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A set of S-asymptotically omega periodic functions in the Stepanov sense

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Abstract. In this paper, we give sufficient conditions for the existence and uniqueness of S -Asymptotically ω-periodic solutions for a nonlinear differential equation with generalized type argument (EGTA) in a Banach space. This is done using the Banach fixed point Theorem.

1. Introduction

This work is concerned with the existence of asymptotically ω-periodic and S asymptotically ω-periodic solutions, valued in a Banach space $\mathbb{X}$, to the following differential equation of generalized type argument (EGTA):

$$\begin{cases} x'(t) = Ax(t) + \sum_{j=1}^{N} A_j x(\varphi_j(t)) + f(t, x(\varphi_{N+1}(t))); \\ x(0) = c_0. \end{cases} \tag{1}$$

Here, $c_0 \in \mathbb{X}$, f is a continuous function on $\mathbb{R}^+ \times \mathbb{X}$, A generates a semi-group exponentially stable and for $j = 1, ..., N+1$, $\varphi_j(t)$ is a measurable real function and Aj is a bounded linear operator. Moreover each function φ_j is subject to the functional relation: $\varphi_j(t+s) = \varphi_j(t) + rs$; $(t,s) \in \mathbb{R} \times \mathbb{S}_j$, where $\mathbb{S}_j$ is a subset of $\mathbb{R}$ and r is a positive integer.

The existence of almost periodic, almost automophic ([5, 15]), asymptotically ω-periodic ([7, 10, 12, 23–25]) and S asymptotically ω-periodic fonctions ([3, 6, 8, 9, 11]) is one of the most attracting topics in the qualitative theory of differential equations, due both to its mathematical interest and to the applications. The study of differential equations with piecewise constant argument (EPCA) is an important subject because these equations have the structure of continuous dynamical systems in intervals of unit length. Therefore they combine the properties of both differential and difference equations. Application of these equation to the problems of biology can be found in [1, 2, 4].

There have been many papers studying EPCA, see for instance [13, 18–22] and the references therein. However, to the best of the authors's knowledge, the theory of differential equation with generalized type

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argument who generalises the concept of differential equation with piecewise constant argument is recent, and there is no paper dealing with the existence of S asymptotically ω-periodic solution of these differential equations. In this paper, we give sufficient conditions for the existence of asymptotically ω-periodic and S asymptotically ω-periodic solution of (1) considering asymptotically ω-periodic functions in the Stepanov sense, S asymptotically ω-periodic functions in the Stepanov sense and the Banach fixed point theorem.
This paper is organized as follows. In Section 2, we recall the concepts of asymptotically ω-periodic functions, asymptotically ω-periodic functions in the Stepanov sense, S asymptotically ω-periodic functions, S asymptotically ω-periodic functions in the Stepanov sense and their basic properties. In Section 3, we present some results showing the existence of functions wich are not asymptotically ω-periodic but asymptotically ω-periodic in the Stepanov sense. In section 4, we study the existence and uniqueness of asymptotically ω-periodic and S asymptotically ω-periodic solution of the equation (1).

2. Preliminaries

Let $\mathbb{X}$ be a Banach space. We denote by $BC(\mathbb{R}^+,\mathbb{X})$ the space of the continuous bounded functions from $\mathbb{R}^+$ into $\mathbb{X}$. Endowed with the norm $\|f\|_\infty := \sup_{t\geq 0}\|f(t)\|$, $BC(\mathbb{R}^+,\mathbb{X})$ is a Banach space. Set $C_0(\mathbb{R}^+,\mathbb{X}) = \{f \in BC(\mathbb{R}^+,\mathbb{X}) : \lim_{t\to\infty} f(t) = 0\}$ and $P_\omega(\mathbb{R}^+,\mathbb{X}) = \{f \in BC(\mathbb{R}^+,\mathbb{X}) : f \textit{ is } \omega\textit{-periodic}\}$.

Definition 2.1. *A function $f \in BC(\mathbb{R}^+,\mathbb{X})$ is said to be asymptotically ω-periodic if it can be expressed as $f = g + h$, where $g \in P_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in C_0(\mathbb{R}^+,\mathbb{X})$. The collection of such functions will be denoted by $AP_\omega(\mathbb{R}^+,\mathbb{X})$.*

Theorem 2.2. *[23] Let $f \in BC(\mathbb{R}^+,\mathbb{X})$ and $\omega > 0$. Then the following statements are equivalent:*

(i) $f \in AP_\omega(\mathbb{R}^+,\mathbb{X})$
(ii) $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *uniform on* $\mathbb{R}^+$*;*
(iii) $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *uniformly on compact subset of* $\mathbb{R}^+$*;*
(iv) $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *is well defined for each* $t \in \mathbb{R}^+$ *and* $g(t) = \lim_{n\to\infty} f(t + n\omega)$ *uniformly on* $[0,\omega]$.

Let $p \in [1,\infty[$. The space $BS^p(\mathbb{R}^+,\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{X}$ such that $f^b \in \mathbb{L}^\infty(\mathbb{R}, L^p([0,1];\mathbb{X}))$, where$f^b$ is the Bochner transform of f defined by $f^b(t,s) := f(t+s), t \in \mathbb{R}^+, s \in [0,1]$. Then $BS^p(\mathbb{R}^+,\mathbb{X})$ is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{\mathbb{L}^\infty(\mathbb{R}^+,L^p)} = \sup_{t\in\mathbb{R}^+}\Big(\int_t^{t+1}\|f(\tau)\|^p d\tau\Big)$$

Obviously, if $p \geq q \geq 1$ we have $L^p(\mathbb{R},\mathbb{X}) \subset BS^p(\mathbb{R},\mathbb{X}) \subset L^p_{loc}(\mathbb{R},\mathbb{X})$ and $BS^p(\mathbb{R},\mathbb{X}) \subset BS^q(\mathbb{R},\mathbb{X})$. Define the subspaces of $BS^p(\mathbb{R}^+,\mathbb{X})$ by

$$S^pP_\omega(\mathbb{R}^+,\mathbb{X}) = \{f \in BS^p(\mathbb{R}^+,\mathbb{X}) : \int_t^{t+1}\|f(s+\omega) - f(s)\|^p ds = 0, t \geq 0\}$$

and

$$BS^p_0(\mathbb{R}^+,\mathbb{X}) = \{f \in BS^p(\mathbb{R}^+,\mathbb{X}) : \lim_{t\to\infty}\int_t^{t+1}\|f(s)\|^p ds = 0\}.$$

Definition 2.3. *[23] A function $f \in BS^p(\mathbb{R}^+,\mathbb{X})$ is called asymptotically ω-periodic in the Stepanov sense if it can be expressed as $f = g + h$, where $g \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ and $h \in BS^p_0(\mathbb{R}^+,\mathbb{X})$. The collection of such functions will be denoted by $S^pAP_\omega(\mathbb{R}^+,\mathbb{X})$.*

Definition 2.4. *[23] A function $f \in BS^p(\mathbb{R}^+\times\mathbb{X})$ with $f(t,x) \in L^p_{loc}(\mathbb{R}^+,\mathbb{X})$ for each $x \in \mathbb{X}$ is said to be asymptotically ω-periodic in the Stepanov sense uniformly on bounded sets of $\mathbb{X}$ if there exists a function $g : \mathbb{R}_+ \times \mathbb{X} \to \mathbb{X}$ with $g(t,x) \in S^pP_\omega(\mathbb{R}^+,\mathbb{X})$ for each $x \in \mathbb{X}$ such that for every bounded set $K \subset \mathbb{X}$ we have*

$$\Big(\int_t^{t+1}\|f(s+n\omega,x) - g(s,x)\|^p ds\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$ uniformly for $x \in K$. The collection of such functions will be denoted by $S^pAP_\omega(\mathbb{R}^+\times\mathbb{X})$.

Theorem 2.5. *[23] Let $f \in L^p_{loc}(\mathbb{R}^+, \mathbb{X})$ and $\omega > 0$. Then the following statements are equivalent:*

(i) $f \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$;

(ii) *There exists a function $g \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ such that $\int_t^{t+1} \|f(s + n\omega) - g(s)\|^p ds \to 0$ as $n \to \infty$ uniformly for $t \in \mathbb{R}^+$;*

(iii) *There exists a function $g \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ such that $\int_t^{t+1} \|f(s + n\omega) - g(s)\|^p ds \to 0$ as $n \to \infty$ pointwise for $t \in \mathbb{R}^+$.*

Lemma 2.6. *[23] Suppose $f \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$, $f = g + h$ where $g \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ and $h \in BS^p_0(\mathbb{R}^+, \mathbb{X})$. Let $\omega = n_0 + \theta$, where $n_0 \in \mathbb{N}$ and $\theta \in (0, 1)$. Then the following statements are true.*

(i) $\int_t^{t+\omega} \|f(s)\|ds \leq (n_0 + 1)\|f\|_{S^p}$ *for each* $t \in \mathbb{R}^+$;

(ii) $\int_t^{t+\omega} \|g(s + m\omega) - g(s)\|ds = 0$ *for each* $t \in \mathbb{R}^+$ *and any* $m \in \mathbb{N}$;

(iii) $\displaystyle\lim_{n\to\infty} \int_t^{t+\omega} \|h(s + n)\|ds = 0$ *uniformly for* $t \in \mathbb{R}^+$.

Proposition 2.7. *[7] Let $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$ where $\omega \in \mathbb{N}^*$. Then the function $t \mapsto u([t + k])$, where $k \in \mathbb{N}$ is Asymptotically ω-periodic in the Stepanov sense but is not Asymptotically ω-periodic.*

Theorem 2.8. *[7] Let $\omega \in \mathbb{N}^*$. Let $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ be a continuous function such that:*

(i) $\forall\ (t, x) \in \mathbb{R} \times \mathbb{X},\ f(t + \omega, x) = f(t, x)$;

(ii) $\exists L_f > 0,\ \forall (t, x) \in \mathbb{R} \times \mathbb{X}$
$$\|f(t, x) - f(t, y)\| \leq L_f\|x - y\|.$$

If $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$, then the function $t \mapsto f(t, u([t]))$ is Asymptotically ω-periodic in the Stepanov sense but is not Asymptotically ω-periodic.

Lemma 2.9. *[7] Let $\omega \in \mathbb{N}^*$. Assume that $f \in S^pAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ and assume that f satisfies a Lipschitz condition in $\mathbb{X}$ uniformly in $t \in \mathbb{R}^+$:*
$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$
for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant. Let $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$. Then the function $F : \mathbb{R}^+ \to \mathbb{X}$ defined by $F(t) = f(t, u([t]))$ is asymptotically ω-periodic in the Stepanov sense.

Now, we give the definition of S Asymptotically ω-periodic functions.

Definition 2.10. *([11]) A function $f \in BC(\mathbb{R}^+, \mathbb{X})$ is called S-asymptotically ω periodic if there exists ω such that $\lim_{t\to\infty}(f(t + \omega) - f(t)) = 0$. In this case we say that ω is an asymptotic period of f and that f is S-asymptotically ω periodic. The set of all such functions will be denoted by $SAP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Definition 2.11. *([11]) A continuous function $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K^* \subset \mathbb{X}$, the set $\{f(t, x) : t \geq 0, x \in K^*\}$ is bounded and*
$$\lim_{t\to\infty}(f(t, x) - f(t + \omega, x)) = 0$$
uniformly in $x \in K^*$.

Definition 2.12. *([11]) A continuous function $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is said to be asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and every bounded set K^*, there exist $L_{\varepsilon,K^*} > 0$ and $\delta_{\varepsilon,K^*} > 0$ such that $\|f(t, x) - f(t, y)\| < \varepsilon$ for all $t \geq L_{\varepsilon,K^*}$ and all $x, y \in K^*$ with $\|x - y\| < \delta_{\varepsilon,K^*}$.*

Lemma 2.13. *([3]) Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, and denote by $B(\mathbb{X}, \mathbb{Y})$, the space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$. Let $A \in B(\mathbb{X}, \mathbb{Y})$. Then when $f \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, we have $Af := [t \mapsto Af(t)] \in SAP_\omega(\mathbb{R}^+, \mathbb{Y})$.*

Lemma 2.14. *([11]) Let $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ be a function which is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u : \mathbb{R}^+ \to \mathbb{X}$ be a S-asymptotically ω-periodic function. Then the Nemytskii operator $\phi(\cdot) := f(\cdot, u(\cdot))$ is a S-asymptotically ω-periodic function.*

Lemma 2.15. *([25]) Assume $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ be a function which is uniformly S-asymptotically ω periodic on bounded sets and satisfies the Lipschitz condtion, that is, there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \forall t \geq 0, \forall x, y \in \mathbb{X}.$$

If $u \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, then the function $f(t, u(t))$ belongs to $SAP_\omega(\mathbb{R}^+, \mathbb{X})$.

Let $p \in [0, \infty[$. The space $BS^p(\mathbb{R}^+, \mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{X}$ such that $f^b \in \mathbb{L}^\infty(\mathbb{R}, L^p([0,1]; \mathbb{X}))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s), t \in \mathbb{R}^+, s \in [0,1]$. It is well-known that $BS^p(\mathbb{R}^+, X)$ is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{\mathbb{L}^\infty(\mathbb{R}^+, L^p)} = \sup_{t \in \mathbb{R}^+} \Big(\int_t^{t+1} \|f(\tau)\|^p d\tau \Big)^{\frac{1}{p}}.$$

It is obvious that $L^p(\mathbb{R}^+, \mathbb{X}) \subset BS^p(\mathbb{R}^+, \mathbb{X}) \subset L^p_{loc}(\mathbb{R}^+)$ and $BS^p(\mathbb{R}^+, \mathbb{X}) \subset BS^q(\mathbb{R}^+, \mathbb{X})$ for $p \geq q \geq 1$. We denote by $BS^p_0(\mathbb{R}^+)$ the subspace of $BS^p(\mathbb{R}^+, \mathbb{X})$ consisting of functions f such that $\int_t^{t+1} \|f(s)\|^p ds \to 0$ when $t \to \infty$. Now we give the definition of S-asymptotically ω-periodic functions in the Stepanov sense.

Definition 2.16. *[12] A function $f \in BS^p(\mathbb{R}^+, \mathbb{X})$ is called S-asymptotically ω-periodic in the Stepanov sense (or S^p-S-asymptotically ω-periodic) if*

$$\lim_{t \to \infty} \int_t^{t+1} \|f(s+\omega) - f(s)\|^p ds = 0.$$

Denote by $S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$ the set of such functions.

Remark 2.1. *It is easy to see that $SAP_\omega(\mathbb{R}^+, X) \subset S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Definition 2.17. *[12] A function $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is said to be uniformly S-asymptotically ω-periodic on bounded sets in the Stepanov sense if for every bounded set $B \subset \mathbb{X}$, there exist positive functions $g_b \in BS^p(\mathbb{R}^+, \mathbb{R})$ and $h_b \in BS^p_0(\mathbb{R}^+, \mathbb{R})$ such that $f(t,x) \leq g_b(t)$ for all $t \geq 0$, $x \in B$ and $\|f(t+\omega, x) - f(t,x)\| \leq h_b(t)$ for all $t \geq 0$, $x \in B$.*

Denote by $S^pSAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ the set of such functions.

Definition 2.18. *[12] A function $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is said to be asymptotically uniformly continuous on bounded sets in the Stepanov sense if for every $\varepsilon > 0$ and every bounded set $B \subset X$, there exist $t_\varepsilon \geq 0$ and $\delta_\varepsilon > 0$ such that*

$$\int_t^{t+1} \|f(s,x) - f(s,y)\|^p ds \leq \varepsilon^p,$$

for all $t \geq t_\varepsilon$ and all $x, y \in B$ with $\|x - y\| \leq \delta_\varepsilon$.

Lemma 2.19. *[12] Assume that $f \in S^pSAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ is an asymptotically uniformly continuous on bounded sets in the Stepanov sense function. Let $u \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, then $v(.) = f(., u(.)) \in S^pSAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$.*

Corollary 2.20. *[8] Let $u \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$ where $\omega \in \mathbb{N}^*$, then the function $t \mapsto u([t])$ is S-asymptotically ω-periodic in the Stepanov sense but is not S-asymptotically ω-periodic.*

Lemma 2.21. *[8] Let $\omega \in \mathbb{N}^*$. Assume $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ be a function which is uniformly S-asymptotically ω periodic on bounded sets and satisfies the Lipschitz condition, that is, there exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \forall t \geq 0, \forall x, y \in \mathbb{X}.$$

If $u \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, then

(1) *the bounded piecewise continuous function* $t \mapsto f(t, u(\left[t\right]))$ *satisfies*

$$\lim_{t\to\infty}(f(t+\omega, u(\left[t+\omega\right])) - f(t, u(\left[t\right])) = 0.$$

(2) *the function* $t \mapsto f(t, u(\left[t\right]))$ *belongs to* $S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$.

(3) *the function* $t \mapsto f(t, u(\left[t\right]))$ *does not belongs to* $SAP_\omega(\mathbb{R}^+, \mathbb{X})$.

Lemma 2.22. *[8] Let* $\omega \in \mathbb{N}^*$. *Assume that* $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ *is uniformly S-asymptotically* ω*-periodic on bounded sets in the Stepanov sense and asymptotically uniformly continuous on bounded sets in the Stepanov sense. Let* $u : \mathbb{R}^+ \to \mathbb{X}$ *be a function in* $SAP_\omega(\mathbb{R}^+, \mathbb{X})$, *and let* $v(t) = f(t, u([t]))$. *Then* $v \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$.

Lemma 2.23. *[23] Let* $\{S(t)\}_{t\geq 0} \subset B(\mathbb{X})$ *be a strongly family of bounded and linear operators such that* $\|S(t)\| \leq \phi(t),\ t \in \mathbb{R}^+$, *where* $\phi \in L^1(\mathbb{R}^+)$ *is nonincreasing. Let* $f \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$, *then* $u(t) = \int_0^t S(t-s)f(s)ds \in AP_\omega(\mathbb{R}^+, \mathbb{X})$.

Lemma 2.24. *[24] Let* $\{S(t)\}_{t\geq 0} \subset B(\mathbb{X})$ *be a strongly family of bounded and linear operators such that* $\|S(t)\| \leq \phi(t),\ t \in \mathbb{R}^+$, *where* $\phi \in L^1(\mathbb{R}^+)$ *is nonincreasing. Let* $f \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$, *then* $u(t) = \int_0^t S(t-s)f(s)ds \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$.

3. Main Results

Definition 3.1. *A solution of* (1) *on* $\mathbb{R}^+$ *is a function* $x(t)$ *that satisfies the conditions:*

(1) $x(t)$ *is continuous on* $\mathbb{R}^+$.
(2) *The derivative* $x'(t)$ *exists at each point* $t \in \mathbb{R}^+$, *with the possible exception of the points* $t_n \in \mathbb{R}^+$ *where one-sided derivatives exists.*
(3) *The equation (1) is satisfied on each interval* $[t_n, t_{n+1}[$ *with* $n \in \mathbb{N}$.

We assume that A generates a semi-group $(T(t))$ in $\mathbb{X}$. Then the function g defined by $g(s) = T(t-s)x(s)$, where x is a solution of (1), is differentiable for $t \geq 0$ and we have:

$$\begin{aligned} \frac{dg(s)}{ds} &= -AT(t-s)x(s) + T(t-s)\frac{d\,x(s)}{ds} \\ &= -AT(t-s)x(s) + T(t-s)Ax(s) \\ &+ \sum_{j=0}^{N} T(t-s)\,A_j x(\varphi_j(s)) + T(t-s)f(s, x(\varphi(s))) \end{aligned}$$

which gives

$$\frac{dg(s)}{ds} = \sum_{j=0}^{N} T(t-s)\,A_j x(\varphi_j(s)) + T(t-s)f(s, x(\varphi(s))). \tag{2}$$

The functions $x(\varphi_j(s))$ are a step functions. Therefore for all $j = 0, ..., N,\ T(t-s)\,A_0 x(\varphi_j(s))$ is integrable on $[0, t[$. By **(H1)**, $f(s, x(\varphi(s)))$ is piecewise continuous. Therefore $f(s, x(\varphi(s)))$ is integrable on $[0, t]$ where $t \in \mathbb{R}^+$. Integrating (2) on $[0, t]$ we obtain that

$$x(t) - T(t)c_0 = \sum_{j=0}^{N} \int_0^t T(t-s)\,A_j x(\varphi_j(s))ds + \int_0^t T(t-s)f(s, x(\varphi(s)))ds.$$

Therefore, we define

Definition 3.2. *We assume* **(H1)** *is satisfied and that A generates a semi-group $T(t)$) in $\mathbb{X}$. The continuous function x given by*

$$x(t) = T(t)c_0 + \sum_{j=0}^{N} \int_0^t T(t-s)\,A_j x(\varphi_j(s))ds + \int_0^t T(t-s) f(s, x(\varphi(s)))ds$$

is called the mild solution of equation (1).

Now we make the following hypothesis.

(H1): A generates an exponentially stable semi-group $T(t)$ in $\mathbb{X}$:

(1) $T(0) = I$ for all $t \geq 0$ where I is the identity operator.
(2) $T(t+s) = T(t)T(s)$ for all $t \geq s \geq 0$.
(3) The map $t \mapsto T(t)x$ is continuous for every fixed $x \in \mathbb{X}$.
(4) There exist $K > 0$ and $a > 0$ such that $\|T(t)\| \leq Ke^{-at}$ for $t \geq 0$.

Let $\mathbb{S}$ denote a subset of $\mathbb{R}$ which is not void and $\{0\}$. For every non zero real number r we consider the function $\theta_r : \mathbb{R} \to \mathbb{R}$ such that for every $(t, s) \in \mathbb{R} \times \mathbb{S}$:

$$\theta_r(t+s) = \theta_r(t) + rs. \tag{3}$$

In particular for all $s \in \mathbb{S}$, we have:

$$\theta_r(s) = rs + \theta_r(0).$$

Definition 3.3. *A subset of $\mathbb{R}$ is said to be r-stable if it is invariant under the homothety of ratio r and center* 0*.*

We give an example of a subset $\mathbb{S}$ of $\mathbb{R}$ which is r-stable with an associated function θ_r.

Example 3.4. *Let $\mathbb{S}$ be a discrete subgroup of $\mathbb{R}$. Then there exists a non negative real α such that $\mathbb{S} = \alpha\mathbb{Z}$ which shows that $\mathbb{S}$ is r-stable for all non zero integer r. Set $\theta_r(t) = [rt/\alpha]\alpha + c$ whith $[.]$ the integer part function and c is some constant. Then it is easily seen that* (3) *is satisfied.*

Definition 3.5. *A function $f : \mathbb{R} \to \mathbb{X}$ is be said to be $\mathbb{S}$-continuous if it is continuous in $\mathbb{R} \setminus \mathbb{S}$, which is refered as a $\mathbb{S}$-continuous function.*

The set of all $\mathbb{S}$-continuous function $f : \mathbb{R}^+ \to \mathbb{X}$ will be denoted by $\mathbb{S}C(\mathbb{R}^+, \mathbb{X})$ and those that are bounded will be denoted by $\mathbb{S}C_b(\mathbb{R}^+, \mathbb{X})$.
Now we make the following hypothesis, where $j \in \{1, ..., N+1\}$:

(H2j): φ_j is an increasing function such that for every $(t, s) \in \mathbb{R} \times \mathbb{S}_j$, $\varphi_j(t+s) = \varphi_j(t) + rs$, where $r \in \mathbb{N}^*$.

(H3j): φ_j is an increasing function such that for every $(t, s) \in \mathbb{R} \times \mathbb{S}_j$, $\varphi_j(t+s) = \varphi_j(t) + s$.

(H4) $f \in S^p AP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ and f satisfies a Lipschitz condition in $\mathbb{X}$ uniformly in $t \in \mathbb{R}^+$:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant.

(H5) $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is uniformly S-asymptotically ω-periodic on bounded sets in the Stepanov sense and asymptotically uniformly continuous on bounded sets in the Stepanov sense.

Lemma 3.6. *Assume that φ is an increasing function which satisfies $\varphi(t+s) = \varphi(t) + rs$ for $(t, s) \in \mathbb{R} \times \mathbb{S}$, where $r > 0$. We assume also that for all $s \in \mathbb{S}$, for all $p \in \mathbb{Z}$, $ps \in \mathbb{S}$. Then $\lim \varphi(t) = +\infty$ as $t \to +\infty$.*

Proof. Let $s \in \mathbb{S} \setminus \{0\}$. Take $t > 0$ and set $p = [t/|s|]$ where $[\cdot]$ denotes the integer part function. Then, we have $p|s| \leq t < (p+1)|s|$ and since φ is increasing it follows that $\varphi(p|s|) \leq \varphi(t) \leq \varphi((p+1)|s|)$ which means that:

$$\varphi(0) + pr|s| \leq \varphi(t) \leq \varphi(0) + (p+1)r|s|.$$

Therefore, since $p \to +\infty$ as $t \to +\infty$, it follows that $\lim_{t\to+\infty} \varphi(t) = +\infty$. □

Proposition 3.7. *Let $\omega \in \mathbb{S}$ and $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$. Assume that for every $(t,s) \in \mathbb{R} \times \mathbb{S}$, $\varphi(t+s) = \varphi(t) + rs$, where $r \in \mathbb{N}^*$. Furthermore, assume that φ a measurable function such that $\lim_{t\to+\infty} \varphi(t) = +\infty$. Then the function $u \circ \varphi$ is asymptotically ω-periodic in the Stepanov sense.*

Proof. Since $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$, we can write $u = v + h$, where $v \in P_\omega(\mathbb{R}^+, \mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$. We observe that:

$$\begin{aligned} v(\varphi(t+\omega)) &= v(\varphi(t) + r\omega) \\ &= v(\varphi(t)). \end{aligned}$$

The function $v \circ \varphi$ is then ω-periodic and since φ is a measurable function, then $v \circ \varphi$ is also a measurable function, thus $v \circ \varphi \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$. Since $h \in C_0(\mathbb{R}^+, \mathbb{X})$ then we can write:

$$(\forall \varepsilon > 0, \exists T > 0, t > T) \implies (\|h(t)\| < \varepsilon).$$

Using $\lim_{t\to+\infty} \varphi(t) = +\infty$, there exists $t_0 > 0$ such that $\varphi(t) > T$ if $t > t_0$. It follows that $\|h(\varphi(t))\| < \varepsilon$ for $t > t_0$ which shows that $\lim\limits_{t\to+\infty} h(\varphi(t)) = 0$. Therefore $h \circ \varphi \in BS_0^p(\mathbb{R}^+, \mathbb{X})$ which proves the proposition.
□

The following corollary is a straightforward consequence of Lemma 3.6.

Lemma 3.8. *Let $\omega \in \mathbb{S}$ and $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$. We assume that B is a limear bounded operator and for every $(t,s) \in \mathbb{R} \times \mathbb{S}$, $\varphi(t+s) = \varphi(t) + rs$, where $r \in \mathbb{N}^*$. φ is increasing function. Then the function $t \mapsto Bu(\varphi(t))$ is Asymptotically ω-periodic in the Stepanov sense.*

Remark 3.9. *The proof of this lemma is similar to the proof of the proposition 3.7*

Proposition 3.10. *Let $\omega \in \mathbb{S}$ and $\phi \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$. We assume that for every $(t,s) \in \mathbb{R} \times \mathbb{S}$, $\varphi(t+s) = \varphi(t) + s$. Furthermore $\lim \varphi(t) = +\infty$ as $t \to +\infty$. Then the function $t \mapsto \phi(\varphi(t))$ is S-asymptotically ω-periodic in the Stepanov sense.*

Proof. We have that $\lim\limits_{t\to\infty} \|\phi(t+\omega) - \phi(t)\| = 0$.
Therefore:

$$(\forall \varepsilon > 0,\ \exists\, T_\varepsilon^0 \in \mathbb{R}^+, \forall\ t > T_\varepsilon^0) \Rightarrow (\|\phi(t+\omega) - \phi(t)\| < \varepsilon).$$

Since φ is increasing function of real numbers and $\lim \varphi(t) = +\infty$ as $t \to +\infty$, then there exists $T > 0$ such that if $t > T$ then $\varphi(t) > T_\varepsilon^0$. We deduce so that

$$(\forall \varepsilon > 0,\ \exists\, T \in \mathbb{R}^+, \forall\ t > T) \Rightarrow (\|\phi(\varphi(t) + \omega) - \phi(\varphi(t))\| < \varepsilon).$$

Since $\omega \in \mathbb{S}$, we observe that

$$(\forall \varepsilon > 0,\ \exists\, T \in \mathbb{R}^+, \forall\ t > T) \Rightarrow (\|\phi(\varphi(t+\omega)) - \phi(\varphi(t))\| < \varepsilon).$$

We have showed that $\lim\limits_{t\to\infty} \phi(\varphi(t+\omega)) - \phi(\varphi(t)) = 0$. Therefore, we can write

$$(\forall \varepsilon^{1/p} > 0, \exists T > 0;\ t \geq T) \Rightarrow (\|\phi(\varphi(t+\omega)) - \phi(\varphi(t))\| < \varepsilon^{1/p}).$$

The function $t \mapsto \phi(\varphi(t))$ is measurable on $\mathbb{R}_+$. Then for $t \geq T$, we have

$$\begin{aligned} \int_t^{t+1} \|\phi(\varphi(s+\omega)) - \phi(\varphi(s))\|^p &\leq \int_t^{t+1} \varepsilon ds \\ &\leq \varepsilon. \end{aligned}$$

Therefore the function $t \mapsto \phi(\varphi(t))$ is S-asymptotically ω-periodic in the Stepanov sense. □

Example 3.11. *Let $\alpha \in \mathbb{R}^+$ and $f_\alpha : [0,\infty[\to \mathbb{R}$ by $f_\alpha(t) = \ln(\alpha+t)$, $t \in [0,\infty[$. Then for any $\omega > 0$, f_α is S-asymptotically ω-periodic but is not asymptotically ω-periodic. In fact, since $f_\alpha(t+\omega) - f_\alpha(t) = \ln(1+\frac{\omega}{t+\alpha})$ then $f_\alpha \in SAP_\omega(\mathbb{R}^+,\mathbb{R})$. We assume that $f_\alpha \in AP_\omega(\mathbb{R}^+,\mathbb{R})$. Then there exist $g \in P_\omega(\mathbb{R},\mathbb{R})$ and $h \in C_0(\mathbb{R}^+,\mathbb{R})$ such that $f_\alpha(t) = g(t) + h(t)$, $t \in \mathbb{R}^+$. Since $g(t) = f_\alpha(t) - h(t)$, we deduce so that $\lim_{t\to+\infty} g(t) = +\infty$. Show that for all $t \in \mathbb{R}^+$, $g(t) = 0$. Let $t_0 \in \mathbb{R}^+$ such that $g(t_0) = c \neq 0$. Let $A > |c|$. Then there exists $T > 0$ such that $g(t) > A$ for every $t > T$. There exist $n \in \mathbb{N}$ such that $t_0 + n\omega > T$. Therefore $g(t_0+n\omega) > A > c$. Since $g(t_0+n\omega) = g(t_0)$, then $c > c$.*

Example 3.12. *Let $\alpha \in \mathbb{R}^+$ and $l_\alpha : [0,\infty[\to \mathbb{R}$ by $l_\alpha(t) = \sqrt{\alpha+t}$, $t \in [0,\infty[$. Then for any $\omega > 0$, l_α is S-asymptotically ω-periodic but is not asymptotically ω-periodic. In fact, since $l_\alpha(t+\omega) - l_\alpha(t) = \frac{\omega}{\sqrt{\alpha+t+\omega}+\sqrt{\alpha+t}}$ then $v_\alpha \in SAP_\omega(\mathbb{R}^+,\mathbb{R})$. We show that $l_\alpha \notin AP_\omega(\mathbb{R}^+,\mathbb{X})$ with the same method that in the last example.*

Example 3.13. *Let $\alpha > 0$ and $v_\alpha : [0,\infty[\to \mathbb{R}$ by $v_\alpha(t) = \frac{1}{(t+\alpha)^n}$, $t \in [0,\infty[$. Then for any $\omega > 0$, l_α is S-asymptotically ω-periodic but is not asymptotically ω-periodic. We observe that*

$$\begin{aligned} v_\alpha(t+\omega) - v_\alpha(t) &= \frac{(t+\alpha)^n - (t+\alpha+\omega)^n}{(t+\alpha+\omega)^n(t+\alpha)^n} \\ &= \frac{(t+\alpha)^n - \sum_{k=0}^n \binom{n}{k}(t+\alpha)^{n-k}\omega^k}{(t+\alpha+\omega)^n(t+\alpha)^n} \\ &= \frac{-\sum_{k=1}^n \binom{n}{k}(t+\alpha)^{n-k}\omega^k}{(t+\alpha+\omega)^n(t+\alpha)^n} \\ &= -\sum_{k=1}^n \frac{\binom{n}{k}\omega^k}{(t+\alpha+\omega)^n(t+\alpha)^k} \end{aligned}$$

and we deduce so that $v_\alpha \in SAP_\omega(\mathbb{R}^+,\mathbb{R})$. We show that $l_\alpha \notin AP_\omega(\mathbb{R}^+,\mathbb{X})$ with the same method that in the remark 2.7 of [16].

Corollary 3.14. *For all $\omega > 0$, there exist an infinity of function in $SAP_\omega(\mathbb{R}^+,\mathbb{R})$ wich are not in $AP_\omega(\mathbb{R}^+,\mathbb{R})$.*

Proof. We have showed in the last examples that for all $\alpha > 0$, f_α, l_α and v_α are S-asymptotically ω-periodic but are not asymptotically ω-periodic □

Example 3.15. *([7],[8]) Let $\omega \in \mathbb{N}^*$ and f an asymptoticallly ω-periodic function (resp. S asymptoticallly ω-periodic function). Then the function $t \to f([t])$ is an asymptoticallly ω-periodic function in the Stepanov sense (resp.S asymptoticallly ω-periodic function in the Stepanov sense). We have in this example that $\varphi(t) = [t]$ and $\mathbb{S} = \mathbb{N}^*$.*

Example 3.16. *We consider $\mathbb{S} = \alpha h\mathbb{N}$, where $\alpha > 0$ and $h > 0$. We consider also*

$$\varphi(t) = [\frac{t}{\alpha h}]\alpha h - i\alpha h,$$

where $i \in \mathbb{N}$. Let $\omega \in \alpha h\mathbb{N}$. If f is an asymptoticallly ω-periodic function (resp. S asymptoticallly ω-periodic function) then the function $t \mapsto f([\frac{t}{\alpha h}]\alpha h - i\alpha h.)$ is an asymptoticallly ω-periodic function in the Stepanov sense (resp.S asymptoticallly ω-periodic function in the Stepanov sense) .

Proof. The function $\varphi(t) = [\frac{t}{\alpha h}]\alpha h - i\alpha h$ is constant on every interval $[n\alpha h, (n+1)\alpha h[$. Let $s = \alpha hn$, where $n \in \mathbb{N}$. We have

$$\begin{aligned} \varphi(t+s) &= \varphi(t + \alpha hn) \\ &= \Big[\frac{t + \alpha hn}{\alpha h}\Big]\alpha h - i\alpha h \\ &= \Big[\frac{t}{\alpha h}\Big]\alpha h - i\alpha h + \alpha hn \\ &= \varphi(t) + s. \end{aligned}$$

Now, show that the function φ is increasing. Assume that $t_0 < t_1$. If there exists n such that $t_0, t_1 \in [n\alpha h, (n+1)\alpha h[$, then $\varphi(t_0) = \varphi(t_1)$. Otherwise there are n_0 and n_1 such that $t_0 \in [n_0\alpha h, (n_0+1)\alpha h[$, $t_1 \in [n_1\alpha h, (n_1+1)\alpha h[$ with $n_0 < n_1$. Then $\varphi(t_0) = (n_0 - i)\alpha h$, $\varphi(t_1) = (n_1 - i)\alpha h$ and $\varphi(t_0) < \varphi(t_1)$. □

Example 3.17. *The article [16] shows that the function $f : [0,\infty[\to \mathbb{R}$ by $f(t) = e^{-t}$, $t \in [0,\infty[$ is for any $\omega > 0$, S-asymptotically ω-periodic but is not asymptotically ω-periodic. Consider the function $m_c : [0,\infty[\to \mathbb{R}$ by $m_c(t) = e^{-[\frac{t}{\omega}]\omega - c}$, where c is a real. Since $f \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, then $m_c \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$. The function m_c is not continuous in n, where $n \in \mathbb{N}$. In fact, $\lim_{t\to\omega n^-} m_c(t) = e^{-(n-1)\omega - c}$ and $\lim_{t\to\omega n^+} m_c(t) = e^{-n\omega - c}$. If $e^{-(n-1)\omega - c} = e^{-n\omega - c}$, then $\omega = 0$. Then the function m_c is a piecewise constinuous function and can't belong to $SAP_\omega(\mathbb{R}^+, \mathbb{R})$. We deduce that for all $c \in \mathbb{R}$, the function m_c is in $S^PSAP_\omega(\mathbb{R}^+, \mathbb{R})$ but is not in $SAP_\omega(\mathbb{R}^+, \mathbb{R})$.*

Example 3.18. *The article [17] shows that the function $f : [0,\infty[\to \mathbb{R}$ by $f(t) = \sin\ln(t+1)$, $t \in [0,\infty[$ is for any $\omega > 0$, S-asymptotically ω-periodic but is not asymptotically ω-periodic. Consider now the function $g_c : [0,\infty[\to \mathbb{R}$ by $g_c(t) = \sin\ln([\frac{t}{\omega}]\omega + c + 1)$,, where c is a real. Since $f \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, then $g_c \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$. Show that $g_c \notin SAP_\omega(\mathbb{R}^+, \mathbb{X})$.*
We observe that

$$\lim_{t\to n\omega^-} \sin\ln([\frac{t}{\omega}]\omega + c + 1) = \sin\ln((n-1)\omega + c + 1)$$

where n is an enteger and

$$\lim_{t\to n\omega^+} \sin\ln([\frac{t}{\omega}]\omega + c + 1) = \sin\ln(n\omega + c + 1).$$

we have that

$$\sin\ln((n-1)\omega + c + 1) = \sin\ln(n\omega + c + 1)$$

if and only if $\ln(n\omega+c+1) = \ln((n-1)\omega+c+1) + 2K\pi$ with $K \in \mathbb{Z}$ or $\ln(n\omega+c+1) = \pi - \ln((n-1)\omega+c+1) + 2K'\pi$ with $K' \in \mathbb{Z}$.

First case: $\ln(n\omega + c + 1) = \ln((n-1)\omega + c + 1) + 2K\pi$ *with* $K \in \mathbb{Z}$.

If $\ln(n\omega+c+1) = \ln((n-1)\omega+c+1)+2K\pi$ then $\frac{n\omega+c+1}{(n-1)\omega+c+1} = e^{2K\pi}$. Consider the following equation $\frac{X+\omega}{X} = e^{2K\pi}$. We have that for all $\mathbb{K} \in \mathbb{Z}^*$, the equation $\frac{X+\omega}{X} = e^{2K\pi}$ has an unique solution $\frac{-\omega}{1-e^{2K\pi}}$. For $K = 0$, the equation $\frac{X+\omega}{X} = e^{2K\pi}$ has not solution. For any enteger n_0, we deduce that if $c \in \mathbb{R}/\{\frac{-\omega}{1-e^{2K\pi}} - (n_0-1)\omega - 1, K \in \mathbb{Z}^*\}$ then for all $K \in \mathbb{Z}$

$$\ln(n_0\omega + c + 1) \neq \ln((n_0-1)\omega + c + 1) + 2K\pi.$$

Second case:$\ln(n\omega + c + 1) = \pi - \ln((n-1)\omega + c + 1) + 2K'\pi$ *with* $K' \in \mathbb{Z}$.

If $\ln(n\omega+c+1) = \pi - \ln((n-1)\omega+c+1) + 2K'\pi$ then $(n\omega+c+1)((n-1)\omega+c+1) = e^{\pi+2K'\pi}$. Consider the following equation $(X+\omega)X = e^{\pi+2K'\pi}$. We observe that the equation $(X+\omega)X = e^{\pi+2K'\pi}$ have two solutions $\frac{-\omega-\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2}$ and $\frac{-\omega+\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2}$.
For any entger n_0, we deduce that if for all $K' \in \mathbb{Z}$, $c \neq \frac{-\omega-\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0-1)\omega - 1$ and $c \neq \frac{-\omega+\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0-1)\omega - 1$ then for all $K' \in \mathbb{Z}$

$$\ln(n_0\omega + c + 1) \neq \pi - \ln((n_0-1)\omega + c + 1) + 2K'\pi.$$

Considering the first and second case, we deduce that if c is a real such that for all $K \in \mathbb{Z}^*$ $c \neq \frac{-\omega}{1-e^{2K\pi}} - (n_0 - 1)\omega - 1$, *such that for all* $K' \in \mathbb{Z}$ $c \neq \frac{-\omega-\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0 - 1)\omega - 1$ *and* $c \neq \frac{-\omega+\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0 - 1)\omega - 1$ *then*

$$\sin \ln((n_0 - 1)\omega + c + 1) \neq \sin \ln(n_0\omega + c + 1).$$

Therefore for all real c such that for all $K \in \mathbb{Z}^*$ $c \neq \frac{-\omega}{1-e^{2K\pi}} - (n_0 - 1)\omega - 1$, *such that for all* $K' \in \mathbb{Z}$ $c \neq \frac{-\omega-\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0 - 1)\omega - 1$ *and* $c \neq \frac{-\omega+\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0 - 1)\omega - 1$ *then the function* g_c *is not continuous in* ωn_0. *Therefore for all real c such that for all* $K \in \mathbb{Z}^*$ $c \neq \frac{-\omega}{1-e^{2K\pi}} - (n_0 - 1)\omega - 1$, *such that for all* $K' \in \mathbb{Z}$ $c \neq \frac{-\omega-\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0 - 1)\omega - 1$ *and* $c \neq \frac{-\omega+\sqrt{\omega^2+4e^{\pi+2K'\pi}}}{2} - (n_0 - 1)\omega - 1$ *then* $g_c \notin SAP_\omega(\mathbb{R}^+, \mathbb{X})$.

Example 3.19. *Let* $\alpha \in \mathbb{R}^+$ *and* $w_\alpha : [0, \infty[\to \mathbb{R}$ *by* $w_\alpha(t) = \ln(\alpha + [\frac{t}{\omega}]\omega), t \in [0, \infty[$. *We have that* $w_\alpha \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{R})$. *For all* $n \in \mathbb{N}$, w_α *is not continuous in* $n\omega$. *Therefore* $w_\alpha \notin SAP_\omega(\mathbb{R}^+, \mathbb{R})$.

Example 3.20. *Let* $\alpha \in \mathbb{R}^+$ *and* $q_\alpha : [0, \infty[\to \mathbb{R}$ *by*

$$q_\alpha(t) = \sqrt{\alpha + [\frac{t}{\omega}]\omega}, t \in [0, \infty[.$$

We have that $q_\alpha \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{R})$. *For all* $n \in \mathbb{N}$, q_α *is not continuous in* $n\omega$. *Therefore* $q_\alpha \notin SAP_\omega(\mathbb{R}^+, \mathbb{R})$.

Corollary 3.21. *For all* $\omega > 0$, *there exist an infinity of function in* $S^pSAP_\omega(\mathbb{R}^+, \mathbb{R})$ *wich are not in* $SAP_\omega(\mathbb{R}^+, \mathbb{R})$.

Proof. For all $\alpha \in \mathbb{R}^+$, w_α and q_α are in $S^pSAP_\omega(\mathbb{R}^+, \mathbb{R})$ but are not in $SAP_\omega(\mathbb{R}^+, \mathbb{R})$. □

Proposition 3.22. *Let* $\omega \in \mathbb{S}$ *and* $u \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$. *We assume that for every* $(t, s) \in \mathbb{R} \times \mathbb{S}$, $\varphi(t + s) = \varphi(t) + s$ *and that B is a linear bounded operator such that* $\|B\| \neq 0$. *Then the function* $t \mapsto Bu(\varphi(t))$ *is S-asymptotically* ω*-periodic in the Stepanov sense.*

Proof. Since $\phi \in SAP_\omega(\mathbb{R}^+, \mathbb{X})$, according to the proof of the proposition 3.10 we have:

$$(\forall \varepsilon > 0, \exists T \in \mathbb{R}^+, \forall t > T) \Rightarrow (\|\phi(\varphi(t + \omega)) - \phi(\varphi(t))\| < \varepsilon).$$

Let $\varepsilon > 0$. Since $\frac{\varepsilon}{\|B\|} > 0$, then

$$(\exists T_1 \in \mathbb{R}^+, \forall t > T_1) \Rightarrow (\|\phi(\varphi(t + \omega)) - \phi(\varphi(t))\| < \frac{\varepsilon}{\|B\|}).$$

Observing that

$$\|B\phi(\varphi(t + \omega)) - B\phi(\varphi(t))\| \leq \|B\| \, \|\phi(\varphi(t + \omega)) - \phi(\varphi(t))\|,$$

we deduce so that if $t > T_1$, then

$$\begin{aligned} \|B\phi(\varphi(t + \omega)) - B\phi(\varphi(t))\| &< \|B\| \tfrac{\varepsilon}{\|B\|} \\ &< \varepsilon \end{aligned}$$

We have showed that $\lim_{t\to\infty} \|B\phi(\varphi(t + \omega)) - B\phi(\varphi(t))\| = 0$. Therefore, we can write:

$$(\forall \varepsilon^{1/p} > 0, \exists T > 0;\ t \geq T) \Rightarrow (\|B\phi(\varphi(t + \omega)) - B\phi(\varphi(t))\| < \varepsilon^{1/p}).$$

The function $t \to Bu(\varphi(t))$ is measurable on $\mathbb{R}_+$. Then for $t \geq T$, we have

$$\begin{aligned} \int_t^{t+1} \|Bu(\varphi(s + \omega)) - Bu(\varphi(s))\|^p &\leq \int_t^{t+1} \varepsilon ds \\ &\leq \varepsilon. \end{aligned}$$

Therefore the function $t \mapsto Bu(\varphi(t))$ is S-asymptotically ω-periodic in the Stepanov sense. □

Lemma 3.23. *Let $\omega \in \mathbb{S}$. Assume that $f \in S^pAP_\omega(\mathbb{R}^+ \times \mathbb{X}, \mathbb{X})$ and assume that f satisfies a Lipschitz condition in $\mathbb{X}$ uniformly in $t \in \mathbb{R}^+$:*

$$||f(t, x) - f(t, y)|| \leq L||x - y||$$

for all $x, y \in \mathbb{X}$ and $t \in \mathbb{R}^+$, where L is a positive constant. We assume also that $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$ and that for every $(t, s) \in \mathbb{R} \times \mathbb{S}$, $\varphi(t + s) = \varphi(t) + rs$, where $r \in \mathbb{N}^*$. Furthermore $\lim \varphi(t) = +\infty$ as $t \to +\infty$. Then the function $F : \mathbb{R}^+ \to \mathbb{X}$ defined by $F(t) = f(t, u(\varphi(t)))$ is asymptotically ω-periodic in the Stepanov sense.

Proof. Since $u \in AP_\omega(\mathbb{R}^+, \mathbb{X})$, we can write $u = v + l$, where $v \in P_\omega(\mathbb{R}^+, \mathbb{X})$ and $l \in C_0(\mathbb{R}^+, \mathbb{X})$. The function $u(\varphi(t)) = v(\varphi(t)) + l(\varphi(t)) \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$ according to the proposition 3.7. In particular, we have $t \mapsto u(\varphi(t)) \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ and $t \mapsto l(\varphi(t)) \in BS^p_0(\mathbb{R}^+, \mathbb{X})$. By theorem 2.5, we obtain

$$\int_t^{t+1} ||u(\varphi(s + n\omega)) - v(\varphi(s))||^p ds \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$.
Denote $K = \overline{\{v(\varphi(t)) : t \in \mathbb{R}^+\}}$; K is a bounded set. Since f is asymptotically ω-periodic in the Stepanov sense uniformly on bounded sets of $\mathbb{X}$, there exists a function $g : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ with $g(t, x) \in S^pP_\omega(\mathbb{R}^+, \mathbb{X})$ for each $x \in \mathbb{X}$ such that for every bounded set $K \subset \mathbb{X}$ we have

$$\Big(\int_t^{t+1} ||f(s + n\omega, x) - g(s, x)||^p\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$ uniformly for $x \in K$.
We observe that

$$\begin{aligned}
&\Big(\int_t^{t+1} ||f(s + n\omega, u(\varphi(s + n\omega)) - g(s, v(\varphi(s)))||^p ds\Big)^{\frac{1}{p}}\\
\leq\ &\Big(\int_t^{t+1} ||f(s + n\omega, u(\varphi(s + n\omega))) - f(s + n\omega, v(\varphi(s)))||^p ds\Big)^{\frac{1}{p}}\\
+\ &\Big(\int_t^{t+1} ||f(s + n\omega, v(\varphi(s))) - g(s, v(\varphi(s)))||^p ds\Big)^{\frac{1}{p}}\\
\leq\ &L\Big(\int_t^{t+1} ||u(\varphi(s + n\omega)) - v(\varphi(s))||^p ds\Big)^{\frac{1}{p}}\\
+\ &\Big(\int_t^{t+1} ||f(s + n\omega, v(\varphi(s))) - g(s, v(\varphi(s)))||^p ds\Big)^{\frac{1}{p}}
\end{aligned}$$

Hence, we deduce so that

$$\Big(\int_t^{t+1} ||f(s + n\omega, u(\varphi(s + n\omega))) - g(s, v(\varphi(s)))||^p ds\Big)^{\frac{1}{p}} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}^+$. By Theorem 2.5, we deduce that $F \in S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$. □

Lemma 3.24. *Let $\omega \in \mathbb{S}$. Assume that $f : \mathbb{R}^+ \times \mathbb{X} \to \mathbb{X}$ is uniformly S-asymptotically ω-periodic on bounded sets in the Stepanov sense and asymptotically uniformly continuous on bounded sets in the Stepanov sense.We assume also that for every $(t, s) \in \mathbb{R} \times \mathbb{S}$, $\varphi(t + s) = \varphi(t) + s$. Furthermore $\lim \varphi(t) = +\infty$ as $t \to +\infty$. Let $u : \mathbb{R}^+ \to \mathbb{X}$ be a function in $SAP_\omega(\mathbb{R}^+, \mathbb{X})$, and let $v(t) = f(t, u(\varphi(t)))$. Then $v \in S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$.*

Proof. Set $B =: \mathcal{R}(u) = \{u(\varphi(t)),\ t \geq 0\} \subset \mathbb{X}$.
Since f is uniformly S-asymptotically ω-periodic on bounded sets in the Stepanov sense, there exist functions $g_B \in BS^p(\mathbb{R}^+, \mathbb{X})$ and $h_B \in BS^p_0(\mathbb{R}^+, \mathbb{X})$ satisfying the properties involved in Definition 2.17 in relation with

the set $B = \mathcal{R}(u)$.
The function v belongs to $BS^p(\mathbb{R}^+, \mathbb{X})$ because

$$\begin{aligned}\int_t^{t+1} \|v(\tau)\|^p d\tau &= \int_t^{t+1} \|f(\tau, u(\varphi(\tau)))\|^p d\tau \\ &\leq \int_t^{t+1} \|g_B(\tau)\|^p d\tau \\ &\leq \sup_{t\geq 0}\Big(\int_t^{t+1} \|g_B(\tau)\|^p d\tau\Big).\end{aligned}$$

Therefore

$$\|v^b\|_{\mathbb{L}^\infty(\mathbb{R}^+, L^p)} \leq \|g_B\|_{S^p}.$$

We have for all $t \geq 0$:

$$\begin{aligned}&\int_t^{t+1} \|f(s+\omega, u(\varphi(s+\omega))) - f(s, u(\varphi(s+\omega)))\|^p ds \\ &\leq \int_t^{t+1} \|h_B(s)\|^p ds.\end{aligned}$$

Note that $h_B \in BS^p_0(\mathbb{R}^+, \mathbb{R})$; this implies that for $\varepsilon > 0$ there exist $t'_\varepsilon > 0$ such that for all $t \geq t'_\varepsilon$ we have

$$\int_t^{t+1} \|h_B(s)\|^p ds \leq \varepsilon^p/2.$$

Thus

$$\begin{aligned}&\int_t^{t+1} \|f(s+\omega, u(\varphi(s+\omega)) - f(s, u(\varphi(s+\omega)))\|^p ds \\ &\leq \varepsilon^p/2 \quad \text{for all } t \geq t'_\varepsilon (*).\end{aligned}$$

Furthermore since f is asymptotically uniformly continuous on bounded sets in the Stepanov sense, thus for all $\varepsilon > 0$, there exists $t_\varepsilon \geq 0$ and $\delta_\varepsilon > 0$ such that

$$\begin{aligned}&\int_t^{t+1} \|f(s, u(\varphi(s+\omega)) - f(s, u(\varphi(s)))\|^p ds \\ &\leq \varepsilon^p/2 \quad \text{for all } \ t \geq t_\varepsilon \quad (**)\end{aligned}$$

because

$$\|u(\varphi(s+\omega)) - u(\varphi(s))\| \leq \delta_\varepsilon.$$

The estimates $(*)$ and $(**)$ lead to

$$\begin{aligned}&\int_t^{t+1} \|v(s+\omega) - v(s)\|^p ds \\ =& \int_t^{t+1} \|f(s+\omega, u(\varphi(s+\omega))) - f(s, u(\varphi(s))\|^p ds \\ \leq& \int_t^{t+1} \|f(s+\omega, u(\varphi(s+\omega))) - f(s, u(\varphi(s+\omega)))\|^p ds \\ +& \int_t^{t+1} \|f(s, u(\varphi(s+\omega))) - f(s, u(\varphi(s))\|^p ds \\ \leq& \ \varepsilon^p/2 + \varepsilon^p/2 = \varepsilon^p.\end{aligned}$$

Therefore for all $\varepsilon > 0$ there exists $T_\varepsilon = Max(t_\varepsilon, t'_\varepsilon) > 0$ such that for all $t \geq T_\varepsilon$ we have

$$\Big(\int_t^{t+1} \|v(s+\omega) - v(s)\|^p ds\Big)^{1/p} \leq \varepsilon.$$

We conclude that $v \in S^p SAP_\omega(\mathbb{R}^+, \mathbb{X})$. □

Lemma 3.25. *Let $j \in \{1, ..., N\}$ and $\omega \in \mathbb{S}_j$. We assume that the hypothesis* (**H1**) *and* (**H2$_\mathbf{j}$**) *are satisfied. We assume that A_j is a bounded linear operator. Then*

$$(\wedge_j \ \phi)(t) = \int_0^t T(t-s) A_j \phi(\varphi_j(s) ds$$

maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.

Proof. According to the lemma 3.8, $t \to A_j\phi(\varphi_j(t))$ is an asymptotically ω-periodic function in the Stepanov sense. Therefore, considering Lemma 2.23, we deduce so that the operator $\wedge_j$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. □

Lemma 3.26. *Let $\omega \in \mathbb{S}$. We assume also that the hypothesis* (**H1**), (**H4**) *and* (**H2$_\mathbf{N+1}$**) *are satisfied. Then*

$$(\wedge \ \phi)(t) = \int_0^t T(t-s) f(s, \phi(\varphi_{N+1}(s)) ds$$

maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.

Proof. According to the proposition (3.7), $s \to \phi_{N+1}(\varphi(s))$ is asymptotically ω-periodic function in the Stepanov sense. According to the lemma 3.23, $t \to f(t, \phi_{N+1}(\varphi(t))$ is an asymptotically ω-periodic function in the Stepanov sense.Therefore, considering Lemma (2.23), we deduce so that the operator $\wedge$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. □

Lemma 3.27. *Let $j \in \{1, ..., N\}$ and $\omega \in \mathbb{S}_j$. We assume that the hypothesis* (**H1**) *and* (**H3$_\mathbf{j}$**) *are satisfied. We assume that A_j is a bounded linear operator. Then*

$$(\wedge_j \ \phi)(t) = \int_0^t T(t-s) A_j \phi(\varphi_j(s) ds$$

maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.

Proof. According to Proposition (3.22), $s \to A_j\phi(\varphi_j(t))$ is an S-asymptotically ω-periodic function in the Stepanov sense. Therefore, considering Lemma (2.24), we deduce so that the operator $\wedge_j$ maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. □

Lemma 3.28. *Let $\omega \in \mathbb{S}_{N+1}$. We assume also that the hypothesis* (**H1**), (**H3$_\mathbf{N+1}$**) *and* (**H5**) *are satisfied. Then*

$$(\wedge \ \phi)(t) = \int_0^t T(t-s) f(s, \phi(\varphi_{N+1}(s)) ds$$

maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.

Proof. According to the proposition 3.10, $s \to \phi(\varphi(s)$ is S-asymptotically ω-periodic function in the Stepanov sense. According to the lemma 3.24, $t \to f(t, \phi_{N+1}(\varphi(t))$ is an S asymptotically ω-periodic function in the Stepanov sense. Therefore, considering Lemma (2.24), we deduce so that the operator $\wedge$ maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. □

Theorem 3.29. *Let $\omega \in \cap_{j=1}^{N+1} \mathbb{S}_j$ and assume that the hypothesis* (**H1**) *and* (**H4**) *(resp.* (**H5**)*) are satisfied. We assume also that for all $j \in \{1, ..., N+1\}$, the hypothesis* ($\mathbf{H2_j}$) *(resp.* ($\mathbf{H3_j}$)*) are satisfied and that A_j is a linear bounded operator. Then* (1) *has a unique Asymptotically ω-periodic (resp. S-Asymptotically ω-periodic) mild solution provided*

$$\Theta := \frac{M\Big(\sum_{j=0}^{N} \|A_j\|_\infty + L\Big)}{a} < 1.$$

Proof. We define the nonlinear operator Γ by the expression

$$\begin{aligned}(\Gamma\phi)(t) &= T(t)c_0 + \sum_{j=1}^{N} \int_0^t T(t-s)A_j\,\phi(\varphi_j(s))ds \\ &\quad + \int_0^t T(t-s)f(s,\phi(\varphi_{N+1}(s)))ds \\ &= T(t)c_0 + \sum_{j=1}^{N} (\wedge_j\phi)(t) + (\wedge\phi)(t).\end{aligned}$$

Here the operators $\wedge_j$ and $\wedge$ are defined as

$$(\wedge_j\phi)(t) \quad = \quad \int_0^t T(t-s)A_j\,\phi(\varphi_j(s))ds$$

and

$$(\wedge\phi)(t) \quad = \quad \int_0^t T(t-s)f(s,\phi(\varphi_{N+1}(s)))ds.$$

According to the hypothesis **(H.1)**, we have $\|T(t)\| \leq Me^{-at}$ and therefore $\lim_{t\to\infty}\|T(t)\| = 0$.
It follows from lemma 3.23 that the function $t \to f(t, \phi(\varphi(t)))$ belongs to $S^pAP_\omega(\mathbb{R}^+, \mathbb{X})$. Finally Lemma 3.26

shows that the operator $\wedge$ maps $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. Similarly, considering Lemma 3.25, we deduce that for all $j \in \{1, ..., N\}$, the operators $\wedge_j$ map $AP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. Therefore the operator Γ maps $AP_\omega(\mathbb{R}^+), \mathbb{X}$ into itself.
According to the lemma 3.24 the function $t \to f(t, \phi(\varphi(t)))$ belongs to $S^pSAP_\omega(\mathbb{R}^+, \mathbb{X})$. According to Lemma 3.28 the operator $\wedge$ maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. Similarly, considering Lemma 3.27, we deduce that for all $j \in \{1, ..., N\}$, the operators $\wedge_j$ maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself. Therefore the operator Γ maps $SAP_\omega(\mathbb{R}^+, \mathbb{X})$ into itself.
Now we have

$$\|(\Gamma\phi)(t) - \Gamma\psi)(t)\| \quad \leq \quad \left\| \sum_{j=1}^{N} \int_0^t T(t-s)A_j\Big(\phi(\varphi_j(s)) - \psi(\varphi_j(s))\Big)ds \right\|$$

$$
\begin{aligned}
&+ \quad \left\| \int_0^t T(t-s)\Big(f(s,\phi(\varphi_{N+1}(s))) - f(s,\psi(\varphi_{N+1}(s)))\Big)ds \right\| \\
&\leq \quad \sum_{j=0}^{N} \int_0^t \|T(t-s)\| \, \|A_j\| \, \|\phi(\varphi_j(s)) - \psi(\varphi_j(s))\| ds \\
&+ \quad \int_0^t \|T(t-s)\| \, \|f(s,\phi(\varphi_{N+1}(s))) - f(s,\psi(\varphi_{N+1}(s)))\| ds \\
&\leq \quad \sum_{j=0}^{N} \|A_j\|_\infty M \int_0^t e^{-a(t-s)} \|\phi(\varphi_j(s)) - \psi(\varphi_j(s))\| ds \\
&+ \quad LM \int_0^t e^{-a(t-s)} \|\phi(\varphi_{N+1}(s)) - \psi(\varphi_{N+1}(s))\| ds \\
&\leq \quad \sum_{j=0}^{N} \|A_j\|_\infty M \int_0^t e^{-a(t-s)} \|\phi - \psi\|_\infty ds \\
&+ \quad LM \int_0^t e^{-a(t-s)} \|\phi - \psi\|_\infty ds \\
&\leq \quad \sum_{j=0}^{N} \|A_j\|_\infty M \frac{1-e^{-at}}{a} \|\phi - \psi\|_\infty + LM \frac{1-e^{-at}}{a} \|\phi - \psi\|_\infty \\
&\leq \quad \frac{M\Big(\sum_{j=0}^{N} \|A_j\|_\infty + L\Big)}{a} \|\phi - \psi\|_\infty .
\end{aligned}
$$

Hence we have :

$$\|\Gamma\phi - \Gamma\psi\|_\infty \leq \|\phi - \psi\|_\infty$$

which proves that Γ is a contraction and we conclude that Γ has a unique fixed point in AP_ω (resp. SAP_ω). The proof is complete. □

4. Application

Consider the following heat equation with Dirichlet conditions:

$$
\begin{cases}
\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - 2u(t,x) + \alpha u(\varphi_1(t),x) + f(t,u(\varphi_2(t),x)), \\
u(t,0) = u(t,\pi) = 0, t \in \mathbb{R}.
\end{cases}
\tag{4}
$$

where f satisfies the hypothesis **(H4)** (resp. **(H5)**). We assume also that for $j = 1,2$, the hypotheses (**H2$_j$**) (resp.(**H3$_j$**) are satisfied.
Take $\mathbb{X} = L^2[0,\pi]$ with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)_2$. Define

$$D(A) = \{u(\cdot) \in \mathbb{X} : u'' \in \mathbb{X}, u' \in \mathbb{X} \textit{is absolutely continuous on}$$

$$[0,\pi], u(0) = u(\pi) = 0\}$$

and $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ by

$$Au = \frac{\partial^2 u(x)}{\partial x^2} - 2u$$

It is well known that A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$ satisfying

$$\|T(t)\| \leq e^{-3t} \textit{ for } t > 0,$$

see [14]. Then (4) can be reformulated as the asbtract problem (1). φ_1 is subject to the functional relation $\varphi_1(t+s) = \varphi_1(t) + s$; $(t, s) \in \mathbb{R} \times \mathbb{S}$, where $\mathbb{S}$ is a subset of $\mathbb{R}$. Considering Theorem 3.29, we claim that

Theorem 4.1. *If $L + |\alpha| < 3$ then the equation (4) admits a unique mild solution $u(t) \in AP_\omega(\mathbb{R}^+)$ (resp. $u(t) \in SAP_\omega(\mathbb{R}^+)$).*

References

[1] D. Altinton *Extension of the logistic equation of differential equation with piecewise constant argument* Int. J. Math. Sci. 28 (2001) 609-619.

[2] M.U.Akhmet, H. Oktem, S.W. Pickl, G.W. Weber *An anticipatory extension of malthusian model, in : D.M Dubois (Ed), CASYS 2005-Seventh International Conference, in : AIP Conference Proceeding* vol. 839, The American Institue of Physics, 2006, p. 260-264.

[3] J. Blot, P. Cieutat and G. M. N'Guérékata *S asymptotically ω-periodic functions and applications to evolution equations. African Diaspora J. Math. 12, 113-121, 2011.*

[4] S. Busenberg, K. Cooke *Vertically transmitted diseases* Models and Dynamics in Biomathematics, vol. 23, (1993) Springer-Verlag

[5] T. Diagana. *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces. Springer, 2013.*

[6] W. Dimbour, J-C Mado. *S-asymptotically ω periodic solution for a nonlinear differentil equation with piecewise constant argument in a Banach space.* Cubo Vol. 16 (2014) p.55-65

[7] W. Dimbour, S. Manou-Abi. *Asymptotically ω-periodic function in the Stepanov sense and its applications for an advanced differential equation with piecewise constant argument in a Banach space.* Mediterranean Journal of Mathematics (2018) 15: 25

[8] W. Dimbour, S. Manou-Abi. *S-asymptotically ω-periodic solution for a nonlinear differential equation with piecewise constant argument via S-asymptotically ω-periodic functions in the Stepanov sense.* Journal of Nonlinear Systems and Applications,Vol. 7(1), 14-20

[9] W. Dimbour, G.Mophou and G.M. N'Guérékata. *S asymptotically ω-periodic solution for partial differential equations with finite delay. Electron.J.Differ.Equa. 2011, 1-12, 2011.*

[10] W. Dimbour, V. Valmorin. *Asymptotically antiperiodic solutions for a nonlinear differential equation with piecewise constant argument in a Banach space.* Applied Mathematics, 7 (2016).

[11] H. R. Henríquez, M. Pierri and P. Táboas. *On S asymptotically ω-periodic function on Banach spaces and applications. J. Math. Anal Appl. 343, 1119-1130, 2008.*

[12] H. R. Henríquez. *Asymptotically periodic solutions of abstract differential equations. NonlinearAnal.80, 135-149, 2013.*

[13] M. Kostic, D. Velinov. *Asymptotically Bloch periodic solutions of abstract fractional nonlinear differential inclusions with piecewise constant argument* Functional Analysis, Approximation and Computation 9 (2017), p. 27-36.

[14] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems.* Progress in Nonlinear Differential Equations and their applications, vol. 16. Birhhauser, Basel, 1995.

[15] G.M.N'Guérékata. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces. Kluwer Academic, New York, London, Moscou, 2001.*

[16] E.R. Oueama-Guengai, G.M. N'Guérékata. *On S-asymptotically ω-periodic and Bloch periodic mild solutions to some fractionnal differential equations in abstract spaces.* Math. Meth. Appl. Sci (2018) 9116-9122

[17] S. Nicola, M. Pierri. *A note on S-asymtotically periodic functionq*, Nonlinear Analysis: Real World Applications, Vol 10 (2009) 2937-2938.

[18] J.Wiener, *A Second-Order delay differential equation with multiple Periodic solutions*, Journal of Mathematical Analysis and Appliction, 229 (1999) 6596-676.

[19] J.Wiener, L.Debnath *Boundary Value Problems for the diffusion equation with piecewise continuous time delay*, Internat.J.Math.and Math.Sci., Vol.20 (1997) 187-195.

[20] J.Wiener, L.Debnath *A survey of partial differential equations with piecewise continuous arguments*, Internat.J.Math.and Math.Sci., Vol.18, No.2 (1995) 209-228.

[21] J.Wiener, V.Lakshmikantham, *Excitability of a second-order delay differential equation*, Nonlinear Analysis, Vol.38 (1999) 1-11.

[22] J.Wiener, *Generalized solutions of functional differential equations*, World Scientific (1999).

[23] R. Xie and C. Zhang. *Criteria of asymptotic ω-periodicity and their applications in a class of fractional differential equations. Advances in Difference Equations, 2015.*

[24] Z.Xia. *Asymptotically periodic of semilinear fractional integro-differential equations. Advances in Differencial Equations, 1-19, 2014* **?.**

[25] Zhong-Hua Wu and Jin Liang. *Asymptotic periodicity for a class of fractional integro-differential equations. J. Nonlinear Sci. Appl. 6, 152-161, 2015.*