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Classical Solutions for a Class Second-Order Fuchsian Equations

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Abstract. In this article we investigate the Cauchy problem for a class second-order Fuchsian equations. We propose a new topological approach to prove the existence of at least one classical solution and at least two nonnegative classical solutions. The arguments are based upon recent theoretical results.

1. Introduction

In this paper, we investigate the IVP for a class of second-order Fuchsian equations

$$\begin{aligned} t^2\partial_t^2 u + 2a(x)t\partial_t u + b(x)u &= f(t,x,u,t\partial_t u,\partial_x u), \quad t>0, \quad x\in\mathbb{R},\\ u(0,x) &= u_0(x), \quad x\in\mathbb{R}, \\ u_t(0,x) &= u_1(x), \quad x\in\mathbb{R}, \end{aligned} \tag{1}$$

where

(H1) $a, b \in \mathcal{C}(\mathbb{R})$, $|a|, |b| \le B$ on $\mathbb{R}$, for some positive constant $B > \frac{1}{2}$.

(H2) $f \in \mathcal{C}(\mathbb{R}^5)$,

$$|f(t,x,u,t\partial_t u,\partial_x u)| \le a_1(t,x)|u|^{p_1} + a_2(t,x)|\partial_t u|^{p_2} + a_3(t,x)|\partial_x u|^{p_3},$$

$(t,x) \in [0,\infty)\times\mathbb{R}$, $a_j \in \mathcal{C}([0,\infty)\times\mathbb{R})$, $0 \le a_j \le B$ on $[0,\infty)\times\mathbb{R}$, $p_j \in \mathbb{R}$, $p_j \ge 0$, $j \in \{1,2,3\}$.

(H3) $u_0, u_1 \in \mathcal{C}^1(\mathbb{R})$, $\frac{1}{2} < u_0 \le B$, $|u_1| \le B$ on $[0,\infty)\times\mathbb{R}$.

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In [3] is developed a well-posedness theory in Sobolev spaces for the class of second-order hyperbolic Fuchsian systems and they are investigated the numerical approximations of these Fuchsian equations when data are imposed on the singularity of the spacetime and one evolves the solution from the singularity.

The aim of this paper is to investigate the IVP (1) for existence of global classical solutions. Suppose

(H4) $g \in C([0,\infty)\times\mathbb{R})$ is a nonnegative function such that

$$4\left(1+t+t^2\right)^3(1+|x|)\int_0^t\left|\int_0^x g(t_1,x_1)dx_1\right|dt_1 \leq A, \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

for some constant $A>0$.

In the last section, we will give an example for a function g that satisfies $(H4)$. Our main result for existence of classical solutions of the IVP (1) is as follows.

Theorem 1.1. *Suppose* $(H1)$-$(H4)$. *Then the IVP* (1) *has at least one solution* $u \in C^2([0,\infty), C^1(\mathbb{R}))$.

Theorem 1.2. *Suppose* $(H1)$-$(H4)$. *Then the IVP* (1) *has at least two nonnegative solutions* $u_1, u_2 \in C^2([0,\infty), C^1(\mathbb{R}))$.

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3 we prove Theorem 1.1. In Section 3, we prove Theorem 1.2. In Section 4, we give an example to illustrate our main results.

2. Preliminary Results

Below, assume that X is a real Banach space. Now, we will recall the definitions of compact and completely continuous mappings in Banach spaces.

Definition 2.1. *Let* $K : M \subset X \to X$ *be a map. We say that K is compact if* $K(M)$ *is contained in a compact subset of X. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

Proposition 2.2. *(Leray-Schauder nonlinear alternative [1]) Let C be a convex, closed subset of a Banach space E,* $0 \in U \subset C$ *where U is an open set. Let* $f : \overline{U} \to C$ *be a continuous, compact map. Then*

(a) *either f has a fixed point in* $\overline{U}$,

(b) *or there exist* $x \in \partial U$, *and* $\lambda \in (0,1)$ *such that* $x = \lambda f(x)$.

To prove our existence result we will use the following fixed point theorem which is a consequence of Proposition 2.2.

Theorem 2.3. *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with* $0 \in U$. *Consider two operators T and S, where*

$$Tx = \varepsilon x, \; x \in \overline{U},$$

for $\varepsilon > 0$ *and* $S : \overline{U} \to E$ *be such that*

(i) $I - S : \overline{U} \to Y$ *continuous, compact and*

(ii) $\left\{x \in \overline{U} : x = \lambda(I-S)x, \quad x \in \partial U\right\} = \emptyset$, *for any* $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \overline{U}$ *such that*

$$Tx^* + Sx^* = x^*.$$

Proof. We have that the operator $\frac{1}{\varepsilon}(I-S):\overline{U}\to Y$ is continuous and compact. Suppose that there exist $x_0\in\partial U$ and $\mu_0\in(0,1)$ such that

$$x_0 = \mu_0\,\frac{1}{\varepsilon}(I-S)x_0,$$

that is

$$x_0 = \lambda_0\,(I-S)x_0,$$

where $\lambda_0=\mu_0\,\frac{1}{\varepsilon}\in\left(0,\,\frac{1}{\varepsilon}\right)$. This contradicts the condition (ii). From Leray-Schauder nonlinear alternative, it follows that there exists $x^*\in\overline{U}$ so that

$$x^* = \frac{1}{\varepsilon}(I-S)x^*, \quad \text{or} \quad \varepsilon\,x^*+Sx^*=x^*, \quad \text{or} \quad Tx^*+Sx^*=x^*.$$

□

Definition 2.4. *Let X and Y be real Banach spaces. A map $K:X\to Y$ is called expansive if there exists a constant $h>1$ for which one has the following inequality*

$$\|Kx-Ky\|_Y\ge h\|x-y\|_X,$$

for any $x,y\in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 2.5. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x\in\mathcal{P}$ for any $\alpha\ge 0$ and for any $x\in\mathcal{P}$,*
2. *$x,-x\in\mathcal{P}$ implies $x=0$.*

Denote $\mathcal{P}^*=\mathcal{P}\backslash\{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1). For its proof, we refer the reader to [4] and [6].

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1,U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1\subset\overline{U}_2\subset U_3$ and $0\in U_1$. Assume that $T:\Omega\to\mathcal{P}$ is an expansive mapping, $S:\overline{U}_3\to E$ is a completely continuous map and $S(\overline{U}_3)\subset(I-T)(\Omega)$. Suppose that $(U_2\setminus\overline{U}_1)\cap\Omega\neq\emptyset$, $(U_3\setminus\overline{U}_2)\cap\Omega\neq\emptyset$, and there exists $u_0\in\mathcal{P}^*$ such that the following conditions hold:*

(i) *$Sx\neq(I-T)(x-\lambda u_0)$, for all $\lambda>0$ and $x\in\partial U_1\cap(\Omega+\lambda u_0)$,*

(ii) *there exists $\epsilon\ge 0$ such that $Sx\neq(I-T)(\lambda x)$, for all $\lambda\ge 1+\epsilon$, $x\in\partial U_2$ and $\lambda x\in\Omega$,*

(iii) *$Sx\neq(I-T)(x-\lambda u_0)$, for all $\lambda>0$ and $x\in\partial U_3\cap(\Omega+\lambda u_0)$.*

Then $T+S$ has at least two non-zero fixed points $x_1,x_2\in\mathcal{P}$ such that

$$x_1\in\partial U_2\cap\Omega \text{ and } x_2\in(\overline{U}_3\setminus\overline{U}_2)\cap\Omega,$$

or

$$x_1\in(U_2\setminus U_1)\cap\Omega \text{ and } x_2\in(\overline{U}_3\setminus\overline{U}_2)\cap\Omega.$$

3. Proof of Theorem 1.1

Let $X = \mathcal{C}^2([0,\infty), \mathcal{C}^1(\mathbb{R}))$ be endowed with the norm

$$\begin{aligned}
\|u\| &= \max\Big\{ \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_t(t,x)|, \\
&\qquad \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_{tt}(t,x)|, \sup_{(t,x)\in[0,\infty)\times\mathbb{R}} |u_x(t,x)|, \Big\},
\end{aligned}$$

provided it exists. For $u \in X$, define the operator

$$\begin{aligned}
S_1u(t,x) &= u(t,x) - u_0(x) - tu_1(x) \\
&+ \int_0^t (t-t_1)\Big((t_1^2-1)\partial_t^2 u(t_1,x) + 2a(x)t_1\partial_t u(t_1,x) + b(x)u(t_1,x) \\
&- f(t_1,x,u(t_1,x),t_1\partial_t u(t_1,x),\partial_x u(t_1,x))\Big)dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned}$$

Lemma 3.1. *Suppose (H1)-(H3). If $u \in X$ satisfies the equation*

$$S_1u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}, \tag{2}$$

then it is a solution of the IVP (1).

Proof. Let $u \in X$ is a solution of the equation (2).

$$\begin{aligned}
0 &= u(t,x) - u_0(x) - tu_1(x) \\
&+ \int_0^t (t-t_1)\Big((t_1^2-1)\partial_t^2 u(t_1,x) + 2a(x)t_1\partial_t u(t_1,x) + b(x)u(t_1,x) \\
&- f(t_1,x,u(t_1,x),t_1\partial_t u(t_1,x),\partial_x u(t_1,x))\Big)dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned} \tag{3}$$

We differentiate (3) with respect to t and we find

$$\begin{aligned}
0 &= \partial_t u(t,x) - u_1(x) \\
&\quad + \int_0^t \Big((t_1^2-1)\partial_t^2 u(t_1,x) + 2a(x)t_1\partial_t u(t_1,x) + b(x)u(t_1,x) \\
&\quad - f(t_1,x,u(t_1,x),t_1\partial_t u(t_1,x),\partial_x u(t_1,x))\Big)dt_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}.
\end{aligned} \tag{4}$$

Now, we differentiate (4) with respect to t and we get

$$\begin{aligned}
0 &= \partial_t^2 u(t,x) + (t^2-1)\partial_t^2 u(t,x) + 2a(x)t\partial_t u(t,x) + b(x)u(t,x) \\
&\quad - f(t,x,u(t,x),t\partial_t u(t,x),\partial_x u(t,x)) \\
&= t^2\partial_t^2 u(t,x) + 2a(x)t\partial_t u(t,x) + b(x)u(t,x) \\
&\quad - f(t,x,u(t,x),t\partial_t u(t,x),\partial_x u(t,x)), \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

i.e., u satisfies the first equation of (1). Now, we put $t = 0$ in (3) and (4) and we arrive at

$$\begin{aligned} 0 &= u(0,x) - u_0(x) \\ 0 &= \partial_t u(0,x) - u_1(x), \quad x \in \mathbb{R}. \end{aligned}$$

Therefore u satisfies (1). This completes the proof. □

Let

$$B_1 = \max\left\{2B, 2B^2, B + B^2 + B^{p_1+1} + B^{p_2+1} + B^{p_3+1}\right\}.$$

Lemma 3.2. *Suppose* (*H*1)-(*H*3)*. For $u \in X$ with $\|u\| \le B$, we have*

$$|S_1 u(t,x)| \le B_1(1 + t + t^2)^2, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. We have

$$\begin{aligned} |S_1 u(t,x)| &= \Big| u(t,x) - u_0(x) - t u_1(x) \\ &\quad + \int_0^t (t - t_1)\Big((t_1^2 - 1)\partial_t^2 u(t_1,x) + 2a(x)t_1\partial_t u(t_1,x) + b(x)u(t_1,x) \\ &\quad - f(t_1, x, u(t_1,x), t_1\partial_t u(t_1,x), \partial_x u(t_1,x))\Big)dt_1\Big| \\ &\le |u(t,x)| + |u_0(x)| + t|u_1(x)| \\ &\quad + \int_0^t (t - t_1)\Big((t_1^2 + 1)|\partial_t^2 u(t_1,x)| + 2|a(x)|t_1|\partial_t u(t_1,x)| + |b(x)||u(t_1,x)| \\ &\quad + |f(t_1, x, u(t_1,x), t_1\partial_t u(t_1,x), \partial_x u(t_1,x))|\Big) \\ &\le 2B + tB + \int_0^t (t - t_1)\Big((t_1^2 + 1)B + 2B^2 t_1 + B^2 \\ &\quad + a_1(t_1,x)|u(t_1,x)|^{p_1} + a_2(t_1,x)|\partial_t u(t_1,x)|^{p_2} + a_3(t_1,x)|\partial_x u(t_1,x)|^{p_3}\Big)dt_1 \\ &\le 2B + tB + t^2\Big((t^2 + 1)B + 2B^2 t + B^2 + B^{p_1+1} + B^{p_2+1} + B^{p_3+1}\Big) \\ &\le B_1 + tB_1 + B_1 t^2(t^2 + t + 1) \le B_1(1 + t + t^2)^2, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{aligned}$$

This completes the proof. □

For $u \in X$, define the operator

$$S_2 u(t,x) = \int_0^t \int_0^x (t - t_1)^2 (x - x_1) g(t_1,x_1) S_1 u(t_1,x_1) dx_1 dt_1, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Lemma 3.3. *Suppose* (*H*1)-(*H*4)*. For $u \in X$, $\|u\| \le B$, we have $\|S_2 u\| \le AB_1$.*

Proof. We have

$$\begin{aligned}
|S_2u(t,x)| &= \Big|\int_0^t \int_0^x (t-t_1)^2(x-x_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\Big| \\
&\le \int_0^t \Big|\int_0^x (t-t_1)^2|x-x_1|g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\Big|dt_1 \\
&\le 2B_1(1+|x|)\int_0^t \Big|\int_0^x (t-t_1)^2 g(t_1,x_1)(1+t_1+t_1^2)^2dx_1\Big|dt_1 \\
&\le 2B_1(1+|x|)(1+t+t^2)^3\int_0^t \Big|\int_0^x g(t_1,x_1)dx_1\Big|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|\partial_x S_2u(t,x)| &= \Big|\int_0^t \int_0^x (t-t_1)^2 g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\Big| \\
&\le \int_0^t \Big|\int_0^x (t-t_1)^2 g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\Big|dt_1 \\
&\le B_1\int_0^t \Big|\int_0^x (t-t_1)^2 g(t_1,x_1)(1+t_1+t_1^2)^2dx_1\Big|dt_1 \\
&\le B_1(1+t+t^2)^3\int_0^t \Big|\int_0^x g(t_1,x_1)dx_1\Big|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
|\partial_t S_2u(t,x)| &= 2\Big|\int_0^t \int_0^x (t-t_1)(x-x_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\Big| \\
&\le 2\int_0^t \Big|\int_0^x (t-t_1)|x-x_1|g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\Big|dt_1 \\
&\le 4B_1(1+|x|)\int_0^t \Big|\int_0^x (t-t_1)g(t_1,x_1)(1+t_1+t_1^2)^2dx_1\Big|dt_1 \\
&\le 4B_1(1+|x|)(1+t+t^2)^3\int_0^t \Big|\int_0^x g(t_1,x_1)dx_1\Big|dt_1 \\
&\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned} |\partial_t^2 S_2 u(t,x)| &= 2\Big|\int_0^t\int_0^x (x-x_1)g(t_1,x_1)S_1u(t_1,x_1)dx_1dt_1\Big| \\ &\le 2\int_0^t\Big|\int_0^x |x-x_1|g(t_1,x_1)|S_1u(t_1,x_1)|dx_1\Big|dt_1 \\ &\le 4B_1(1+|x|)\int_0^t\Big|\int_0^x g(t_1,x_1)(1+t_1+t_1^2)^2dx_1\Big|dt_1 \\ &\le 4B_1(1+|x|)(1+t+t^2)^2\int_0^t\Big|\int_0^x g(t_1,x_1)dx_1\Big|dt_1 \\ &\le AB_1, \quad (t,x)\in[0,\infty)\times\mathbb{R}. \end{aligned}$$

Consequently $\|S_2u\| \le AB_1$. This completes the proof. □

Lemma 3.4. *Suppose* (*H*1)-(*H*3) *and let* $g \in \mathcal{C}([0,\infty)\times\mathbb{R})$ *be a nonnegative function. If* $u \in X$ *satisfies the equation*

$$S_2u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}, \tag{5}$$

then u is a solution to the IVP (1).

Proof. We differentiate three times with respect to t and two times with respect to x the equation (5) and we find

$$g(t,x)S_1u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R},$$

whereupon

$$S_1u(t,x) = 0, \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

Hence by using Lemma 3.1, we conclude that u is a solution to the IVP (1). This completes the proof. □

Below, suppose

(H5) $\epsilon \in (0,1)$, A and B satisfy the inequalities $\epsilon B_1(1+A) < 1$ and $AB_1 < 1$.

Let $\tilde{\tilde{Y}}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $\tilde{Y} = \tilde{\tilde{Y}} \cup \{u_0, u_1\}$, $Y = \overline{\overline{\tilde{Y}}}$ and

$$U = \Big\{u \in Y : \|u\| < B, \quad \text{if} \quad u \not\equiv 0 \quad \text{then} \quad u(0,x) > \frac{1}{2}, \quad x \in \mathbb{R}^n\Big\}.$$

For $u \in \overline{U}$ and $\epsilon > 0$, define the operators

$$Tu(t,x) = \epsilon u(t,x),$$

$$Su(t,x) = u(t,x) - \epsilon u(t,x) - \epsilon S_2u(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}.$$

For $u \in \overline{U}$, we have

$$\|(I-S)u\| = \|\epsilon u + \epsilon S_2 u\| \le \epsilon\|u\| + \epsilon\|S_2u\| \le \epsilon B_1 + \epsilon AB_1.$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \partial U$ so that

$$u = \lambda(I-S)u,$$

or

$$u = \lambda\epsilon\,(u + S_2 u), \tag{6}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $S_2u(0,x) = 0$, we get

$$u(0,x) = \lambda\epsilon(u(0,x) + S_2u(0,x)) = \lambda\epsilon u(0,x), \quad x \in \mathbb{R}^n,$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently

$$\left\{u \in \overline{U} : u = \lambda_1(I - S)u,\ u \in \partial U\right\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.3, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t,x) &= Tu^*(t,x) + Su^*(t,x) \\ &= \epsilon u^*(t,x) + u^*(t,x) - \epsilon u^*(t,x) - \epsilon S_2 u^*(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}, \end{aligned}$$

whereupon

$$S_2u^*(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

From here, u^* is a solution to the problem (1). From here, it follows that u is a solution to the IVP (1). This completes the proof.

4. Proof of Theorem 1.2

Let X be the space used in the previous section. Suppose

(H6) Let $m > 0$ be large enough and A, B, r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L, \quad AB_1 < \frac{L}{5}.$$

Let

$$\widetilde{P} = \left\{u \in X : u \geq 0 \quad \text{on} \quad [0,\infty) \times \mathbb{R}\right\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1v(t) &= (1 + m\epsilon)v(t) - \epsilon\frac{L}{10}, \\ S_3v(t) &= -\epsilon S_2v(t) - m\epsilon v(t) - \epsilon\frac{L}{10}, \end{aligned}$$

$t \in [0,\infty)$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \left\{v \in \mathcal{P} : \|v\| < r\right\}, \\ U_2 &= \mathcal{P}_L = \left\{v \in \mathcal{P} : \|v\| < L\right\}, \\ U_3 &= \mathcal{P}_{R_1} = \left\{v \in \mathcal{P} : \|v\| < R_1\right\}, \\ R_2 &= R_1 + \frac{A}{m}B_1 + \frac{L}{5m}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \left\{v \in \mathcal{P} : \|v\| \leq R_2\right\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

   $$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|,$$

   whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.
2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

   $$\|S_3 v\| \leq \varepsilon\|S_2 v\| + m\varepsilon\|v\| + \varepsilon\frac{L}{10} \leq \varepsilon\Big(AB_1 + mR_1 + \frac{L}{10}\Big).$$

   Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.
3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

   $$v_2 = v_1 + \frac{1}{m}S_2 v_1 + \frac{L}{5m}.$$

   Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[t_0, \infty)$. We have $v_2 \geq 0$ on $[t_0, \infty)$ and

   $$\|v_2\| \leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m} \leq R_1 + \frac{A}{m}B_1 + \frac{L}{5m} = R_2.$$

   Therefore $v_2 \in \Omega$ and

   $$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10},$$

   or

   $$(I - T_1)v_2 = -\varepsilon m v_2 + \varepsilon\frac{L}{10} = S_3 v_1.$$

   Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.
4. Assume that for any $u_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ or $x \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda u_0)$ such that

   $$S_3 x = (I - T_1)(x - \lambda u_0).$$

   Then

   $$-\epsilon S_2 x - m\epsilon x - \epsilon\frac{L}{10} = -m\epsilon(x - \lambda u_0) + \epsilon\frac{L}{10} \quad \text{or} \quad -S_2 x = \lambda m u_0 + \frac{L}{5}.$$

   Hence,

   $$\|S_2 x\| = \left\|\lambda m u_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

   This is a contradiction.
5. Suppose that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

   $$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{7}$$

   In particular, for $\epsilon_1 > \frac{2}{5m}$, we have $x_1 \in \partial\mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (7) holds. Since $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

   $$\Big(\frac{2}{5m} + 1\Big)L < \lambda_1 L = \lambda_1\|x_1\| \leq R_1.$$

   Moreover,

   $$-\epsilon S_2 x_1 - m\epsilon x_1 - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon x_1 + \epsilon\frac{L}{10}, \quad \text{or} \quad S_2 x_1 + \frac{L}{5} = (\lambda_1 - 1)m x_1.$$

   From here,

   $$2\frac{L}{5} \geq \left\|S_2 x_1 + \frac{L}{5}\right\| = (\lambda_1 - 1)m\|x_1\| = (\lambda_1 - 1)mL, \quad \text{and} \quad \frac{2}{5m} + 1 \geq \lambda_1,$$

   which is a contradiction.

Therefore all conditions of Theorem 1.2 hold. Hence, the IVP (1) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1 \quad \text{or} \quad r < \|u_1\| < L < \|u_2\| < R_1.$$

5. An Example

Below, we will illustrate our main results. Let

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 4,$$

and

$$R_1 = B = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{5B_1}, \quad \epsilon = \frac{1}{5B_1(1+A)}.$$

Then

$$B_1 = \max\left\{20, \quad 2\cdot 10^2, \quad 10 + 10^2 + 10^3 + 10^4 + 10^5\right\} = 10 + 10^2 + 10^3 + 10^4 + 10^5,$$

and

$$AB_1 = \frac{1}{5} < B, \quad \epsilon B_1(1+A) < 1,$$

i.e., $(H5)$ holds. Next,

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L, \quad AB_1 < \frac{L}{5}.$$

i.e., $(H6)$ holds. Take

$$h(s) = \log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty}(1 + s + s^2)h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty}(1 + s + s^2)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2)^3\left(\frac{1}{44\sqrt{2}}\log\frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s\to\pm 1} l(s) = \frac{\pi}{2}$ and by [7] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}}\log\frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1+s^{44})(1+s+s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t,x) = Q(t)Q(x), \quad t \in [0,\infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C > 0$ such that

$$4\left(1+t+t^2\right)^3 (1+|x|) \int_0^t \left| \int_0^x g_1(t_1,x_1)dx_1 \right| dt_1 \leq C, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Let

$$g(t,x) = \frac{A}{C} g_1(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

Then

$$4\left(1+t+t^2\right)^3 (1+|x|) \int_0^t \left| \int_0^x g(t_1,x_1)dx_1 \right| dt_1 \leq A, \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

i.e., $(H4)$ holds. Therefore for the IVP

$$\begin{aligned} t^2\partial_t^2 u + \tfrac{2}{1+x^2} t\partial_t u + \tfrac{1}{1+x^4} u &= \tfrac{1}{1+t^4} u^2 + \tfrac{1}{1+t^6} (t\partial_t u)^3 + \tfrac{1}{1+x^8} (\partial_x u)^4, \quad t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = \partial_t u(0,x) &= \tfrac{1}{(1+x^2)^8}, \quad x \in \mathbb{R}, \end{aligned}$$

are fulfilled all conditions of Theorem 1.1 and Theorem 1.2.

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