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Relatively regular semi-Fredholm operators and essentially Saphar decomposition

Snežana Č. Živković-Zlatanović[a]

[a] *University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia*

Abstract. We characterize left (right) Weyl, left (right) Fredholm and left (right) Browder operators by means of essentially Saphar decompositions and improve some results from [18] and [20]. Some relationships between different parts of the spectrum of bounded linear operator pencils are established.

1. Introduction

For an infinite dimensional complex Banach space X, $L(X)$ is the Banach algebra of all bounded linear operators acting on X. For $T \in L(X)$ let $\alpha(T)$ denote the dimension of the kernel $N(T)$ and let $\beta(T)$ denote the dimension of the range $R(T)$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, then we write $T = T_M \oplus T_N$ and say that T is completely reduced by the pair (M, N), denoting this by $(M, N) \in Red(T)$. An operator $T \in L(X)$ is said to be Saphar if T is relatively regular and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. An operator $T \in L(X)$ is essentially Saphar if and only if there exists $(M, N) \in Red(T)$ such that T_M is Saphar, T_N is nilpotent and $\dim N < \infty$.

Linear operator pencils have the form $T - \lambda S$, where $\lambda \in \mathbb{C}$, T and S are two bounded linear operators acting on a Banach space. In this paper we characterize left (right) Weyl, left (right) Fredholm and left (right) Browder operators by means of essentially Saphar decompositions using various types of spectra of bounded linear operator pencils. Furthermore we get that boundaries of the essential spectra of operator pencils are contained in the essentially Saphar spectrum of operator pencils and that their connected hulls coincide. As applications of these results we establish some relationships between various parts of the spectrum of operator pencils.

For $T, S \in L(X)$, where S is invertible and commutes with T, one of the characterizations is that T is left (right) Weyl if and only if T is essentially Saphar and 0 is not an interior point of the S-upper (S-lower) semi-B-Weyl spectrum, while T is left (right) Fredholm if and only if T is essentially Saphar and 0 is not an interior point of the S-essentially upper Drazin spectrum (the S-essentially descent spectrum). From [18, Theorems 5 and 6] it follows that $T \in L(X)$ is left (right) Browder if and only if T is left (right) Fredholm and 0 is not

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Email address: mladvlad@mts.rs (Snežana Č. Živković-Zlatanović)

an accumulation point of the S-left (S-right) spectrum of T, $\sigma_l(T,S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not left invertible}\}$ ($\sigma_r(T,S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not right invertible}\}$), where $S \in L(X)$ is invertible and commutes with T. In this paper we prove that the condition that T is left (right) Fredholm in the previous equivalence can be replaced by the weaker condition that T is essentially Saphar, while the condition that 0 is not an accumulation point of the S-left (S-right) spectrum of T can be replaced by the condition that 0 is not an interior point of the S-point (S-compression) spectrum, or by the condition that 0 is not an interior point of the S-upper Drazin (S-descent) spectrum. Also this shows that in characterizations of left (right) Browder operators given in [20, Theorem 3.1] ([20, Theorem 3.2]) the condition that the operator T is left (right) Fredholm can be replaced by a weaker condition that the operator T is essentially Saphar. These characterizations allow us to prove that the boundary of the S-left (right) Browder spectrum, as well as the boundary of S-left (right) Weyl and the boundary of S-left (right) Fredholm spectrum is contained in the S-essentially Saphar spectrum. In that way we improve Theorem 10 in [18] for the case of relatively regular semi-Fredholm operators. Furthermore using the consept of essentially Saphar decompositions allows us to give an alternative more direct proof of Theorem 3.5 in [20]. In [20, Theorem 3.5 (2)] it is stated that the connected hulls of before mentioned S-spectra coincide with the connected hull of $\sigma_{\Phi_{l,r}}(T,S) = \{\lambda \in \mathbb{C} : T - \lambda S \notin \Phi_{l,r}(X)\}$, where $\Phi_{l,r}(X)$ is the set of all relatively regular semi-Fredholm operators. We get more than this: the connected hulls of all these S-spectra coincide with the connected hull of the S-essentially Saphar spectrum. As applications of results regarding boundaries of S-essential spectra we establish some new relationships between various parts of the S-spectrum, especially regarding isolated points of the S-essential spectra.

This paper is divided into five sections. In the second section we set up terminology and give some preliminary results. In the third section we provide some new characterizations of left (right) Weyl, left (right) Fredholm and left (right) Browder by means of essentially Saphar decompositions. The forth section is dedicated to numerous relationships between different parts of the S-spectrum.

2. Basic notation and preliminary results

For $T, S \in L(X)$, $S \neq 0$, the corresponding S-spectra are defined as

$$\begin{aligned}
\sigma(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not invertible}\} - \text{the } S - \text{spectrum of } T,\\
\sigma_l(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not left invertible}\} - \text{the } S - \text{left spectrum of } T,\\
\sigma_r(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not right invertible}\} - \text{the } S - \text{right spectrum of } T,\\
\sigma_p(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not injective}\} - \text{the } S - \text{point spectrum of } T,\\
\sigma_{cp}(T,S) &= \{\lambda \in \mathbb{C} : T - \lambda S \text{ does not have dense range}\} - \text{the } S - \text{compression}\\
&\quad\ \text{spectrum of } T.
\end{aligned}$$

An operator $T \in L(X)$ is called upper (lower) semi-Fredholm, or $T \in \Phi_+(X)$ ($T \in \Phi_-(X)$), if $\alpha(T) < \infty$ and $R(T)$ is closed ($\beta(T) < \infty$). The set of semi-Fredholm operators is $\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$. The index for $T \in \Phi_\pm(X)$ is defined by $i(T) = \alpha(T) - \beta(T)$. The set of Fredholm operators is $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$.

An operator $T \in L(X)$ is said to be upper (lower) semi-Weyl, or $T \in \mathcal{W}_+(X)$ ($T \in \mathcal{W}_-(X)$), if $T \in \Phi_+(X)$ and $i(T) \leq 0$ ($T \in \Phi_-(X)$ and $i(T) \geq 0$). The set of Weyl operators is $\mathcal{W}(X) = \mathcal{W}_+(X) \cap \mathcal{W}_-(X) = \{T \in \Phi(X) : i(T) = 0\}$.

A closed subspace M of X is complemented if there is a closed subspace N of X such that $X = M \oplus N$. An operator $T \in L(X)$ is relatively regular (or g-invertible) if there exists $S \in L(X)$ such that $TST = T$. It is well-known that T is relatively regular if and only if $R(T)$ and $N(T)$ are complemented subspaces of X.

An operator $T \in L(X)$ is called left (right) Fredholm, or $T \in \Phi_l(X)$ ($T \in \Phi_r(X)$), if T is relatively regular upper (lower) semi-Fredholm. Set $\Phi_{l,r}(X) = \Phi_l(X) \cup \Phi_r(X)$. An operator $T \in L(X)$ is left (right) Weyl, or $T \in \mathcal{W}_l(X)$ ($T \in \mathcal{W}_r(X)$), if T is upper (lower) semi-Weyl and relatively regular.

For $T \in L(X)$ and $n \in \mathbb{N}_0$ we set

$$\alpha_n(T) = \dim N(T^{n+1})/N(T^n) \ \text{ and } \ \beta_n(T) = \dim R(T^n)/R(T^{n+1}).$$

From [12, Lemmas 3.1 and 3.2] it follows that $\alpha_n(T) = \dim(N(T) \cap R(T^n))$ and $\beta_n(T) = \operatorname{codim}(R(T) + N(T^n))$.

For $n \in \mathbb{N}_0$, set $k_n(T) = \dim(R(T^n) \cap N(T))/(R(T^{n+1}) \cap N(T))$ [9]. Equivalently, $k_n(T) = \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n))$. Let $k(T) = \sum_{n=0}^{\infty} k_n(T)$.

An operator $T \in L(X)$ has uniform descent for $n \geq d$ if there exists $d \in \mathbb{N}_0$ such that $k_n(T) = 0$ for each $n \geq d$. $T \in L(X)$ is quasi-Fredholm of degree d if there exists a $d \in \mathbb{N}_0$ such that T has uniform descent for $n \geq d$ and $R(T^n)$ is closed for each $n \geq d$. An operator $T \in L(X)$ is quasi-Fredholm if it is quasi-Fredholm of some degree d. It is said that $T \in L(X)$ has topological uniform descent for (TUD for brevity) $n \geq d$ [9] if there exists $d \in \mathbb{N}_0$ for which T has uniform descent for $n \geq d$ and if $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$.

The ascent $a(T)$ of T is the smallest $n \in \mathbb{N}$ such that $\alpha_n(T) = 0$. If such n does not exist, then $a(T) = \infty$. The descent $d(T)$ is the smallest $n \in \mathbb{N}$ such that $\beta_n(T) = 0$. If such n does not exist, then $d(T) = \infty$. The essential ascent $a_e(T)$ is the smallest $n \in \mathbb{N}$ such that $\alpha_n(T) = 0$.If such n does not exist, then $a_e(T) = \infty$. The essential descent $d_e(T)$ of T is the smallest $n \in \mathbb{N}$ such that $\beta_n(T) < \infty$. If such n does not exist, then $d_e(T) = \infty$.

An operator $T \in L(X)$ is upper (lower) semi-Browder if it is upper (lower) semi-Fredholm of finite ascent (descent), and then we write $T \in \mathcal{B}_+(X)$ ($T \in \mathcal{B}_-(X)$). The set of Browder operators is $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. T is left (right) Browder if $T \in L(X)$ is relatively regular upper (lower) semi-Browder and then we write $T \in \mathcal{B}_l(X)$ ($T \in \mathcal{B}_r(X)$) [18].

For $S \in L(X)$ such that $S \neq 0$ and $H = \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}, \Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}$, the corresponding S-spectrum of $T \in L(X)$ is defined by

$$\sigma_H(T, S) = \{\lambda \in \mathbb{C} : \ T - \lambda S \notin H(X)\}.$$

We recall that the sets $\Phi_+(X), \Phi_-(X), \Phi_l(X), \Phi_{l,r}(X), \Phi_r(X), \mathcal{W}_+(X), \mathcal{W}_-(X), \mathcal{W}_l(X), \mathcal{W}_r(X)$ are open in $L(X)$ ([6], Theorems 4.2.1, 4.2.2), ([5], Chapter 5.2, Theorem 6), as well $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ ([13], Satz 4) and consequently, $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$ are also open in $L(X)$. Hence $\sigma_H(T, S)$ is closed for each $H = \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}, \Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}$.

An operator $T \in L(X)$ is said to be upper (lower) Drazin invertible if $a(T) < \infty$ and $R(T^{a(T)+1})$ is closed ($d(T) < \infty$ and $R(T^{d(T)})$ is closed). If $a_e(T) < \infty$ and $R(T^{a_e(T)+1})$ is closed ($d_e(T) < \infty$ and $R(T^{d_e(T)})$ is closed), then T is called *essentially upper (lower) Drazin invertible*.

If $T, S \in L(X)$ such that $S \neq 0$, the S-upper Drazin spectrum of T, the S-lower Drazin spectrum of T, the S-essentially upper Drazin spectrum of T, the S-essentially lower Drazin spectrum of T are denoted as $\sigma_{D_+}(T, S)$, $\sigma_{D_-}(T, S)$, $\sigma^e_{D_+}(T, S)$, $\sigma^e_{D_-}(T, S)$, respectively. If $S \neq 0$ commutes with T, then the spectra $\sigma_{D_+}(T, S)$, $\sigma_{D_-}(T, S)$, $\sigma^e_{D_+}(T, S)$, $\sigma^e_{D_-}(T, S)$ are closed [15, (C), p. 135, p. 139].

The S-descent spectrum of T and the S-essentially descent spectrum of T are defined as

$$\begin{aligned} \sigma_{dsc}(T, S) &= \{\lambda \in \mathbb{C} : d(T - \lambda S) = \infty\}, \\ \sigma^e_{dsc}(T, S) &= \{\lambda \in \mathbb{C} : d_e(T - \lambda S) = \infty\}. \end{aligned}$$

If $T \in L(X)$ has finite descent or essential descent, then T has TUD. For $T, S \in L(X)$ such that S is invertible and $TS = ST$, from [9, Theorem 4.7] it follows that $\sigma_{dsc}(T, S)$ and $\sigma^e_{dsc}(T, S)$ are closed.

For $T \in L(X)$ and $n \in \mathbb{N}_0$ let T_n denote the restriction of T to to $R(T^n)$ [2]. If there exist an integer n for which the range space $R(T^n)$ is closed and T_n is upper semi-Weyl (lower semi-Weyl), then T is said to be a upper semi-B-Weyl (lower semi-B-Weyl) operator.

For $T, S \in L(X)$, $S \neq 0$, the S-upper semi-B-Weyl spectrum of T is

$$\sigma_{B\mathcal{W}_+}(T, S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not upper semi-B-Weyl}\},$$

and the S-lower semi-B-Weyl spectrum of T is

$$\sigma_{B\mathcal{W}_-}(T, S) = \{\lambda \in \mathbb{C} : T - \lambda S \text{ is not lower semi-B-Weyl}\}.$$

The quasinilpotent part of an operator $T \in L(X)$ is the set:

$$H_0(T) = \{x \in X : \lim_{n \to \infty} \|T^n x\|^{1/n} = 0\}.$$

The analytical core of T is the set:

$$K(T) = \{x \in X : \text{there exist } \delta > 0 \text{ and a sequence } (x_n)_n \text{ in } X \text{ such that } Tx_1 = x,\ Tx_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n \in \mathbb{N}.\}.$$

An operator $T \in L(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies $(T-\lambda I)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$, is the function $f \equiv 0$.

If M, N are vector subspace of X, then we write $M \overset{e}{\subset} N$ (M is *essentially contained* in N) if there exists a finite-dimensional subspace F of X such that $M \subset N + F$.

For $T \in L(X)$ we say that it is *Kato* if $R(T)$ is closed and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. An operator $T \in L(X)$ is said to be *Saphar* if it is a relatively regular Kato operator. For $T \in L(X)$ it is said to be *essentially Kato* if $R(T)$ is closed and $N(T) \overset{e}{\subset} \cap_{n=1}^{\infty} R(T^n)$. An operator $T \in L(X)$ is *essentially Saphar* if T is a relatively regular essentially Kato operator [16, p. 233].

For $K \subset \mathbb{C}$, the boundary of K, the set of accumulation points of K, the set of interior points of K and the set of isolated points of K are denoted respectively by ∂K, acc K, int K and iso K. For a compact subset K of $\mathbb{C}$, the complement of the unbounded component of $\mathbb{C} \setminus K$ is called the *connected hull* of K and denoted by ηK [10, Definition 7.10.1]. If $H, K \subset \mathbb{C}$ are compact, then:

$$\partial H \subset K \subset H \Longrightarrow \partial H \subset \partial K \subset K \subset H \subset \eta K = \eta H\ . \tag{2.1}$$

Lemma 2.1. *Let $T \in L(X)$ and let there exists a pair $(M, N) \in Red(T)$. Then the following statements hold:*
(i) *T is g-invertible if and only if T_M and T_N are g-invertible.*
(ii) *T is left (right) Fredholm if and only if T_M and T_N are left (right) Fredholm, and in that case $i(T) = i(T_M) + i(T_N)$.*
(iii) *If T_M and T_N are left (right) Weyl, then T is left (right) Weyl.*

Proof. For (i) and (ii) see [19, Lemma 2.1]. The assertion (iii) follows from (ii). □

Lemma 2.2. *For $T \in L(X)$ let there exists a pair $(M, N) \in Red(T)$. Then T is Saphar if and only if T_M and T_N are Saphar.*

Proof. According to [15, p. 143], T is Kato if and only if T_M and T_N are Kato. Applying Lemma 2.1 (i) we get the assertion. □

Lemma 2.3. *Let $T \in L(X)$. Then T is essentially Saphar if and only if there exists $(M, N) \in Red(T)$ such that* dim$N < \infty$, *T_N is nilpotent and T_M is Saphar.*

Proof. It follows from [17, Theorem 2.1] and Lemma 2.1 (i). □

Lemma 2.4. *[8, Lemma 2.2] Let $T \in L(X)$ be of Kato type of degree d, i.e. there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and $T_N^d = 0$. Then T is quasi-Fredholm of degree d and for $n \ge d$ it holds*

$$R(T) + N(T^n) = R(T_M) \oplus N$$

and

$$N(T) \cap R(T^n) = N(T_M).$$

Lemma 2.5. *Let $T,\ S \in Ł(X)$, $TS = ST$ and let S be invertible. The spectra $\sigma_{B\mathcal{W}_+}(T, S)$ and $\sigma_{B\mathcal{W}_-}(T, S)$ are closed.*

Proof. It follows from [3, Proposition 2.5 and Corollary 3.2]. □

Theorem 2.6. *[20, Theorem 2.1] Let $T,\ S \in Ł(X)$, $TS = ST$ and let S be invertible. If $T \in \Phi_\pm(X)$, then there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, $0 < |\lambda| < \epsilon$ implies that $T - \lambda S \in \Phi_\pm(X)$ and*

$$\alpha(T - \lambda S) = \alpha(T) - k(T), \tag{2.2}$$
$$\beta(T - \lambda S) = \beta(T) - k(T), \tag{2.3}$$
$$i(T - \lambda S) = i(T). \tag{2.4}$$

Lemma 2.7. *Let E and F be sets of the complex plane. Then:*
(i) *If* $\partial F \subset E \subset F$, *then* $\text{iso } F \subset \text{iso } E$.
(ii) *If* $\partial F \subset E$ *and F is closed, then* $\partial F \cap \text{iso } E \subset \text{iso } F$.

Proof. See [7, Lemma 2.2]. □

3. Left and right Weyl, Fredholm and Browder operators

In this section we are concerned with various characterizations of left (right) Weyl, left (right) Fredholm and left (right) Browder operators by means of essentially Saphar decompositions.

Theorem 3.1. *Let* $T, S \in L(X)$, *and let S be invertible and commute with T. Then the following statements are equivalent:*
(i) *T is left Weyl;*
(ii) *T is essentially Saphar and* $0 \notin \text{acc } \sigma_{\mathcal{W}_l}(T, S)$;
(iii) *T is essentially Saphar and* $0 \notin \text{int } \sigma_{\mathcal{W}_l}(T, S)$;
(iv) *T is essentially Saphar and* $0 \notin \text{acc } \sigma_{\mathcal{W}_+}(T, S)$;
(v) *T is essentially Saphar and* $0 \notin \text{int } \sigma_{\mathcal{W}_+}(T, S)$.
(vi) *T is essentially Saphar and* $0 \notin \text{acc } \sigma_{B\mathcal{W}_+}(T, S)$;
(vii) *T is essentially Saphar and* $0 \notin \text{int } \sigma_{B\mathcal{W}_+}(T, S)$.

Proof. (i)$\Longrightarrow$(ii): Let T be left Weyl. Then T is left Fredholm, and hence according to [16, Theorem 16.21] there exists $(M, N) \in Red(T)$ such that $\dim N < \infty$, T_N is nilpotent and T_M is Kato. From Lemma 2.1 (i) it follows that T_M is Saphar. Lemma 2.3 provides that T is essentially Saphar. Using the fact that $\Phi_l(X)$ is open and Theorem 2.6 we conclude that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$, $0 < |\lambda| < \epsilon$ implies that $T - \lambda S \in \mathcal{W}_l(X)$. Consequently, $0 \notin \text{acc } \sigma_{\mathcal{W}_l}(T, S)$.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$(v)$\Longrightarrow$(vii), (ii)$\Longrightarrow$(iv)$\Longrightarrow$(vi)$\Longrightarrow$(vii) It is clear.

(vii)$\Longrightarrow$(i) Suppose that T is essentially Saphar and $0 \notin \text{int } \sigma_{B\mathcal{W}_+}(T, S)$. From Lemma 2.3 it follows that there exist $(M, N) \in Red(T)$ and $d \in \mathbb{N}_0$ such that $\dim N < \infty$, $T_N^d = 0$ and T_M is Saphar. Lemma 2.4 provides that T is quasi-Fredholm of degree d, and hence T has TUD for $n \geq d$. According to [9, Theorem 4.7] it follows that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ the following implication holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \begin{array}{c} \alpha_n(T - \lambda S) = \alpha_d(T), \\ \beta_n(T - \lambda S) = \beta_d(T), \text{ for every } n \in \mathbb{N}_0. \end{array} \tag{3.1}$$

As $\sigma_{B\mathcal{W}_+}(T, S)$ is closed, from $0 \notin \text{int } \sigma_{B\mathcal{W}_+}(T, S)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ and $T - \lambda S$ is an upper semi-B-Weyl operator. Therefore, there exists $n \in \mathbb{N}_0$ such that $R((T - \lambda S)^n)$ is closed and $(T - \lambda S)_n : R((T - \lambda S)^n) \to R((T - \lambda S)^n)$ is upper semi-Weyl. Consequently,

$$\alpha_n(T - \lambda S) = \dim(N(T - \lambda S) \cap R((T - \lambda S)^n)) = \alpha((T - \lambda S)_n) < \infty \tag{3.2}$$

and

$$\beta_n(T - \lambda S) = \dim(R((T - \lambda S)^n)/R((T - \lambda S)^{n+1})) = \beta((T - \lambda S)_n),$$

which implies

$$\alpha_n(T - \lambda S) - \beta_n(T - \lambda S) = \alpha((T - \lambda S)_n) - \beta((T - \lambda S)_n) \leq 0. \tag{3.3}$$

Now from (3.1), (3.2) and (3.3) we conclude that

$$\alpha_d(T) < \infty \text{ and } \alpha_d(T) \leq \beta_d(T). \tag{3.4}$$

According to Lemma 2.4 we have that

$$R(T) + N(T^d) = R(T_M) \oplus N$$

and

$$N(T) \cap R(T^d) = N(T_M),$$

and hence

$$\alpha_d(T) = \dim(N(T) \cap R(T^d) = \alpha(T_M), \quad \beta_d(T) = \operatorname{codim}(R(T) + N(T^n)) = \beta(T_M).$$

Now from (3.4) we obtain that $\alpha(T_M) < \infty$ and $\alpha(T_M) \leq \beta(T_M)$. Since T_M is relatively regular, it follows that T_M is left Weyl. As $\dim N < \infty$, we have that T_N is Weyl. According to Lemma 2.1 (iii) it follows that T is left Weyl. □

Theorem 3.2. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then the following statements are equivalent:*

(i) *T is right Weyl;*

(ii) *T is essentially Saphar and* $0 \notin \operatorname{acc} \sigma_{\mathcal{W}_r}(T, S)$*;*

(iii) *T is essentially Saphar and* $0 \notin \operatorname{int} \sigma_{\mathcal{W}_r}(T, S)$*;*

(iv) *T is essentially Saphar and* $0 \notin \operatorname{acc} \sigma_{\mathcal{W}_-}(T, S)$*;*

(v) *T is essentially Saphar and* $0 \notin \operatorname{int} \sigma_{\mathcal{W}_-}(T, S)$*.*

(vi) *T is essentially Saphar and* $0 \notin \operatorname{acc} \sigma_{B\mathcal{W}_-}(T, S)$*;*

(vii) *T is essentially Saphar and* $0 \notin \operatorname{int} \sigma_{B\mathcal{W}_-}(T, S)$*.*

Proof. Similarly to the proof of Theorem 3.1. □

Theorem 3.3. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then the following statements are equivalent:*

(i) *T is left Fredholm;*

(ii) *T is essentially Saphar and* $0 \notin \operatorname{acc} \sigma_{\Phi_l}(T, S)$*;*

(iii) *T is essentially Saphar and* $0 \notin \operatorname{int} \sigma_{\Phi_l}(T, S)$*;*

(iv) *T is essentially Saphar and* $0 \notin \operatorname{acc} \sigma_{\Phi_+}(T, S)$*;*

(v) *T is essentially Saphar and* $0 \notin \operatorname{int} \sigma_{\Phi_+}(T, S)$*;*

(vi) *T is essentially Saphar and* $0 \notin \operatorname{acc} \sigma^e_{D_+}(T, S)$*;*

(vii) *T is essentially Saphar and* $0 \notin \operatorname{int} \sigma^e_{D_+}(T, S)$*.*

Proof. (i)⟹(ii): It follows from the proof of the implication (i)⟹(ii) in Theorem 3.1.

(ii)⟹(iii)⟹(v)⟹(vii), (ii)⟹(iv)⟹(vi)⟹(vii) It is clear.

(vii)⟹(i) Suppose that T is essentially Saphar and $0 \notin \operatorname{int} \sigma^e_{D_+}(T, S)$. Then T has TUD for $n \geq d$, for some $d \in \mathbb{N}_0$, and according to [9, Theorem 4.7] it follows that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ the implication (3.1) holds. Since $\sigma^e_{D_+}(T, S)$ is closed, from $0 \notin \operatorname{int} \sigma^e_{D_+}(T, S)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ and $a_e(T - \lambda S) < \infty$. Hence there is an $n \in \mathbb{N}_0$ such that $\alpha_n(T - \lambda S) < \infty$. From (3.1) it follows that $\alpha_d(T) < \infty$. As in the proof of Theorem 3.1 we conclude that $\alpha(T_M) = \alpha_d(T)$, and so $\alpha(T_M) < \infty$. Consequently, T_M is left Fredholm. From $\dim N < \infty$, we have that T_N is Fredholm. Now according to Lemma 2.1 (ii) it follows that T is left Fredholm. □

Theorem 3.4. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then the following statements are equivalent:*

(i) *T is right Fredholm;*

(ii) *T is essentially Saphar and* $0 \notin \text{acc}\ \sigma_{\Phi_r}(T, S)$*;*

(iii) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma_{\Phi_r}(T, S)$*;*

(iv) *T is essentially Saphar and* $0 \notin \text{acc}\ \sigma_{\Phi_-}(T, S)$*;*

(v) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma_{\Phi_-}(T, S)$*;*

(vi) *T is essentially Saphar and* $0 \notin \text{acc}\ \sigma^e_{D_-}(T, S)$*;*

(vii) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma^e_{D_-}(T, S)$*;*

(viii) *T is essentially Saphar and* $0 \notin \text{acc}\ \sigma^e_{dsc}(T, S)$*;*

(ix) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma^e_{dsc}(T, S)$*.*

Proof. Similarly to the proof of Theorem 3.3 and Theorem 3.1. □

In the following two theorems we give some characterizations of left and right Browder operators by means of essentially Saphar decompositions.

Theorem 3.5. *Let* $T, S \in L(X)$*, and let S be invertible and commute with T. The following statements are equivalent:*

(i) *T is left Browder;*

(ii) *T is essentially Saphar and* $a(T) < \infty$*;*

(iii) *T is essentially Saphar and T has SVEP at* 0*;*

(iv) *T is essentially Saphar and there exists* $p \in \mathbb{N}$ *such that* $H_0(T) = N(T^p)$*;*

(v) *T is essentially Saphar and* $H_0(T)$ *is closed;*

(vi) *T is essentially Saphar and* $H_0(T) \cap K(T) = \{0\}$*;*

(vii) *T is essentially Saphar and* $H_0(T) \cap K(T)$ *is closed;*

(viii) *T is essentially Saphar and* $N(T^\infty) \cap R(T^\infty) = \{0\}$*.*

(ix) *T is essentially Saphar and* $0 \notin \text{acc}\,\sigma_l(T, S)$*,*

(x) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma_l(T, S)$*,*

(xi) *T is essentially Saphar and* $0 \notin \text{acc}\,\sigma_{\mathcal{B}_l}(T, S)$*,*

(xii) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma_{\mathcal{B}_l}(T, S)$*,*

(xiii) *T is essentially Saphar and* $0 \notin \text{acc}\,\sigma_{D_+}(T, S)$*,*

(xiv) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma_{D_+}(T, S)$*.*

(xv) *T is essentially Saphar and* $0 \notin \text{acc}\,\sigma_p(T, S)$*,*

(xvi) *T is essentially Saphar and* $0 \notin \text{int}\,\sigma_p(T, S)$*,*

Proof. (i)⟹(ii): Let T be left Browder. Then $a(T) < \infty$, T is left Fredholm, and hence T is essentially Saphar.

(ii)⟹(i): Let T be essentially Saphar and let $a(T) < \infty$. Then according to Lemma 2.3 there exists $(M, N) \in Red(T)$ such that $\dim N < \infty$, T_N is nilpotent and T_M is Saphar. From $a(T) < \infty$ it follows that $a(T_M) < \infty$, and so from [1, Lemma 1.19 (i)] it follows that $N(T_M) \cap R(T^{a(T_M)}) = \{0\}$. Since T_M is Saphar, we have that $N(T_M) \cap R(T^{a(T_M)}) = N(T_M)$, and hence $N(T_M) = \{0\}$. Thus T_M is left invertible. Now according to [18, Theorem 5] we conclude that T is left Browder.

(ii)⟺(iii): Suppose that T is essentially Saphar. Then T has TUD for $n \geq d$, and from [4, Theorem 2.5] it follows that $a(T) < \infty$ if and only if T has SVEP at 0.

(iii)⟺(iv)⟺(v)⟺(vi)⟺(vii)⟺(viii): It follows from [1, Theorem 2.79], [1, Corollary 2.66], [11, Theorem 3.2].

(i)⟹(ix): Suppose that T is left Browder. Then T is essentially Saphar. According to the equivalence between (5.1) and (5.2) in [18, Theorem 5] there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ satisfying $|\lambda| < \epsilon$ it follows that $T - \lambda S$ is left invertible. Consequently, $0 \notin \text{acc}\ \sigma_l(T, S)$.

The implications (ix)$\Longrightarrow$(x)$\Longrightarrow$(xvi), (ix)$\Longrightarrow$(xv)$\Longrightarrow$(xvi), (ix)$\Longrightarrow$(x)$\Longrightarrow$(xii)$\Longrightarrow$(xiv), (ix)$\Longrightarrow$(xi) $\Longrightarrow$(xiii)$\Longrightarrow$(xiv) are clear.

(xiv)$\Longrightarrow$(ii): Suppose that T is essentially Saphar and $0 \notin \operatorname{int} \sigma_{D_+}(T,S)$. Then there is a $d \in \mathbb{N}_0$ such that T has TUD for $n \geq d$. From [9, Theorem 4.7] it follows that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$ it holds:

$$0 < |\lambda| < \epsilon \Longrightarrow \alpha_n(T - \lambda S) \quad = \quad \alpha_d(T), \text{ for every } n \in \mathbb{N}_0. \tag{3.5}$$

As $\sigma_{D_+}(T,S)$ is closed, from $0 \notin \operatorname{int} \sigma_{D_+}(T,S)$ it follows that there exists a $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $T - \mu S$ is upper Drazin invertible. Hence there is $n \in \mathbb{N}_0$ such that $\alpha_n(T - \mu S) = 0$. Now according to (3.5) we obtain that $\alpha_d(T) = 0$, and hence $a(T) < \infty$.

(xvi)$\Longrightarrow$(ii): Suppose that T is essentially Saphar and $0 \notin \operatorname{int} \sigma_p(T,S)$. As in the previous part of the proof we conclude that there exists an $\epsilon > 0$ such that (3.5) holds. Since $0 \notin \operatorname{int} \sigma_p(T,S)$, there exists a $\mu \in \mathbb{C}$ such that $|\mu| < \epsilon$ and $T - \mu S$ is injective. If $\mu = 0$, then T is injective and $a(T) < \infty$. If $\mu \neq 0$, then $0 < |\mu| < \epsilon$ and from $\alpha_0(T - \mu S) = 0$ according to (3.5) it follows that $\alpha_d(T) = 0$, and hence $a(T) < \infty$. □

Theorem 3.6. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then the following statements are equivalent:*
(i) *T is right Browder;*
(ii) *T is essentially Saphar and $d(T) < \infty$;*
(iii) *T is essentially Saphar and T' has SVEP at 0;*
(iv) *T is essentially Saphar and there exists $q \in \mathbb{N}$ such that $K(T) = R(T^q)$;*
(v) *T is essentially Saphar and $H_0(T) + K(T) = X$;*
(vi) *T is essentially Saphar and $X = N(T^\infty) + R(T^\infty)$*
(vii) *T is essentially Saphar and $H_0(T) + K(T)$ is norm dense in X;*
(viii) *T is essentially Saphar and $N(T^\infty) + R(T^\infty)$ is norm dense in X.*
(ix) *T is essentially Saphar and $0 \notin \operatorname{acc} \sigma_r(T,S)$;*
(x) *T is essentially Saphar and $0 \notin \operatorname{int} \sigma_r(T,S)$;*
(xi) *T is essentially Saphar and $0 \notin \operatorname{acc} \sigma_{cp}(T,S)$;*
(xii) *T is essentially Saphar and $0 \notin \operatorname{int} \sigma_{cp}(T,S)$;*
(xiii) *T is essentially Saphar and $0 \notin \operatorname{acc} \sigma_{\mathcal{B}_r}(T,S)$;*
(xiv) *T is essentially Saphar and $0 \notin \operatorname{int} \sigma_{\mathcal{B}_r}(T,S)$;*
(xv) *T is essentially Saphar and $0 \notin \operatorname{acc} \sigma_{D_-}(T,S)$;*
(xvi) *T is essentially Saphar and $0 \notin \operatorname{int} \sigma_{D_-}(T,S)$;*
(xvii) *T is essentially Saphar and $0 \notin \operatorname{acc} \sigma_{dsc}(T,S)$;*
(xviii) *T is essentially Saphar and $0 \notin \operatorname{int} \sigma_{dsc}(T,S)$.*

Proof. (i)$\Longrightarrow$(ii): Let T be right Browder. Then $d(T) < \infty$ and T is right Fredholm. As in the proof of the implication (i)$\Longrightarrow$(ii) in Theorem 3.1 we obtain that T is essentially Saphar.

(ii)$\Longrightarrow$(i): Suppose that T is essentially Saphar and $d(T) < \infty$. Then there exists $(M,N) \in Red(T)$ such that $\dim N < \infty$, T_N is nilpotent and T_M is Saphar. From $d(T) < \infty$ it follows that $d(T_M) < \infty$, and so from [1, Lemma 1.19 (ii)] it follows that $N(T_M^{d(T_M)}) + R(T_M) = X$. Since T_M is Kato, we have that $N(T_M^{a(T_M)}) \subset R(T_M)$. Consequently, $R(T_M) = X$, and hence T_M is right invertible. From [18, Theorem 6] it follows that T is right Browder.

(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi)$\Longleftrightarrow$(vii)$\Longleftrightarrow$(viii): It follows from [1, Theorem 2.80], [1, Corollary 2.74] and [11, Theorem 3.4].

(i)$\Longrightarrow$(ix): It follows from the equivalence between (6.1) and (6.2) in [18, Theorem 6].

The implications (ix)$\Longrightarrow$(x)$\Longrightarrow$(xii), (ix)$\Longrightarrow$(xi)$\Longrightarrow$(xii), (ix)$\Longrightarrow$(x)$\Longrightarrow$(xiv)$\Longrightarrow$(xvi)$\Longrightarrow$ (xviii) and (ix)$\Longrightarrow$(xiii)$\Longrightarrow$(xv)$\Longrightarrow$(xvii)$\Longrightarrow$(xviii) are clear.

(xii)$\Longrightarrow$(ii): Suppose that T is essentially Saphar and $0 \notin \operatorname{int}\sigma_{cp}(T,S)$. Then there exists $(M,N) \in Red(T)$ such that $\dim N < \infty$, T_N is nilpotent and T_M is Saphar. Since T_M is Kato, we have that $R(T_M)$ is closed, and since $\dim R(T_N) < \infty$, we have that $R(T) = R(T_M) \oplus R(T_N)$ is closed. There is a $d \in \mathbb{N}_0$ such that T has TUD for $n \geq d$ and from [9, Theorem 4.7] it follows that there exists an $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$,

$$0 < |\lambda| < \epsilon \Longrightarrow \begin{array}{c} R(T-\lambda S) \text{ is closed,} \\ \beta_n(T-\lambda S) = \beta_d(T), \text{ for every } n \in \mathbb{N}_0. \end{array} \tag{3.6}$$

Since $0 \notin \operatorname{int}\sigma_{cp}(T,S)$, there exists a $\mu \in \mathbb{C}$ such that $|\mu| < \epsilon$ and $R(T-\mu S)$ is dense. As $R(T-\mu S)$ is closed, it follows that $R(T-\mu S) = X$. If $\mu = 0$, then $R(T) = X$ and $d(T) < \infty$. If $0 < |\mu| < \epsilon$, then from $\beta_0(T-\mu S) = 0$ according to (3.6) it follows that $\beta_d(T) = 0$, and hence $d(T) < \infty$.

(xviii)$\Longrightarrow$(ii): Suppose that T is essentially Saphar and $0 \notin \operatorname{int}\sigma_{dsc}(T,S)$. As in the previous part of the proof we conclude that there exists an $\epsilon > 0$ such that (3.6) holds. As $0 \notin \operatorname{int}\sigma_{dsc}(T,S)$ and $\sigma_{dsc}(T,S)$ is closed, there exists a $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $d(T-\mu S) < \infty$. It implies that there is $n \in \mathbb{N}_0$ such that $\beta_n(T-\mu S) = 0$, and so from $0 < |\mu| < \epsilon$ according to (3.6) it follows that $\beta_d(T) = 0$. Thus $d(T) < \infty$. □

4. Relationships between various parts of the S-spectrum

If $\mathcal{K} \subset L(X)$ the commutant of $\mathcal{K}$ is defined by

$$\operatorname{comm}(\mathcal{K}) = \{A \in L(X) : AB = BA \text{ for every } B \in \mathcal{K}\}.$$

The commutant of $T \in L(X)$ is $\operatorname{comm}(T) = \operatorname{comm}(\mathcal{K})$ with $\mathcal{K} = \{T\}$, and the double commutant is the commutant of the commutant:

$$\operatorname{comm}^2(T) = \operatorname{comm}(\operatorname{comm}(T)).$$

Theorem 4.1. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. If T is essentially Saphar, then there exists $\epsilon > 0$ such that $T - \lambda S$ is Saphar for each λ such that $0 < |\lambda| < \epsilon$.*

Proof. Suppose that T is essentially Saphar. There exists $(M,N) \in Red(T)$, such that $T = T_M \oplus T_N$, T_M is Saphar, $\dim N < \infty$ and T_N is nilpotent. If $M = \{0\}$, then T is nilpotent. Since $TS = ST$, from [16, Thorem 2.11] it foollows that

$$\sigma(T-\lambda S) \subset \sigma(T) - \lambda\sigma(S) = -\lambda\sigma(S), \text{ for every } \lambda \in \mathbb{C}. \tag{4.1}$$

As $0 \notin \sigma(S)$, from (4.1) it follows that $T - \lambda S$ is invertible for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Therefore, $T - \lambda S$ is Saphar for every $\lambda \neq 0$.

Suppose that $M \neq \{0\}$. Let $P \in L(X)$ be the projection such that $R(P) = M$ and $N(P) = N$. Then $TP = PT$, and hence $SP = PS$, which implies that $(M,N) \in Red(S)$.

From [16, Corollary 12.4 and Lemma 13.6] it follows that there exists an $\epsilon > 0$ such that for $|\lambda| < \epsilon$, $T_M - \lambda S_M$ is Saphar. Since T_N is nilpotent and S_N is invertible and commutes with T_N, as in the previous part of the proof we can conclude that $T_N - \lambda S_N$ is invertible for all $\lambda \neq 0$. Thus $T_N - \lambda S_N$ is Saphar for all $\lambda \neq 0$. Lemma 2.2 provides that $T - \lambda S$ is Saphar for each λ such that $0 < |\lambda| < \epsilon$. □

Corollary 4.2. *Let $T, S \in L(X)$, and let S be invertible and $S \in \operatorname{comm}^2(T)$. Then*

(i) *$\sigma_{eS}(T,S)$ is closed;*

(ii) *The set $\sigma_S(T,S) \setminus \sigma_{eS}(T,S)$ consists of at most countably many points.*

Proof. (i): It follows from Theorem 4.1.

(ii): Suppose that $\lambda_0 \in \sigma_S(T,S) \setminus \sigma_{eS}(T,S)$. Then $T - \lambda_0 S$ is essentially Saphar and applying Theorem 4.1 we get that there exists $\epsilon > 0$ such that $T - \lambda S$ is Saphar for each λ such that $0 < |\lambda - \lambda_0| < \epsilon$. This implies that $\lambda_0 \in \operatorname{iso}\sigma_S(T,S)$. Therefore, $\sigma_S(T,S) \setminus \sigma_{eS}(T,S) \subset \operatorname{iso}\sigma_S(T,S)$, which implies that $\sigma_S(T,S) \setminus \sigma_{eS}(T,S)$ is at most countable. □

The following theorem is an improvement of [18, Theorem 10] and [20, Theorem 3.5]. The proof presented here is more direct than the proof of the inclusions (1) in [20, Theorem 3.5].

Theorem 4.3. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then there are inclusions:*

(i)

$$\begin{array}{ccccccccc} & & \partial\sigma_{\mathcal{B}_l}(T,S) & \subset & \partial\sigma_{\mathcal{W}_l}(T,S) & \subset & \partial\sigma_{\Phi_l}(T,S) & & \\ & \subset & & \subset & & \subset & & \subset & \\ \partial\sigma_{\mathcal{B}}(T,S) & \subset & \partial\sigma_{\mathcal{W}}(T,S) & \subset & \partial\sigma_{\Phi}(T,S) & \subset & & & \partial\sigma_{\Phi_{l,r}}(T,S) \subset \partial\sigma_{eS}(T,S), \\ & \subset & & \subset & & \subset & & \subset & \\ & & \partial\sigma_{\mathcal{B}_r}(T,S) & \subset & \partial\sigma_{\mathcal{W}_r}(T,S) & \subset & \partial\sigma_{\Phi_r}(T,S) & & \end{array}$$

Furthermore, if S is invertible and $S \in \mathrm{comm}^2(T)$, then:

(ii) $\eta\sigma_{eS}(T,S) = \eta\sigma_H(T,S)$ *for each* $H = \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}$.

(iii) *The set $\sigma_H(T,S)$ consists of $\sigma_{eS}(T,S)$ and possibly some holes in $\sigma_{eS}(T,S)$ for each* $H = \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}$.

Proof. Since the following inclusions hold

$$\begin{array}{ccccccccc} & & \sigma_{\Phi_l}(T,S) & \subset & \sigma_{\mathcal{W}_l}(T,S) & \subset & \sigma_{\mathcal{B}_l}(T,S) & & \\ & \subset & & \subset & & \subset & & \subset & \\ \sigma_{eS}(T,S) \subset \sigma_{\Phi_{l,r}}(T,S) & \subset & & \subset & \sigma_{\Phi}(T,S) & \subset & \sigma_{\mathcal{W}}(T,S) & \subset & \sigma_{\mathcal{B}}(T,S), \\ & \subset & & \subset & & \subset & & \subset & \\ & & \sigma_{\Phi_r}(T,S) & \subset & \sigma_{\mathcal{W}_r}(T,S) & \subset & \sigma_{\mathcal{B}_r}(T,S) & & \end{array}$$

according to (2.1), for the inclusion (i) it is enough to prove that

$$\partial\sigma_H(T,S) \subset \sigma_{eS}(T,S) \ \text{ for each } \ H = \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}. \tag{4.2}$$

From the equivalence (i)$\Longleftrightarrow$(xii) in Theorem 3.5 it follows that

$$\begin{aligned} \lambda \notin \sigma_{\mathcal{B}_l}(T,S) \quad &\Longleftrightarrow \quad T - \lambda S \in \mathcal{B}_l(X) \\ &\Longleftrightarrow \quad T - \lambda S \text{ is essentially Saphar } \wedge\ 0 \notin \operatorname{int}\sigma_{\mathcal{B}_l}(T - \lambda S, S) \\ &\Longleftrightarrow \quad \lambda \notin \sigma_{eS}(T,S) \ \wedge\ \lambda \notin \operatorname{int}\sigma_{\mathcal{B}_l}(T,S), \end{aligned}$$

and hence $\sigma_{\mathcal{B}_l}(T,S) = \sigma_{eS}(T,S) \cup \operatorname{int}\sigma_{\mathcal{B}_l}(T,S)$. Similarly, applying Theorems 3.6, 3.1, 3.2, 3.3, 3.4, for each $H = \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}$, we obtain that

$$\sigma_H(T,S) = \sigma_{eS}(T,S) \cup \operatorname{int}\sigma_H(T,S). \tag{4.3}$$

As $\sigma_H(T,S)$ is closed, it follows that $\partial\sigma_H(T,S) \subset \sigma_H(T,S)$, which together with (4.3) implies (4.2).

Suppose that $S \in \mathrm{comm}^2(T)$. The set $\sigma_H(T,S)$ is closed for each $H = eS, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}, \Phi_l, \Phi_r, \Phi, \Phi_{l,r}, \mathcal{B}_l, \mathcal{B}_r, \mathcal{B}$ and from $\sigma_H(T,S) \subset \sigma(T,S) = \sigma(TS^{-1})$ it follows that $\sigma_H(T,S)$ is compact. Now (ii) and (iii) follow from (2.1), (4.2) and (4.3). □

Theorem 4.4. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then*

(i) iso $\sigma_{\Phi_l}(T,S) \subset$ iso $\sigma_{\Phi}(T,S) \cup \operatorname{int}\sigma^e_{dsc}(T,S)$;

(ii) iso $\sigma_{\Phi_r}(T,S) \subset$ iso $\sigma_{\Phi}(T,S) \cup \operatorname{int}\sigma^e_{D_+}(T,S)$;

(iii) iso $\sigma_{\mathcal{W}_l}(T,S) \subset$ iso $\sigma_{\mathcal{W}}(T,S) \cup \operatorname{int}\sigma_{B\mathcal{W}_-}(T,S)$;

(iv) iso $\sigma_{\mathcal{W}_r}(T,S) \subset$ iso $\sigma_{\mathcal{W}}(T,S) \cup \operatorname{int}\sigma_{B\mathcal{W}_+}(T,S)$;

(v) iso $\sigma_{\mathcal{B}_l}(T,S) \subset$ iso $\sigma_{\mathcal{B}}(T,S) \cup \operatorname{int}\sigma_{cp}(T,S)$;

(vi) iso $\sigma_{\mathcal{B}_r}(T,S) \subset$ iso $\sigma_{\mathcal{B}}(T,S) \cup \operatorname{int}\sigma_p(T,S)$;

(vii) iso $\sigma_{\mathcal{W}_l}(T,S) \subset$ iso $\sigma_{\mathcal{B}}(T,S) \cup \operatorname{int}\sigma_{dsc}(T,S)$;

(viii) iso $\sigma_{\mathcal{W}_r}(T,S) \subset$ iso $\sigma_{\mathcal{B}}(T,S) \cup \operatorname{int}\sigma_{D_+}(T,S)$.

Proof. (i): Let $\lambda_0 \in \text{iso}\, \sigma_{\Phi_l}(T,S) \setminus \text{int}\, \sigma^e_{dsc}(T,S)$. Then there exists a sequence (λ_n), $\lambda_n \to \lambda_0$ as $n \to \infty$, such that $d_e(T-\lambda_n S) < \infty$ and $T-\lambda_n S$ is left Fredholm for all $n \in \mathbb{N}$. Thus $\alpha(T-\lambda_n S) < \infty$, and so $a_e(T-\lambda_n S) = 0$, $n \in \mathbb{N}$. From [16, Lemma 22.11] it follows that $d_e(T-\lambda_n S) = a_e(T-\lambda_n S) = 0$, and so $\beta(T-\lambda_n S) < \infty$, $n \in \mathbb{N}$. Therefore, $T-\lambda_n S$ is Fredholm for every $n \in \mathbb{N}$ and hence $\lambda_0 \in \partial\sigma_\Phi(T,S)$. From Theorem 4.3 we have that $\partial\sigma_\Phi(T,S) \subset \sigma_{\Phi_l}(T,S)$ and from $\lambda_0 \in \text{iso}\, \sigma_{\Phi_l}(T,S) \cap \partial\sigma_\Phi(T,S)$ according to Lemma 2.7 (ii) we conclude that $\lambda_0 \in \text{iso}\, \sigma_\Phi(T,S)$.

(ii): It follows from the inclusion $\partial\sigma_\Phi(T,S) \subset \sigma_{\Phi_r}(T,S)$, similarly to the proof of (i).

(iii): According to Theorem 4.3 we have that

$$\partial\sigma_{\mathcal{W}}(T,S) \subset \sigma_{\mathcal{W}_l}(T,S). \tag{4.4}$$

Let $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{W}_l}(T,S) \setminus \text{int}\, \sigma_{B\mathcal{W}_-}(T,S)$. Then there exists a sequence (λ_n), $\lambda_n \to \lambda_0$ as $n \to \infty$, such that $T-\lambda_n S$ is left Weyl and lower semi-B-Weyl. Let $n \in \mathbb{N}$ be arbitrary. Then $\alpha(T-\lambda_n S) < \infty$, $\alpha(T-\lambda_n S) \leq \beta(T-\lambda_n S)$ and there is a $d_n \in \mathbb{N}_0$ such that $(T-\lambda_n S)_{d_n}$ is lower semi-Weyl. Hence $\beta_{d_n}(T-\lambda_n S) = \beta((T-\lambda_n S)_{d_n})) < \infty$, and so $d_e(T-\lambda_n S) < \infty$. As $\alpha(T-\lambda_n S) < \infty$, we have that $a_e(T-\lambda_n S) = 0$. According to [16, Lemma 22.11] it follows that $d_e(T-\lambda_n S) = a_e(T-\lambda_n S) = 0$, and so $\beta(T-\lambda_n S) < \infty$. Therefore, $T-\lambda_n S$ is Fredholm. Since $T-\lambda_n S$ is lower semi-B-Weyl, it is lower semi-Weyl [3, Proposition 2.1], and since $T-\lambda_n S$ is upper semi-Weyl, we conclude that it is Weyl. Hence $\lambda_0 \in \partial\sigma_{\mathcal{W}}(T,S)$. From (4.4) and $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{W}_l}(T,S) \cap \partial\sigma_{\mathcal{W}}(T,S)$ according to Lemma 2.7 (ii) we conclude that $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{W}}(T,S)$.

(iv): It follows from $\partial\sigma_{\mathcal{W}}(T,S) \subset \sigma_{\mathcal{W}_r}(T,S)$, similarly to the proof of (iii).

(v): Suppose that $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{B}_l}(T,S) \setminus \text{int}\, \sigma_{cp}(T,S)$. Then there exists a sequence (λ_n), $\lambda_n \to \lambda_0$ as $n \to \infty$, such that $T-\lambda_n S$ is left Browder and $R(T-\lambda_n S)$ is dense in X for all $n \in \mathbb{N}$. Since $T-\lambda_n S$ is relatively regular, its range is closed, and hence $T-\lambda_n S$ is surjective, that is $d(T-\lambda_n S) = 0$. As $a(T-\lambda_n S) < \infty$, from [1, Theorem 1.20] it follows that $a(T-\lambda_n S) = d(T-\lambda_n S) = 0$, and so $T-\lambda_n S$ is invertible for all $n \in \mathbb{N}$. It implies that $\lambda_0 \in \partial\sigma_{\mathcal{B}}(T,S)$. From Theorem 4.3 we have that $\partial\sigma_{\mathcal{B}}(T,S) \subset \sigma_{\mathcal{B}_l}(T,S)$ and hence from $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{B}_l}(T,S) \cap \partial\sigma_{\mathcal{B}}(T,S)$ according to Lemma 2.7 (ii) we conclude that $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{B}}(T,S)$.

(vi): It follows from $\partial\sigma_{\mathcal{B}}(T,S) \subset \sigma_{\mathcal{B}_r}(T,S)$, similarly to the proof of (v).

(vii): Let $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{W}_l}(T,S) \setminus \text{int}\, \sigma_{dsc}(T,S)$. Then there exists a sequence (λ_n), $\lambda_n \to \lambda_0$ as $n \to \infty$, such that $d(T-\lambda_n S) < \infty$ and $T-\lambda_n S$ is left Weyl for all $n \in \mathbb{N}$. Consequently, $\alpha(T-\lambda_n S) \leq \beta(T-\lambda_n S)$, $n \in \mathbb{N}$. Further, we have $\beta(T-\lambda_n S) \leq \alpha(T-\lambda_n S)$ by [1, Theorem 1.22, part (ii)], so $\alpha(T-\lambda_n S) = \beta(T-\lambda_n S) < \infty$. We conclude from [1, Theorem 1.22, part (iv)] that $a(T-\lambda_n S)$ is finite, hence that $T-\lambda_n S$ is Browder for all $n \in \mathbb{N}$. Thus $\lambda_0 \in \partial\sigma_{\mathcal{B}}(T,S)$. According to Theorem 4.3 we have that $\partial\sigma_{\mathcal{B}}(T,S) \subset \sigma_{\mathcal{W}_l}(T,S)$, and hence from $\lambda_0 \in \partial\sigma_{\mathcal{B}}(T,S) \cap \text{iso}\, \sigma_{\mathcal{W}_l}(T,S)$ according to Lemma 2.7 (ii) we conclude that $\lambda_0 \in \text{iso}\, \sigma_{\mathcal{B}}(T,S)$.

(viii): It can be proved by using the inclusion $\partial\sigma_{\mathcal{B}}(T,S) \subset \sigma_{\mathcal{W}_r}(T,S)$, similarly to the proof of (vii). □

Corollary 4.5. *Let $T, S \in L(X)$, and let S be invertible and commute with T. Then*

(i) $\sigma_\Phi(T,S) = \sigma_{\Phi_l}(T,S) \cup \text{int}\, \sigma^e_{dsc}(T,S)$;

(ii) $\sigma_\Phi(T,S) = \sigma_{\Phi_r}(T,S) \cup \text{int}\, \sigma^e_{D_+}(T,S)$;

(iii) $\sigma_{\mathcal{W}}(T,S) = \sigma_{\mathcal{W}_l}(T,S) \cup \text{int}\, \sigma_{B\mathcal{W}_-}(T,S)$;

(iv) $\sigma_{\mathcal{W}}(T,S) = \sigma_{\mathcal{W}_r}(T,S) \cup \text{int}\, \sigma_{B\mathcal{W}_+}(T,S)$;

(v) $\sigma_{\mathcal{B}}(T,S) = \sigma_{\mathcal{B}_l}(T,S) \cup \text{int}\, \sigma_{cp}(T,S)$;

(vi) $\sigma_{\mathcal{B}}(T,S) = \sigma_{\mathcal{B}_r}(T,S) \cup \text{int}\, \sigma_p(T,S)$.

Proof. (v): From Theorem 3.6 it follows that $\sigma_{\mathcal{B}_r}(T,S) = \sigma_{eS}(T,S) \cup \text{int}\, \sigma_{cp}(T,S)$ and hence $\sigma_{\mathcal{B}_l}(T,S) \cup \text{int}\, \sigma_{cp}(T,S) \subset \sigma_{\mathcal{B}_l}(T,S) \cup \sigma_{\mathcal{B}_r}(T,S) = \sigma_{\mathcal{B}}(T,S)$. To prove the converse inclusion suppose that $\lambda_0 \in \sigma_{\mathcal{B}}(T,S)$ and $\lambda_0 \notin \sigma_{\mathcal{B}_l}(T,S) \cup \text{int}\, \sigma_{cp}(T,S)$. As in the proof of the inclusion (v) in Theorem 4.4 we obtain that $\lambda_0 \in \partial\sigma_{\mathcal{B}}(T,S)$ and since $\partial\sigma_{\mathcal{B}}(T,S) \subset \sigma_{\mathcal{B}_l}(T,S)$ we get $\lambda_0 \in \sigma_{\mathcal{B}_l}(T,S)$, which is a contradiction.

For each of the remaining equalities (inclusions) we proceed in a similar way as in the proof of the corresponding inclusion in Theorem 4.4. □

Corollary 4.6. *Let $T, S \in L(X)$, and let S be invertible and and commute with T. Then*
(i) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\Phi_l}(T,S) \cup \text{int}\,\sigma^e_{D_+}(T,S)$;
(ii) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\Phi_r}(T,S) \cup \text{int}\,\sigma^e_{\text{dsc}}(T,S)$;
(iii) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\mathcal{W}_l}(T,S) \cup \text{int}\,\sigma_{B\mathcal{W}_+}(T,S)$;
(iv) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\mathcal{W}_r}(T,S) \cup \text{int}\,\sigma_{B\mathcal{W}_-}(T,S)$;
(v) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\mathcal{B}_l}(T,S) \cup \text{int}\,\sigma_p(T,S)$;
(vi) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\mathcal{B}_l}(T,S) \cup \text{int}\,\sigma_{D_+}(T,S)$;
(vii) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\mathcal{B}_r}(T,S) \cup \text{int}\,\sigma_{cp}(T,S)$;
(viii) $\text{iso}\,\sigma_{eS}(T,S) \subset \text{iso}\,\sigma_{\mathcal{B}_r}(T,S) \cup \text{int}\,\sigma_{dsc}(T,S)$.

Proof. (i): Suppose that $\lambda_0 \in \text{iso}\,\sigma_{eS}(T,S) \setminus \text{int}\,\sigma^e_{D_+}(T,S)$. Then there exists a sequence (λ_n), $\lambda_n \to \lambda_0$ as $n \to \infty$, such that $T - \lambda_n S$ is essentially Saphar and $\lambda_n \notin \sigma^e_{D_+}(T,S)$, and hence $\lambda_n \notin \text{int}\,\sigma^e_{D_+}(T,S)$, that is $0 \notin \text{int}\,\sigma^e_{D_+}(T - \lambda_n S, S)$ for all $n \in \mathbb{N}$. From Theorem 3.3 it follows that $T - \lambda_n S$ is left Fredholm for all $n \in \mathbb{N}$. Consequently, $\lambda_0 \in \partial\,\sigma_{\Phi_l}(T,S) \cap \text{iso}\,\sigma_{eS}(T,S)$. Since $\partial\,\sigma_{\Phi_l}(T,S) \subset \sigma_{eS}(T,S)$, from Lemma 2.7 (ii) it follows that $\lambda_0 \in \text{iso}\,\sigma_{\Phi_l}(T,S)$.

The inclusion (ii)-(viii) can be proved similarly by using Theorems 3.4, 3.1, 3.2, 3.5, 3.6 and Theorem 4.3. □

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