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Closedness of the set of all linear preservers of DSS-weak majorization on $\ell^p(I)$

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Abstract. It is provided that the set of all linear preservers of DSS-weak majorization on $\ell^p(I)$, $p \in [1, \infty)$ is closed under the norm topology, where I is an arbitrary non-empty set.

1. Introduction

Theory of majorization is a nice tool for developing new mathematical inequalities [1, 8, 10, 18] and it plays an important role in various branches of science [9] especially in quantum mechanics [19, 20]. In recent years, there is a big progress towards developing extensions of the most important majorization relations on sequence spaces [12, 21, 22] and discrete Lebesgue spaces [2, 5–7, 13, 15]. Also, linear preserver problems of majorization relations are discussed in [4, 11, 14, 16, 17].

DSS-weak majorization relations on $\ell^p(I)$ and their linear preservers are studied in [4, 16]. The aim of the paper is to present the constructive proof that the set of all linear preservers of DSS-weak majorization ($\prec\!\!\prec_w$) is norm-closed in the set of all bounded linear operators on $\ell^p(I)$, where I is an arbitrary not-empty set and $p \in [1, \infty)$. When I is finite, this conclusion will be obtained using the compactness of $DSS(\ell^p(I))$. When I is an infinite set, Example 2.2 shows that $DSS(\ell^p(I))$ is not compact. Closedness in this case will be proved using Theorem 2.3.

In the sequel, the set I will be an arbitrary not-empty set and $p \in [1, \infty)$, unless otherwise stated. The Banach space $\ell^p(I)$ contains all functions $f : I \longrightarrow \mathbb{R}$ that satisfy $\sum_{i \in I} |f(i)|^p < \infty$ and it is equipped with standard p-norm. The positive cone of the Banach space $\ell^p(I)$ is defined by

$$\ell^p(I)^+ := \{f \in \ell^p(I) : f(i) \geq 0, \forall i \in I\}.$$

The support of the function $f \in \ell^p(I)$ is a subset of I defined by

$$\text{supp}(f) := \{i \in I \mid f(i) \neq 0\},$$

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which is clearly at most a countable set. We recall that each function $f \in \ell^p(I)$ may be represented in the following form $f = \sum_{i\in I} f(i)e_i$ using Kronecker delta functions δ_{ij}, where $e_i(j) = \delta_{ij}$, $i \in I$ and $e_i : I \longrightarrow \mathbb{R}$.

We will consider operators defined on discrete Lebesgue spaces $\ell^p(I)$, where $p \in [1,\infty)$. A bounded linear operator $A : \ell^p(I) \to \ell^p(I)$ may be represented by a matrix $[a_{ij}]_{i,j\in I}$ which may be finite or infinite depends on cardinality of the set I. If we define matrix elements with $a_{ij} = \langle Ae_j, e_i\rangle$, $\forall i, j \in I$, where the map $\langle\cdot,\cdot\rangle : \ell^p(I) \times \ell^q(I) \longrightarrow \mathbb{R}$ defined by $\langle f, g\rangle = \sum\limits_{i\in I} f(i)g(i)$ is called the dual pairing, we get the matrix representation of the operator A in the following way:

$$Af(i) = \sum_{j\in I} a_{ij} f(j), \quad \forall i \in I, \tag{1}$$

that is,

$$Af = \sum_{i\in I}\left(\sum_{j\in I} a_{ij} f(j)\right) e_i.$$

Definition 1.1. *[2, Definition 2.1][13, Definition 3.1] Let $A : \ell^p(I) \longrightarrow \ell^p(I)$, $p \in [1,\infty)$ be a bounded linear operator, where I is a non-empty set. The operator A is called:*

- positive, *if $Af \in \ell^p(I)^+$, for each $f \in \ell^p(I)^+$;*
- doubly stochastic, *if A is positive,*
$$\forall i \in I \quad \sum_{j\in I}\langle Ae_j, e_i\rangle = 1, \quad \textit{and} \quad \forall j \in I \quad \sum_{i\in I}\langle Ae_j, e_i\rangle = 1;$$
- doubly substochastic, *if A is positive,*
$$\forall i \in I \quad \sum_{j\in I}\langle Ae_j, e_i\rangle \le 1, \quad \textit{and} \quad \forall j \in I \quad \sum_{i\in I}\langle Ae_j, e_i\rangle \le 1;$$
- a permutation, *if there exists a bijection $\theta : I \longrightarrow I$ for which $Ae_j = e_{\theta(j)}$, for each $j \in I$;*
- partial permutation *for sets $I_1 \subseteq I$ and $I_2 \subseteq I$, if there exists a bijection $\theta : I_1 \longrightarrow I_2$ such that*
$$Ae_j = \begin{cases} e_{\theta(j)}, & j \in I_1, \\ 0, & \textit{otherwise}. \end{cases}$$

The set of all doubly substochastic operators and the set of all partial permutations on $\ell^p(I)$ we will denote by $DSS(\ell^p(I))$ and $pP(\ell^p(I))$, respectively.

Definition 1.2. *[4, Definition 2.1] Let $f, g \in \ell^p(I)$. Function f is DSS-weakly majorized by g, which is denoted by $f \prec\!\!\prec_w g$, if $f = Dg$ for some doubly substochastic operator $D \in DSS(\ell^p(I))$. Similarly, function f is majorized by g, which is denoted by $f \prec g$, if $f = Dg$ for some doubly stochastic operator $D \in DS(\ell^p(I))$.*

Definition 1.3. *[4, 16] A bounded linear map $T : \ell^p(I) \to \ell^p(I)$, $p \in [1,\infty)$ is said to preserve DSS-weak majorization if for arbitrary chosen functions $f, g \in \ell^p(I)$ relation $f \prec\!\!\prec_w g$ implies $Tf \prec\!\!\prec_w Tg$. The set of all bounded linear preservers of DSS-weak majorization on $\ell^p(I)$ we will denote by $\mathcal{P}_{dw}(\ell^p(I))$.*

Theorem 1.4. [4, Theorem 2.9] Let I be an infinite set and let $f, g \in \ell^p(I)$, $p \in [1.\infty)$. The following conditions are equivalent:

i) $f \prec\!\!\prec_w g$ and $g \prec\!\!\prec_w f$;

ii) There are two partial permutations $P_1, P_2 \in pP(\ell^p(I))$ for sets $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ such that $f = P_1 g$ and $g = P_2 f$.

In the sequel, we will consider the map $P_\theta : \ell^p(I) \to \ell^p(I)$ defined by

$$P_\theta(f) := \sum_{k \in I} f(k) e_{\theta(k)}, \quad f \in \ell^p(I), \tag{2}$$

where $\theta : I \to I$ is a one-to-one function and I is an infinite set. Obviously, P_θ is a bounded linear operator on $\ell^p(I)$ with norm $\|P_\theta\| = 1$. Moreover, if θ is a surjective, then P_θ is a permutation.

Let I be an infinite set. The shape of linear preserves of standard majorization ($\prec$) is presented in [2, Theorem 4.9.] The result [4, Theorem 3.6] states that set of all linear preservers of standard majorization ($\prec$) and DSS-weak majorization ($\prec\!\prec_w$) coincide when $p \in (1, \infty)$. Linear preservers of DSS-weak majorization ($\prec\!\prec_w$), when $p = 1$ are characterized in [16, Theorem 2.3]. All these results are collected and presented in the next theorem.

Theorem 1.5. *[2, 4, 16] Let I be an infinite set and let $p \in [1, \infty)$. Suppose that $T : \ell^p(I) \to \ell^p(I)$ is a bounded linear operator. The following statements are equivalent:*

i) $T \in \mathcal{P}_{dw}(\ell^p(I))$;

ii) $Te_j \prec\!\prec_w Te_k$ *and* $Te_k \prec\!\prec_w Te_j$, $\forall k, j \in I$, *and for each* $i \in I$ *there is at most one* $j \in I$ *such that* $\langle Te_j, e_i \rangle \neq 0$;

iii) $T = \sum_{k \in I_0} \lambda_k P_{\theta_k}$, *where* $I_0 \subseteq I$ *is at most countable,* $(\lambda_k)_{k \in I_0} \in \ell^p(I_0)$, *and for all* $k \in I_0$, θ_k *belongs to the family* $\Theta = \{\theta_i : I \to I \mid i \in I_0\}$ *of one-to-one maps with mutually disjoint images* ($\theta_i(I) \cap \theta_j(I) = \emptyset$ *for all* $i, j \in I_0$ *with* $i \neq j$).

Corollary 1.6. [14, Corollary 4.1] Let I be an arbitrary finite non-empty set and let $p \in [1, \infty)$. The set $DSS(\ell^p(I))$ is a compact set.

We may prove Corollary 1.6 using the different approach in regards to the original proof of this result. Namely, when I is finite set, the unit ball of bounded linear operators on $\ell^p(I)$ is compact, hence the closed subset $DSS(\ell^p(I))$ is compact, too.

2. Closedness of $\mathcal{P}_{dw}(\ell^p(I))$

Theorem 2.1. *Let I be a finite set and let $p \in [1, \infty)$. The set $\mathcal{P}_{dw}(\ell^p(I))$ is a norm-closed subset of the set of all bounded linear operators on $\ell^p(I)$.*

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of operators $T_n \in \mathcal{P}_{dw}(\ell^p(I))$. Suppose that this sequence converges in norm to a bounded linear map $T : \ell^p(I) \to \ell^p(I)$. Let $f \prec\!\prec_w g$. Then $T_n f \prec\!\prec_w T_n g$, $\forall n \in \mathbb{N}$. It follows that there exists a sequence $(D_n)_{n \in \mathbb{N}}$, $D_n \in DSS(\ell^p(I))$, such that

$$D_n T_n g = T_n f. \tag{3}$$

Since the set $DSS(\ell^p(I))$ is compact by Corollary 1.6, we get that there exists a subsequence $(D_{n_j})_{j \in \mathbb{N}}$, $D_{n_j} \in DSS(\ell^p(I))$ and there is $D \in DSS(\ell^p(I))$ such that

$$\lim_{j \to \infty} D_{n_j} = D.$$

Clearly, $D_{n_j} T_{n_j} g = T_{n_j} f$, $\forall j \in \mathbb{N}$ by (3). Therefore,

$$Tf = \lim_{j \in \mathbb{N}} T_{n_j} f = \lim_{j \in \mathbb{N}} D_{n_j} T_{n_j} g = DTg,$$

which implies $Tf \prec\!\prec_w Tg$. $\square$

When I is an infinite set the next example shows that $DSS(\ell^p(I))$ is not compact set.

Example 2.2. Let $\mathbb{D}_n = \{d^n_{ij} \in \mathbb{R} : i, j \in \mathbb{N}\}$, $n \in \mathbb{N}$ be a sequence of families (i.e. infinite matrices) of real numbers defined by

$$d^n_{ij} := \begin{cases} \frac{1}{2}, & i = j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{N}$ the family $\mathbb{D}_n$ satisfies

$$\sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} |d^n_{ij}| = \frac{1}{2} < \infty \text{ and } \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |d^n_{ij}| = \frac{1}{2} < \infty$$

then using [15, Corollary 3.1] (when $I = \mathbb{N}$) these families may be considered as bounded linear operators on $\ell^p(\mathbb{N})$ for every $p \in [1, \infty)$ defined by

$$D_n f := \sum_{i\in\mathbb{N}} \left(\sum_{j\in\mathbb{N}} d^n_{ij} f(j) \right) e_i. \tag{4}$$

Since, above supremums are equal $\frac{1}{2}$, then using [15, Theorem 3.4] we get that

$$\|D_n\| \leq \max\{\sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} |d^n_{ij}|, \ \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |d^n_{ij}|\} = \frac{1}{2}.$$

On the other hand, $D_n e_n = \frac{1}{2} e_n$ implies $\|D_n\| \geq \frac{1}{2}$. Hence, $\|D_n\| = \frac{1}{2}$. Similarly if $k > m$ than

$$\|(D_k - D_m)e_k\| = \frac{1}{2}.$$

Clearly, the sequence $(D_n)_{n\in\mathbb{N}}$ is not convergent in norm and there is no convergent subsequence of $(D_n)_{n\in\mathbb{N}}$. It follows that $DSS(\ell^p(I))$ is not compact.

Thus, the closedness of the set $\mathcal{P}_{dw}(\ell^p(I))$ cannot be provided using above approach presented in Theorem 2.1 when I is an infinite set. We claim that $\mathcal{P}_{dw}(\ell^p(I))$ is closed and it will be proved using different technique presented below.

Theorem 2.3. *Let I be an infinite set, let $p \in [1, \infty)$. Suppose that $(f_n)_{n\in\mathbb{N}}$, $f_n \in \ell^p(I)$ and $(g_n)_{n\in\mathbb{N}}$, $g_n \in \ell^p(I)$ are two convergent sequences such that*

$$\lim_{n\to\infty} f_n = f \in \ell^p(I) \text{ and } \lim_{n\to\infty} g_n = g \in \ell^p(I).$$

If $f_n \lll_w g_n$ and $g_n \lll_w f_n$, $\forall n \in \mathbb{N}$, then there exists a partial permutation $P \in pP(\ell^p(I))$ for sets $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$, such that

$$f = Pg.$$

Moreover,

$$f \lll_w g \text{ and } g \lll_w f.$$

Proof. Since $f, g \in \ell^p(I)$, we have $\operatorname{card}(\operatorname{supp}(f)) \leq \aleph_0$ and $\operatorname{card}(\operatorname{supp}(g)) \leq \aleph_0$ which implies $\operatorname{card}(|f(I)|) \leq \aleph_0$ and $\operatorname{card}(|g(I)|) \leq \aleph_0$ (where $|f(J)| := \{|f(j)| : j \in J\}$, for some $J \subseteq I$). Because of this, we may define a (strictly) decreasing sequence $(l_n)_{n\in\mathbb{N}_0}$ of non-negative real numbers which converge to zero defined by

$$l_0 > \sup_{i\in I}\{|f(i)|, |g(i)| \mid i \in I\} \text{ and } l_n \notin |f(I)| \cup |g(I)|, \ n \in \mathbb{N}. \tag{5}$$

For any $h \in \ell^p(I)$, we define sets $I_h^n := \{i \in I : l_n < |h(i)| < l_{n-1}\}$, $\forall n \in \mathbb{N}$. It is easy to conclude that all sets I_h^n are finite, mutually disjoint and $\bigcup_{n\in\mathbb{N}} I_h^n = \operatorname{supp}(h)$.

Choose an arbitrary $m \in \mathbb{N}$. We claim that there is $N_f \in \mathbb{N}$ such that $I_f^m = I_{f_n}^m$, for each $n > N_f$.

There exists a real number $\delta > 0$ such that

$$l_m < l_m + \delta < |f(j)| < l_{m-1} - \delta < l_{m-1}$$

for each $j \in I_f^m$, because the set I_f^m is finite. Using $\lim_{n\to\infty} f_n = f$ and

$$\|f - f_n\|^p = \sum_{i\in I} |f(i) - f_n(i)|^p \geq |f(j) - f_n(j)|^p \quad \text{for each} \quad j \in I_f^m$$

we obtain that there is a $n_1 \in \mathbb{N}$ such that $|f(j) - f_n(j)| \leq \|f - f_n\| < \delta$, $\forall n > n_1$, $\forall j \in I_f^m$. Now, we obtain

$$l_m < |f(j)| - \delta < |f_n(j)| < |f(j)| + \delta < l_{m-1}$$

for each $j \in I_f^m$. Thus, $j \in I_{f_n}^m$, so $I_f^m \subseteq I_{f_n}^m$, $\forall n > n_1$.

Similarly as above, since all sets I_f^m are finite and using (5), we get that there exists a real number $\epsilon > 0$ such that

$$(l_m - \epsilon, l_{m-1} + \epsilon) \bigcap \left| f\left(\operatorname{supp} f \setminus I_f^m\right) \right| = \emptyset.$$

Also, there is a $n_2 \in \mathbb{N}$ such that $f_n(i) \in (f(i) - \epsilon/2, f(i) + \epsilon/2)$, $\forall i \in I$, $\forall n > n_2$. Hence, we conclude

$$(l_m, l_{m-1}) \bigcap \left| f_n\left(\operatorname{supp} f \setminus I_f^m\right) \right| = \emptyset,$$

whenever $n > n_2$. Therefore, if $i \in I_{f_n}^m$ that is $|f_n(i)| \in (l_m, l_{m-1})$, then using the above fact we obtain $i \notin \operatorname{supp} f \setminus I_f^m$, that is $i \in I_f^m$. Thus, for each $n > n_2$ we have $I_{f_n}^m \subseteq I_f^m$. Finally,

$$I_{f_n}^m = I_f^m, \quad \forall n > N_f := \max\{n_1, n_2\}.$$

Analogously, there exists $N_g \in \mathbb{N}$ such that $I_{g_n}^m = I_g^m$, $\forall n > N_g$. Therefore,

$$I_{f_n}^m = I_f^m \;\; \textit{and} \;\; I_{g_n}^m = I_g^m, \quad \text{for every} \;\; n > M := \max\{N_f, N_g\}. \tag{6}$$

Fix $n > M$ such that $|f_n(k) - f(k)| < \xi/2$ and $|g_n(k) - g(k)| < \xi/2$, $\forall k \in I$. Since, $f_n \lll_w g_n$ and $g_n \lll_w f_n$ we get that there are two partial permutations $P_1, P_2 \in pP(\ell^p(I))$ for sets $\operatorname{supp}(f_n)$ and $\operatorname{supp}(g_n)$ such that $f_n = P_1 g_n$ and $g_n = P_2 f_n$, by Theorem 1.4. Hence, there exists a bijection

$$\omega : \operatorname{supp}(f_n) \to \operatorname{supp}(g_n)$$

such that $f_n(i) = g_n(\omega(i))$, for each $i \in \operatorname{supp}(f_n)$. Using above argument, we have $g_n(\omega(k)) = f_n(k) \in (l_m, l_{m-1})$, $\forall k \in I_{f_n}^m = I_f^m$. Hence, $\omega(k) \in I_{g_n}^m = I_g^m$. If we assume that $i = \omega^{-1}(j) \in \operatorname{supp}(f_n) \setminus I_{f_n}^m$ for some $j \in I_g^m = I_{g_n}^m$, then $f_n(i) \notin (l_m, l_{m-1})$, so $f_n(i) \neq g_n(\omega(i))$ which is a contradiction with definition of ω. Using above argumments, the map

$$\omega_m : I_f^m \to I_g^m$$

defined by $\omega_m(i) := \omega(i)$, $\forall i \in I_f^m$ is a bijection from I_f^m to I_g^m. If $f(k) = g(j)$ holds $\forall k \in I_f^m$ and $\forall j \in I_g^m$, it is clear that $f(k) = g(\omega_m(k))$, $\forall k \in I_f^m$. Suppose that $f(k) \neq g(j)$ for some $k \in I_f^m$ and $j \in I_g^m$ and let

$$\xi := \min\{|f(k) - g(j)| : k \in I_f^m, j \in I_g^m, f(k) \neq g(j)\} > 0. \tag{7}$$

Using definitions of the maps ω and ω_m we have $f_n(k) = g_n(\omega(k)) = g_n(\omega_m(k))$ for every $k \in I_f^m$, therefore

$$\begin{aligned} |f(k) - g(\omega_m(k))| &\leq |f(k) - f_n(k)| + |f_n(k) - g(\omega_m(k))| \\ &= |f(k) - f_n(k)| + |g_n(\omega_m(k)) - g(\omega_m(k))| < \xi. \end{aligned}$$

Hence, $f(k) = g(\omega_m(k))$, $\forall k \in I_f^m$, by (7).

We recall that the above argument is true for each $m \in \mathbb{N}$. The sets I_f^m are mutually disjoint and $\bigcup_{m\in\mathbb{N}} I_f^m = \text{supp}(f)$, hence that the map $\Omega : \text{supp}(f) \to \text{supp}(g)$ defined by $\Omega(k) := \omega_m(k)$, whenever $k \in I_f^m$ is a bijection and $f(k) = g(\Omega(k))$, $\forall k \in \text{supp}(f)$. Therefore, there exists a partial permutation $P \in pP(\ell^p(I))$ for sets $\text{supp}(f)$ and $\text{supp}(g)$, corresponding to Ω, such that

$$f = Pg.$$

In the similar way we may provide that there is a partial permutation Q for sets $\text{supp}(f)$ and $\text{supp}(g)$ to be $g = Qf$. Clearly, $f \lll_w g$ and $g \lll_w f$ by Theorem 1.4.

□

Theorem 2.4. *Let I be an infinite set, and let $p \in [1, \infty)$. The set $\mathcal{P}_{dw}(\ell^p(I))$ is a norm-closed subset of the set of all bounded linear operators on $\ell^p(I)$.*

Proof. Let $p \in [1, \infty)$, and let $(T_k)_{k\in\mathbb{N}}$ be a sequence of maps $T_k \in \mathcal{P}_{dw}(\ell^p(I))$, which converges in norm to a bounded linear map $T : \ell^p(I) \to \ell^p(I)$. Clearly, $T_k e_i \lll_w T_k e_j$ and $T_k e_j \lll_w T_k e_i$. Since, $\lim_{k\to\infty} T_k e_i = Te_i$ and $\lim_{k\to\infty} T_k e_j = Te_j$ we get

$$Te_i \lll_w Te_j \ \textit{ and } \ Te_j \lll_w Te_i,$$

by Theorem 2.3.

Suppose that $\langle Te_{j_1}, e_i\rangle \neq 0$ and $\langle Te_{j_2}, e_i\rangle \neq 0$ for some $i, j_1, j_2 \in I$. Since $T_k \to T$, we get $\langle T_m e_{j_1}, e_i\rangle \neq 0$ and $\langle T_m e_{j_2}, e_i\rangle \neq 0$, for some $m \in \mathbb{N}$, which is in contradiction with $T_m \in \mathcal{P}_{dw}(\ell^p(I))$, by Theorem 1.5. Thus, for every $i \in I$ there is at most one $j \in I$ such that $\langle Te_j, e_i\rangle \neq 0$. Thus, $T \in \mathcal{P}_{dw}(\ell^p(I))$, by Theorem 1.5. □

Summary 2.5. *Thus, in this paper we provide the constructive proof that the set $\mathcal{P}_{dw}(\ell^p(I))$ of all linear preservers of DSS-weak majorization ($\lll_w$) is a norm-closed subset of the set of all bounded linear operators on $\ell^p(I)$, where $p \in]1, \infty)$. We recall that for each $p \in (1, \infty)$ linear preservers of standard majorization ($\prec$) and DSS-weak majorization ($\lll_w$) coincide when I is an infinite set, by [4, Theorem 3.6]. Also, in this case, the set of all linear preservers of standard majorization ($\prec$) is norm-closed by [3, Theorem 2.4], therefore closedness of the set $\mathcal{P}_{dw}(\ell^p(I))$ follows directly as a corollary. However, when $p = 1$, the above fact do not hold because linear preservers of DSS-weak majorization ($\lll_w$) on $\ell^1(I)$ do not have the same form as preservers of standard majorization ($\prec$) or weak majorization relations ($\prec_w$) and ($\prec^w$), when I is an infinite set. Thus, in this case, Theorem 2.4 cannot be obtained as a consequence of previously published results.*

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