 1$, *and* (3.3) *if* $|\lambda| > 1$.

Proof. We will consider the case $|\lambda| < 1$, only. Let $\epsilon > 0$ be fixed. Then there exists a trigonometric polynomial $p(\cdot)$ such that

$$\limsup_{l\to+\infty} \mathbb{F}(l)\left[\sum_{j\in[-l,l]\cap\mathbb{Z}} \|f(j)-p(j)\|^p\right]^{1/p} < \epsilon.$$

The sequence $j \mapsto u_p(j) \equiv \sum_{v=0}^{+\infty} \lambda^v p(j-v-1)$, $j \in \mathbb{Z}$ is almost periodic, as simply approved. Keeping in mind the assumptions on the weight sequence $\nu(\cdot)$ and the function $\mathbb{F}_1(\cdot)$, we can repeat verbatim the argumentation contained in the proof of [16, Theorem 7] in order to see that

$$\limsup_{l\to+\infty} \mathbb{F}_1(l)\left[\sum_{j\in[-l,l]\cap\mathbb{Z}} \bigl\|u(j)-u_p(j)\bigr\|^p \nu^p(j)\right]^{1/p} < \epsilon. \tag{3.6}$$

Now we can find a trigonometric polynomial $P(\cdot)$ such that $\|P_1(j) - u_p(j)\| \leq \epsilon$ for all $j \in \mathbb{Z}$. Using this estimate, (3.5) and (3.6), we simply get

$$\limsup_{l\to+\infty} \mathbb{F}_1(l)\left[\sum_{j\in[-l,l]\cap\mathbb{Z}} \|u(j)-P(j)\|^p \nu^p(j)\right]^{1/p} < \text{Const.}\cdot\epsilon.$$

□

The existence and uniqueness of metrically generalized ρ-almost periodic solutions for the difference equation

$$u(k+1) = A(k)u(k) + f(k), \quad k \geq 0; \quad u(0) = u_0,$$

where $(A(k))_{k\geq 0}$ is a sequence of closed linear operators obeying certain properties, can be considered similarly as above; cf. also [17, Subsection 3.1] and [18].

2. On the abstract fractional difference equation

$$\Delta^\alpha u(k) = Au(k+1) + f(k), \quad k \in \mathbb{Z}. \tag{3.7}$$

Let us recall that E. Alvarez, S. Díaz and C. Lizama have analyzed, in [2], the existence and uniqueness of (N,λ)-periodic solutions of the abstract fractional difference equation (3.7), where A is a closed linear operator on X, $0 < \alpha < 1$ and $\Delta^\alpha u(k)$ denotes the Caputo fractional difference operator of order α. If A is a closed linear operator on X such that $1 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A, and $\|(\mathrm{I}-A)^{-1}\| < 1$, then [2, Theorem 3.4] shows that A generates a discrete (α,α)-resolvent sequence $\{S_{\alpha,\alpha}(v)\}_{v\in\mathbb{N}_0}$ such that $\sum_{v=0}^{+\infty} \|S_{\alpha,\alpha}(v)\| < +\infty$. Furthermore, if $(f_k)_{k\in\mathbb{Z}}$ is a bounded sequence, then the function

$$u(k) = \sum_{l=-\infty}^{k-1} S_{\alpha,\alpha}(k-1-l)f(l), \quad k \in \mathbb{Z} \tag{3.8}$$

is a mild solution of (3.7).

In [16, Subsection 4.2], we have particularly considered the existence and uniqueness of equi-Weyl-$(\mathbb{F},1,T)$-almost periodic solutions of (3.7), where $T \in L(X)$. Here we will only note that we can similarly analyze the existence and uniqueness of equi-Weyl-$(\mathbb{F},\mathcal{P},T)$-almost periodic solutions of (3.7), where $T \in L(X)$ and for each $l > 0$ and $J \in P_l$ the pseudometric on $P_{l,J}$ is given by (2.2) with $p = 1$ and the sequence $\nu : \mathbb{Z} \to [0,\infty)$ satisfying that there exists a bounded sequence $\psi : \mathbb{Z} \to (0,\infty)$ such that $\nu(x+y) \leq \nu(x)\psi(y)$ for all $x,\ y \in \mathbb{Z}$.

3. Some multi-dimensional analogues of (3.1)**.** In this part, we will only note that the following multi-dimensional linear difference equations

$$u\bigl(k_1+1, k_2+1, ..., k_n+1\bigr) = \lambda_1\lambda_2...\lambda_n \cdot u\bigl(k_1, k_2, ..., k_n\bigr) + F\bigl(k_1, k_2, ..., k_n\bigr),$$

$(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$, where $\lambda_1,\ \lambda_2, ...,\ \lambda_n$ are certain complex numbers satisfying $\max(|\lambda_1|, |\lambda_2|, ..., |\lambda_n|) < 1$, and

$$u\big(k_1 + 1, ..., k_n + 1\big) = \lambda u\big(k_1, ..., k_n\big) + F\big(k_1, ..., k_n\big), \quad \big(k_1, ..., k_n\big) \in \mathbb{Z}^n,$$

where $\lambda \in \mathbb{C}$ and $|\lambda| < 1$, are considered in [16, Subsection 4.3]. Similar conclusions can be given for the metrically generalized ρ-almost periodic solutions of these problems. Moreover, the existence and uniqueness of metrically generalized ρ-almost periodic solutions for the difference equation

$$u(k, m) = A(k, m)u(k - 1, m - 1) + f(k, m), \quad k,\ m \in \mathbb{N},$$

subjected with the initial conditions

$$u(k, 0) = u_{k,0}; \quad u(0, m) = u_{0,m}, \quad k,\ m \in \mathbb{N}_0,$$

can be considered similarly as in [17, Subsection 3.2]. Details can be left to the interested readers.

Concerning the almost periodic solutions of the abstract impulsive Volterra integro-differential equations, we will only note that an attempt should be made to extend the result of [9, Theorem 8] for metrically (equi-)Weyl-p-almost periodic sequences $(y_k)_{k\in\mathbb{N}}$.

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