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Some coupled fixed point results under contractive type conditions in cone S_b-metric spaces

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Abstract. The goal of this paper is to prove some coupled fixed point results under contractive type conditions in the setting of cone S_b-metric spaces. Also, we provide some consequences of the main results. Furthermore, we give an illustrative example in support of the established result. Our results extend the results of Sabetghadam *et al.* [18] (Fixed Point Theory Appl., Volume 2009, Article ID 125426, 8 pages) from cone metric space to the setting of cone S_b-metric space.

1. Introduction

In 1922, the Polish mathematician Stephen Banach [3] established an important metric fixed point result regarding a contraction mapping, known as the Banach contraction principle (in short BCP). This principle is considered as one of the most remarkable results in analysis. It confirms the existence and uniqueness of fixed point of certain self-maps on metric spaces. This result (BCP) has been generalized in various directions. There is a great number of generalization of the Banach contraction principle. The underlying metric space can be generalized in many ways. Bakhtin in [2] introduced the concept of b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the well-known Banach contraction mapping principle in metric spaces. Huang and Zhang [8] in 2007 introduced the concept of cone metric space by replacing the set of real numbers by a general Banach space $\mathbb{E}$ which is partially ordered with respect to a cone $\mathbb{P} \subset \mathbb{E}$ and proved some fixed point theorems for contractive mappings in normal cone metric space.

In 2011, Hussain and Shah [9] introduced the concept of cone b-metric space as a generalization of b-metric space and cone metric spaces. Also they improved some recent results about *KKM* mappings in cone b-metric spaces. In the meantime, Guo and Lakshmikantham [7] introduced the concept of coupled fixed point. Bhaskar and Lakshmikantham [4] introduced coupled fixed point for partially ordered metric spaces. Later, a lot of authors such as Ciric and Lakshmikantham [5], Olaleru *et al.* [14], Samet [22], Loung and Thuan [13] give different generalization of these theorems.

In 2009, Sabetghadam *et al.* [18] considered the corresponding definition of coupled fixed point for mappings on cone metric spaces and established some coupled fixed point theorems.

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In 2011, Aydi [1] proved some coupled fixed point theorems via various contractive type conditions in the setting of partial metric spaces. Recently, in the context of partial metric spaces, Kim *et al.* [11] proved some general coupled fixed point theorems for weak compatible mappings (see, also [10], [20], [21]).

On the other hand, Sedghi *et al.* [23] generalized metric space to S-metric space. In addition, he has proved that the results of d-metric spaces can be derived from SMS results if we consider $d(u, v) = \mathbb{S}(u, u, v)$. Souayah and Mlaiki [27] introduced the concept of S_b-metric space. Dhamodharan and Krishnakumar [6] also further extended S-metric space to cone S-metric space. In 2018, K. Anthony Singh and M. R. Singh [25] generalized the notion of cone S-metric space to cone S_b-metric space and proved some fixed point theorems. Recently, Saluja [19] proved some fixed point results under contractive type mappings in the setting of cone S_b-metric spaces.

Motivated by the works of Sabetghadam *et al.* [18], we extend the results of [18] by defining coupled fixed point to mappings on cone S_b-metric space and prove some coupled fixed point theorems on said space. The results obtained in this paper extend and generalize several results from the existing literature. Also, an example is given to validate the established result.

2. Preliminaries

The following definitions and properties will be needed in the sequel.

Definition 2.1. *(see [8]) Let $\mathbb{E}$ be a real Banach space and $\mathbb{P}$ be a subset of $\mathbb{E}$. Then $\mathbb{P}$ is called a cone if and only if the following conditions hold:*
(C1) $\mathbb{P}$ *is closed, nonempty and* $\mathbb{P} \neq \{0\}$;
(C2) $a\,r + b\,s \in \mathbb{P}$ *for all* $r, s \in \mathbb{P}$ *where* a, b *are nonnegative real numbers;*
(C3) $\mathbb{P} \cap (-\mathbb{P}) = \{0\}$.

Definition 2.2. *(see [8]) Let $\mathbb{P}$ be a cone in real Banach space $\mathbb{E}$, we define a partial ordering $\leq$ in $\mathbb{E}$ with respect to $\mathbb{P}$ by $a \leq b \Leftrightarrow b - a \in \mathbb{P}$. We shall write $r < s$ to indicate that $r \leq s$ but $r \neq s$, while $r \ll s$ will stand for $s - r \in int\,(\mathbb{P})$.*

Remark 2.3. *(see [29]) If $int\,(\mathbb{P}) \neq \emptyset$, then $\mathbb{P}$ is called a solid cone.*

Definition 2.4. *(see [8]) The cone $\mathbb{P}$ is called normal if there is a number $K > 0$ such that for all $r, s \in \mathbb{P}$, $0 \leq r \leq s$ implies $\|r\| \leq K\|s\|$.*

The least positive number K satisfying the inequality $\|r\| \leq K\|s\|$ is called the normal constant of cone.

The cone $\mathbb{P}$ is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{r_n\}$ is a sequence such that $r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots \leq s$ for some $s \in \mathbb{E}$, then there is $r \in \mathbb{E}$ such that $\|r_n - r\| \to 0$ as $n \to \infty$. Equivalently, the cone $\mathbb{P}$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose $\mathbb{E}$ is a Banach space, $\mathbb{P}$ is a cone in $\mathbb{E}$ with $int(\mathbb{P}) \neq \emptyset$ and $\leq$ is partial ordering in $\mathbb{E}$ with respect to $\mathbb{P}$.

Example 2.5. *(see [12]) Let $K > 1$ be given. Consider the real vector space*

$$\mathbb{E} = \Big\{a\,r + b : a, b \in \mathbb{R};\; r \in \Big[1 - \frac{1}{K}, 1\Big]\Big\}$$

with supremum norm and the cone

$$\mathbb{P} = \Big\{a\,r + b \in \mathbb{E} : a \geq 0, b \geq 0\Big\}$$

in $\mathbb{E}$. The cone $\mathbb{P}$ is regular and so normal.

Definition 2.6. *(see [8, 30]) Let $\Xi \neq \emptyset$ be a set. Suppose that the mapping $d \colon \Xi \times \Xi \to \mathbb{E}$ satisfies:*
(CM_1) $0 \leq d(p_1, p_2)$ *for all* $p_1, p_2 \in \Xi$ *with* $p_1 \neq p_2$ *and* $d(p_1, p_2) = 0 \;\Leftrightarrow\; p_1 = p_2$;
(CM_2) $d(p_1, p_2) = d(p_2, p_1)$ *for all* $p_1, p_2 \in \Xi$;
(CM_3) $d(p_1, p_2) \leq d(p_1, p_3) + d(p_3, p_2)$ *for all* $p_1, p_2, p_3 \in \Xi$.
Then d is called a cone metric [8] on Ξ and (Ξ, d) is called a cone metric space [8] or simply CMS.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $\mathbb{E} = \mathbb{R}$ and $\mathbb{P} = [0, +\infty)$.

Lemma 2.7. *(see [17]) Every regular cone is normal.*

Example 2.8. *(see [8]) Let* $\mathbb{E} = \mathbb{R}^2$, $\mathbb{P} = \{(p_1, p_2) \in \mathbb{R}^2 : p_1 \geq 0, p_2 \geq 0\}$, $\Xi = \mathbb{R}$ *and* $d\colon \Xi \times \Xi \to \mathbb{E}$ *defined by* $d(p_1, p_2) = (|p_1 - p_2|, \alpha|p_1 - p_2|)$, *where* $\alpha \geq 0$ *is a constant. Then* (Ξ, d) *is a cone metric space with normal cone* $\mathbb{P}$ *where* $K = 1$.

Example 2.9. *(see [16]) Let* $\mathbb{E} = \ell^2$, $\mathbb{P} = \{\{r_n\}_{n\geq 1} \in \mathbb{E} : r_n \geq 0, \text{for all } n\}$, (Ξ, D) *a metric space, and* $d\colon \Xi \times \Xi \to \mathbb{E}$ *defined by* $d(p_1, p_2) = \{D(p_1, p_2)/2^n\}_{n\geq 1}$. *Then* (Ξ, d) *is a cone metric space.*

Clearly, the above examples show that the class of cone metric spaces contains the class of metric spaces.

Definition 2.10. *(see [15, 23]) Let* $\Xi \neq \emptyset$ *be a set and* $\mathbb{S}\colon \Xi^3 \to [0, \infty)$ *be a function satisfying the following conditions for all* p_1, p_2, p_3, $t \in \Xi$:

$(S1)$ $\mathbb{S}(p_1, p_2, p_3) \geq 0$;

$(S2)$ $\mathbb{S}(p_1, p_2, p_3) = 0$ *if and only if* $p_1 = p_2 = p_3$;

$(S3)$ $\mathbb{S}(p_1, p_2, p_3) \leq \mathbb{S}(p_1, p_1, t) + \mathbb{S}(p_2, p_2, t) + \mathbb{S}(p_3, p_3, t)$.

Then the function $\mathbb{S}$ *is called an S-metric on* Ξ *and the pair* $(\Xi, \mathbb{S})$ *is called an S-metric space or simply SMS.*

Example 2.11. (1) *(see [28]) Let* Ξ *be a nonempty set and* d *be the ordinary metric on* Ξ. *Then* $\mathbb{S}(p_1, p_2, p_3) = d(p_1, p_3) + d(p_2, p_3)$ *is an S-metric on* Ξ.

(2) *(see [23]) Let* $\Xi = \mathbb{R}^n$ *and* $\|.\|$ *a norm on* Ξ, *then* $\mathbb{S}(p_1, p_2, p_3) = \|p_2 + p_3 - 2p_1\| + \|p_2 - p_3\|$ *is an S-metric on* Ξ.

(3) *(see [23]) Let* $\Xi = \mathbb{R}^n$ *and* $\|.\|$ *a norm on* Ξ, *then* $\mathbb{S}(p_1, p_2, p_3) = \|p_1 - p_3\| + \|p_2 - p_3\|$ *is an S-metric on* Ξ.

Definition 2.12. *(see [27]) Let* Ξ *be a nonempty set and* $b \geq 1$ *be a given real number. A function* $\mathbb{S}_b\colon \Xi^3 \to [0, \infty)$ *is said to be* S_b*-metric on* Ξ *if and only if for all* $u, v, z, m \in \Xi$, *the following conditions are satisfied:*

(S_b1) $\mathbb{S}_b(u, v, z) = 0$ *if and only if* $u = v = z$;

(S_b2) $\mathbb{S}_b(u, v, z) \leq b\Big[\mathbb{S}_b(u, u, m) + \mathbb{S}_b(v, v, m) + \mathbb{S}_b(z, z, m)\Big]$.

The pair $(\Xi, \mathbb{S}_b)$ is called an S_b-metric space.

Remark 2.13. *Note that the class of* S_b*-metric spaces is larger than the class of S-metric spaces. Indeed every S-metric space is an* S_b*-metric space with* $b = 1$. *But, the converse need not be always true.*

Example 2.14. *(see [24]) Let* $(\Xi, \mathbb{S})$ *be an S-metric space and* $\mathbb{S}_*(u, v, z) = \Big\{\mathbb{S}(u, v, z)\Big\}^p$, *where* $p > 1$ *is a real number. Then* $\mathbb{S}_*$ *is an* S_b*-metric on* Ξ *with* $b = 2^{2(p-1)}$.

Example 2.15. *(see [25]) Let* $\Xi = \mathbb{R}$ *and let the function* $\mathbb{S}\colon \Xi^3 \to \mathbb{R}$ *be defined as* $\mathbb{S}(u, v, z) = |u - z| + |v - z|$. *Then* $\mathbb{S}$ *is an S-metric on* Ξ. *Therefore, the function* $\mathbb{S}_b(u, v, z) = \Big\{\mathbb{S}(u, v, z)\Big\}^2 = \Big\{|u - z| + |v - z|\Big\}^2$ *is an* S_b*-metric on* Ξ *with* $b = 2^{2(2-1)} = 4$.

Definition 2.16. *(see [6]) Suppose that* $\mathbb{E}$ *is a real Banach space,* $\mathbb{P}$ *is a cone in* $\mathbb{E}$ *with int* $\mathbb{P} \neq \emptyset$ *and* $\leq$ *is partial ordering with respect to* $\mathbb{P}$. *Let* Ξ *be a nonempty set and let the function* $\mathbb{S}\colon \Xi^3 \to \mathbb{E}$ *satisfy the following conditions:*

(CS_1) $\mathbb{S}(u, v, z) \geq 0$;

(CS_2) $\mathbb{S}(u, v, z) = 0 \Leftrightarrow u = v = z$;

(CS_3) $\mathbb{S}(u, v, z) \leq \mathbb{S}(u, u, m) + \mathbb{S}(v, v, m) + \mathbb{S}(z, z, m), \forall\, u, v, z, m \in \Xi$.

Then the function $\mathbb{S}$ is called a cone S-metric on Ξ and the pair $(\Xi, \mathbb{S})$ is called a cone S-metric space or simply CSMS.

Example 2.17. *(see [6]) Let* $\mathbb{E} = \mathbb{R}^2$, $\mathbb{P} = \{(p_1, p_2) \in \mathbb{R}^2 : p_1, p_2 \geq 0\}$, $\Xi = \mathbb{R}$ *and* d *be the ordinary metric on* Ξ. *Then the function* $\mathbb{S}\colon \Xi^3 \to \mathbb{E}$ *defined by* $\mathbb{S}(u, v, z) = \big(d(u, z) + d(v, z), \alpha(d(u, z) + d(v, z))\big)$, *where* $\alpha > 0$ *is a cone S-metric on* Ξ.

Lemma 2.18. *(see [6]) Let* $(\Xi, \mathbb{S})$ *be a cone S-metric space. Then we have* $\mathbb{S}(u, u, v) = \mathbb{S}(v, v, u)$.

K. Anthony Singh and M. R. Singh [25] (2018) introduced the notion of cone S_b-metric space as follows:

Definition 2.19. *(see [25]) Suppose that* $\mathbb{E}$ *is a real Banach space,* $\mathbb{P}$ *is a cone in* $\mathbb{E}$ *with int* $\mathbb{P} \neq \emptyset$ *and* $\leq$ *is partial ordering in* $\mathbb{E}$ *with respect to* $\mathbb{P}$. *Let* Ξ *be a nonempty set and let the function* $\mathcal{S}\colon \Xi^3 \to \mathbb{E}$ *satisfy the following conditions:*

(CS_b1) $\mathcal{S}(u_1, u_2, u_3) \geq 0$;

(CS_b2) $\mathcal{S}(u_1, u_2, u_3) = 0 \Leftrightarrow u_1 = u_2 = u_3$;

(CS_b3) $\mathcal{S}(u_1, u_2, u_3) \leq b\Big[\mathcal{S}(u_1, u_1, r) + \mathcal{S}(u_2, u_2, r) + \mathcal{S}(u_3, u_3, r)\Big]$;

for all $u_1, u_2, u_3, r \in \Xi$, where $b \geq 1$ is a constant. Then the function $\mathcal{S}$ is called a cone S_b-metric on Ξ and the pair $(\Xi, \mathcal{S})$ is called a cone S_b-metric space or simply CS_bMS.

We note that cone S_b-metric spaces are generalizations of cone S-metric spaces since every cone S-metric space is a cone S_b-metric space with $b = 1$.

Example 2.20. *(see [25]) Let* $\mathbb{E} = \mathbb{R}^2$, *the Euclidean plane and* $\mathbb{P} = \{(r, s) \in \mathbb{R}^2 : r, s \geq 0\}$, *a normal cone in* $\mathbb{E}$. *Let* $\Xi = \mathbb{R}$ *and* $\mathbb{S}\colon \Xi^3 \to \mathbb{E}$ *be such that* $\mathbb{S}(u, v, w) = \big(\alpha\, \mathbb{S}_0(u, v, w), \beta\, \mathbb{S}_0(u, v, w)\big)$, *where* $\alpha, \beta > 0$ *are constants and* $\mathbb{S}_0$ *is an* S_b*-metric on* Ξ. *Then* $\mathbb{S}$ *is a cone* S_b*-metric on* Ξ.

In particular, the function $\mathbb{S}_0(u, v, w) = \Big\{|u - w| + |v - w|\Big\}^2$ for all $u, v, w \in \Xi$ is an S_b-metric on Ξ with $b = 4$.

Therefore, the function $\mathbb{S}(u, v, w) = \Big[\Big(|u - w| + |v - w|\Big)^2, \frac{1}{4}\Big(|u - w| + |v - w|\Big)^2\Big]$ for all $u, v, w \in \Xi$ is a cone S_b-metric on Ξ with $b = 4$.

Definition 2.21. *(see [25]) Let* $(\Xi, \mathcal{S})$ *be a cone* S_b*-metric space.*

(i) *A sequence* $\{u_n\}$ *in* Ξ *converges to* $u \in \Xi$ *if and only if* $\mathcal{S}(u_n, u_n, u) \to 0$ *as* $n \to \infty$, *that is, there exists* $n_0 \in N$ *such that for all* $n \geq n_0$, $\mathcal{S}(u_n, u_n, u) \ll \varepsilon$ *for each* $\varepsilon \in \mathbb{E}$ *with* $0 \ll \varepsilon$. *We denote this by* $\lim_{n\to\infty} u_n = u$ *or* $u_n \to u$ *as* $n \to \infty$.

(ii) *A sequence* $\{u_n\}$ *in* Ξ *is called a Cauchy sequence if* $\mathcal{S}(u_n, u_n, u_m) \to 0$ *as* $n, m \to \infty$, *that is, there exists* $n_0 \in N$ *such that for all* $n, m \geq n_0$, $\mathcal{S}(u_n, u_n, u_m) \ll \varepsilon$ *for each* $\varepsilon \in \mathbb{E}$ *with* $0 \ll \varepsilon$.

(iii) *The cone* S_b*-metric space* $(\Xi, \mathcal{S})$ *is called complete if every Cauchy sequence is convergent.*

Lemma 2.22. *(see [25]) Let* $(\Xi, \mathcal{S})$ *be a cone* S_b*-metric space,* $\mathbb{P}$ *be a normal cone with normal constant K. Then a sequence* $\{u_n\}$ *in* Ξ *converges to* u *if and only if* $\mathcal{S}(u_n, u_n, u) \to 0$ *as* $n \to \infty$.

Lemma 2.23. *(see [25]) Let* $(\Xi, \mathcal{S})$ *be a cone* S_b*-metric space,* $\mathbb{P}$ *be a normal cone with normal constant K. Let* $\{u_n\}$ *be a sequence in* Ξ. *If the sequence* $\{u_n\}$ *converges to* u_1 *and* u_2, *then* $u_1 = u_2$, *that is, the limit of a convergent sequence is unique.*

Lemma 2.24. *(see [25]) Let* $(\Xi, \mathcal{S})$ *be a cone* S_b*-metric space,* $\mathbb{P}$ *be a normal cone with normal constant K. Then a sequence* $\{u_n\}$ *in* Ξ *is a Cauchy sequence if and only if* $\mathcal{S}(u_n, u_n, u_m) \to 0$ *as* $n, m \to \infty$.

Lemma 2.25. *(see [25]) Let* $(\Xi, \mathcal{S})$ *be a cone* S_b*-metric space,* $\mathbb{P}$ *be a normal cone with normal constant K. Let* $\{u_n\}$ *be a sequence in* Ξ. *If the sequence* $\{u_n\}$ *converges to* u, *then* $\{u_n\}$ *is a Cauchy sequence, that is, every convergent sequence is a Cauchy sequence.*

Lemma 2.26. *(see [26]) Let* $(\Xi, \mathcal{S})$ *be a cone* S_b*-metric space. Then we have*

(1) $\mathcal{S}(u, u, v) \leq b\, \mathcal{S}(v, v, u)$,

(2) $\mathcal{S}(u, u, v) \leq 2b\mathcal{S}(u, u, m) + b\mathcal{S}(v, v, m) \leq 2b\mathcal{S}(u, u, m) + b^2\mathcal{S}(m, m, v)$ *for all* $u, v, m \in \Xi$.

3. Main Results

Now, we are in a position to state and prove our main results.

First, we give the corresponding definition of coupled fixed point in cone S_b.metric space.

Definition 3.1. *(see [1]) An element $(u, v) \in \Xi \times \Xi$ is said to be a coupled fixed point of the mapping $f\colon \Xi \times \Xi \to \Xi$ if $f(u, v) = u$ and $f(v, u) = v$.*

Example 3.2. *Let $\Xi = [0, +\infty)$ and $f\colon \Xi \times \Xi \to \Xi$ defined by $f(u, v) = \frac{u+v}{3}$ for all $u, v \in \Xi$. One can easily see that f has a unique coupled fixed point* $(0, 0)$.

Example 3.3. *Let $\Xi = [0, +\infty)$ and $f\colon \Xi \times \Xi \to \Xi$ defined by $f(u, v) = \frac{u+v}{2}$ for all $u, v \in \Xi$. Then we see that f has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.*

Theorem 3.4. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p, q), F(p, q), F(u, v)) \leq \delta\, \mathcal{H}(p, q, u, v), \tag{1}$$

where

$$\begin{aligned} \mathcal{H}(p, q, u, v) \quad = \quad & \max\Big\{\mathcal{S}(p, p, u), \mathcal{S}(F(p, q), F(p, q), p), \mathcal{S}(F(p, q), F(p, q), u), \\ & \mathcal{S}(F(u, v), F(u, v), u)\Big\}, \end{aligned}$$

for all $p, q, u, v \in \Xi$ and $\delta \in \left(0, \frac{1}{b}\right)$ is a constant. Then F has a unique coupled fixed point.

Proof. Let us take $x_0, y_0 \in \Xi$ be arbitrary points and set

$$p_1 = F(p_0, q_0),\ q_1 = F(q_0, p_0), \ldots,\ p_{n+1} = F(p_n, q_n),\ q_{n+1} = F(q_n, p_n).$$

If $p_{n_0} = p_{n_0+1}$, $q_{n_0} = q_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then

$$p_{n_0} = p_{n_0+1} = F(p_{n_0}, q_{n_0}),\ q_{n_0} = q_{n_0+1} = F(q_{n_0}, p_{n_0}).$$

Thus, (p_{n_0}, q_{n_0}) is a coupled fixed point of F.

Now, we assume that either $p_n \neq p_{n+1} = F(p_n, q_n)$ or $q_n \neq q_{n+1} = F(q_n, p_n)$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} & \mathcal{S}(F(p_{n-1}, q_{n-1}), F(p_{n-1}, q_{n-1}), F(p_n, q_n)) = \mathcal{S}(p_n, p_n, p_{n+1}) > 0, \text{ or} \\ & \quad \mathcal{S}(F(q_{n-1}, p_{n-1}), F(q_{n-1}, p_{n-1}), F(q_n, p_n)) = \mathcal{S}(q_n, q_n, q_{n+1}) > 0, \end{aligned}$$

for all $n \in \mathbb{N}$. Let $G_n = \mathcal{S}(p_n, p_n, p_{n+1})$ and $H_n = \mathcal{S}(q_n, q_n, q_{n+1})$. Then from equation (1) and using Lemma 2.26, we obtain

$$\begin{aligned} G_n \quad & = \quad \mathcal{S}(p_n, p_n, p_{n+1}) = \mathcal{S}(F(p_{n-1}, q_{n-1}), F(p_{n-1}, q_{n-1}), F(p_n, q_n)) \\ & \leq \quad \delta\, \mathcal{H}(p_{n-1}, q_{n-1}, p_n, q_n), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \mathcal{H}(p_{n-1}, q_{n-1}, p_n, q_n) \quad & = \quad \max\Big\{\mathcal{S}(p_{n-1}, p_{n-1}, p_n), \mathcal{S}(F(p_{n-1}, q_{n-1}), F(p_{n-1}, q_{n-1}), p_{n-1}), \\ & \qquad \mathcal{S}(F(p_{n-1}, q_{n-1}), F(p_{n-1}, q_{n-1}), p_n), \mathcal{S}(F(p_n, q_n), F(p_n, q_n), p_n)\Big\} \\ & = \quad \max\Big\{\mathcal{S}(p_{n-1}, p_{n-1}, p_n), \mathcal{S}(p_n, p_n, p_{n-1}), \mathcal{S}(p_n, p_n, p_n), \\ & \qquad \mathcal{S}(p_{n+1}, p_{n+1}, p_n)\Big\} \\ & = \quad \max\Big\{\mathcal{S}(p_{n-1}, p_{n-1}, p_n), b\mathcal{S}(p_{n-1}, p_{n-1}, p_n), 0, b\mathcal{S}(p_n, p_n, p_{n+1})\Big\} \\ & = \quad \max\Big\{\mathcal{S}(p_{n-1}, p_{n-1}, p_n), b\mathcal{S}(p_{n-1}, p_{n-1}, p_n), b\mathcal{S}(p_n, p_n, p_{n+1})\Big\} \\ & = \quad \max\{G_{n-1}, bG_{n-1}, bG_n\}. \end{aligned}$$

Now, we have the following cases.

Case(a) If $\max\{G_{n-1}, bG_{n-1}, bG_n\} = b\,G_n$, then from equation (2), we obtain

$$G_n \leq b\delta\, G_n,$$

which is a contradiction, since $\delta < 1/b$ and $b \geq 1$.

Case(b) If $\max\{G_{n-1}, bG_{n-1}, bG_n\} = b\,G_{n-1}$, then from equation (2), we obtain

$$G_n \leq b\delta\, G_{n-1}, \tag{3}$$

for all $n \in \mathbb{N}$.

Case(c) If $\max\{G_{n-1}, bG_{n-1}, bG_n\} = G_{n-1}$, then from equation (2), we obtain

$$G_n \leq \delta\, G_{n-1}, \tag{4}$$

for all $n \in \mathbb{N}$.

If we take $\beta = \max\{b\delta, \delta\}$, then $\beta < 1$ since $b \geq 1$ and $\delta < 1/b$. Hence from equations (3) and (4), we obtain

$$G_n \leq \beta\, G_{n-1}, \tag{5}$$

for all $n \in \mathbb{N}$.

Continuing in the same way, we obtain

$$0 \leq G_n \leq \beta\, G_{n-1} \leq \beta^2\, G_{n-2} \leq \cdots \leq \beta^n G_0, \tag{6}$$

for all $n \in \mathbb{N}$.

Similarly, one can obtain

$$0 \leq H_n \leq \beta\, H_{n-1} \leq \beta^2\, H_{n-2} \leq \cdots \leq \beta^n H_0, \tag{7}$$

for all $n \in \mathbb{N}$.

Let $L_n = G_n + H_n$, then from equations (6) and (7), we get

$$0 \leq L_n \leq \beta\, L_{n-1} \leq \beta^2\, L_{n-2} \leq \cdots \leq \beta^n L_0, \tag{8}$$

for all $n \in \mathbb{N}$.

If $L_0 = 0$, then (p_0, q_0) is a coupled fixed point F. So, we assume that $L_0 > 0$.

Then for any $n, m \in \mathbb{N}$ with $m > n$, using Lemma 2.26 and condition (CS_b3), we have

$$\begin{aligned}
\mathcal{S}(p_n, p_n, p_m) &\leq b\Big[2\mathcal{S}(p_n, p_n, p_{n+1}) + \mathcal{S}(p_m, p_m, p_{n+1})\Big] \\
&\leq 2b\mathcal{S}(p_n, p_n, p_{n+1}) + b^2\mathcal{S}(p_{n+1}, p_{n+1}, p_m) \\
&\leq 2b\mathcal{S}(p_n, p_n, p_{n+1}) + 2b^3\mathcal{S}(p_{n+1}, p_{n+1}, p_{n+2}) \\
&\quad + b^4\mathcal{S}(p_{n+2}, p_{n+2}, p_m) \\
&\leq 2b\mathcal{S}(p_n, p_n, p_{n+1}) + 2b^3\mathcal{S}(p_{n+1}, p_{n+1}, p_{n+2}) \\
&\quad + 2b^5\mathcal{S}(p_{n+2}, p_{n+2}, p_{n+3}) + \ldots \\
&\quad + b^{2(m-n-1)}\mathcal{S}(p_{m-1}, p_{m-1}, p_m) \\
&\leq 2b\Big\{\mathcal{S}(p_n, p_n, p_{n+1}) + b^2\mathcal{S}(p_{n+1}, p_{n+1}, p_{n+2}) \\
&\quad + b^4\mathcal{S}(p_{n+2}, p_{n+2}, p_{n+3}) + \ldots \\
&\quad + b^{2(m-n-1)}\mathcal{S}(p_{m-1}, p_{m-1}, p_m)\Big\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathcal{S}(q_n,q_n,q_m) \leq 2b\Big\{&\mathcal{S}(q_n,q_n,q_{n+1}) + b^2\mathcal{S}(q_{n+1},q_{n+1},q_{n+2})\\ &+ b^4\mathcal{S}(q_{n+2},q_{n+2},q_{n+3}) + \dots\\ &+ b^{2(m-n-1)}\mathcal{S}(q_{m-1},q_{m-1},q_m)\Big\}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathcal{S}(p_n,p_n,p_m) + \mathcal{S}(q_n,q_n,q_m) &\leq 2b\Big\{\mathcal{S}(p_n,p_n,p_{n+1}) + \mathcal{S}(q_n,q_n,q_{n+1})\\ &\quad + b^2[\mathcal{S}(p_{n+1},p_{n+1},p_{n+2}) + \mathcal{S}(q_{n+1},q_{n+1},q_{n+2})] + \dots\\ &\quad + b^{2(m-n-1)}[\mathcal{S}(p_{m-1},p_{m-1},p_m) + \mathcal{S}(q_{m-1},q_{m-1},q_m)]\Big\}\\ &= 2b\Big\{L_n + b^2L_{n+1} + b^4L_{n+2} + \cdots + b^{2(m-n-1)}L_{m-1}\Big\}\\ &\leq 2b\Big\{\beta^n + b^2\beta^{n+1} + b^4\beta^{n+2} + \cdots + b^{2(m-n-1)}\beta^{m-1}\Big\}L_0\\ &\leq 2b\beta^n\Big\{1 + b^2\beta + b^4\beta^2 + \cdots + b^{2(m-n-1)}\beta^{m-n-1}\Big\}L_0\\ &= 2b\beta^n\Big\{1 + b^2\beta + (b^2\beta)^2 + \cdots + (b^2\beta)^{m-n-1}\Big\}L_0\\ &\leq \Big(\frac{2b\beta^n}{1-b^2\beta}\Big)L_0.\end{aligned}$$

This implies that

$$\|\mathcal{S}(p_n,p_n,p_m) + \mathcal{S}(q_n,q_n,q_m)\| \leq \Big(\frac{2b\beta^nK}{1-b^2\beta}\Big)\|L_0\| \to 0 \text{ as } n,m\to\infty,$$

since $0<\beta<1$. Again, the above inequality implies that

$$\mathcal{S}(p_n,p_n,p_m)\to 0 \text{ and } \mathcal{S}(q_n,q_n,q_m)\to 0 \text{ as } n,m\to\infty.$$

This shows that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in Ξ. Since by hypothesis $(\Xi,\mathcal{S})$ is complete S_b-metric space, so their exist $p_1,q_1\in\Xi$ such that $p_n\to p_1$ and $q_n\to q_1$ as $n\to\infty$. Now, we show that (p_1,q_1) is a coupled fixed point of F.

For this, using inequality (1), Lemma 2.26 and condition (CS_b3), we have

$$\begin{aligned}\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1) &\leq 2b\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_{n+1}) + b\mathcal{S}(p_1,p_1,p_{n+1})\\ &= 2b\mathcal{S}(F(p_1,q_1),F(p_1,q_1),F(p_n,q_n)) + b\mathcal{S}(p_1,p_1,p_{n+1})\\ &\leq 2b\delta\mathcal{H}(p_1,q_1,p_n,q_n) + b\mathcal{S}(p_1,p_1,p_{n+1}),\end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathcal{H}(p_1,q_1,p_n,q_n) &= \max\Big\{\mathcal{S}(p_1,p_1,p_n),\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1),\\ &\qquad \mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_n),\mathcal{S}(F(p_n,q_n),F(p_n,q_n),p_1)\Big\}\\ &= \max\Big\{\mathcal{S}(p_1,p_1,p_n),\mathcal{S}(p_1,p_1,p_1),\mathcal{S}(p_1,p_1,p_n),\mathcal{S}(p_{n+1},p_{n+1},p_1)\Big\}\\ &= \max\Big\{\mathcal{S}(p_1,p_1,p_n),0,\mathcal{S}(p_1,p_1,p_n),\mathcal{S}(p_{n+1},p_{n+1},p_1)\Big\},\end{aligned}$$

taking the limit as $n\to\infty$, we get

$$\mathcal{H}(p_1,q_1,p_n,q_n)\to 0 \text{ as } n\to\infty.$$

Using this in inequality (9), we obtain

$$\begin{aligned}\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1) &\leq 2b\delta.0 + b\mathcal{S}(p_1,p_1,p_{n+1})\\ &= b\mathcal{S}(p_1,p_1,p_{n+1}).\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1)\| &\leq Kb\|\mathcal{S}(p_1,p_1,p_{n+1})\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence

$$\|\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1)\| = 0.$$

Thus, $\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1) = 0$ and so $F(p_1,q_1) = p_1$. Similarly, we can show that $F(q_1,p_1) = q_1$. This shows that (p_1,q_1) is a coupled fixed point of F.

Now, we show the uniqueness of the coupled fixed point of F. Assume that (p_2,q_2) is another coupled fixed point of F such that $(p_1,q_1) \neq (p_2,q_2)$. Then using equation (1) and Lemma 2.26, we have

$$\begin{aligned}\mathcal{S}(p_1,p_1,p_2) &= \mathcal{S}(F(p_1,q_1),F(p_1,q_1),F(p_2,q_2))\\ &\leq \delta\mathcal{H}(p_1,q_1,p_2,q_2), \qquad (10)\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}(p_1,q_1,p_2,q_2) &= \max\Big\{\mathcal{S}(p_1,p_1,p_2),\mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_1),\\ &\qquad \mathcal{S}(F(p_1,q_1),F(p_1,q_1),p_2),\mathcal{S}(F(p_2,q_2),F(p_2,q_2),p_1)\Big\}\\ &= \max\Big\{\mathcal{S}(p_1,p_1,p_2),\mathcal{S}(p_1,p_1,p_1),\mathcal{S}(p_1,p_1,p_2),\mathcal{S}(p_2,p_2,p_1)\Big\}\\ &= \max\Big\{\mathcal{S}(p_1,p_1,p_2),0,\mathcal{S}(p_1,p_1,p_2),b\mathcal{S}(p_1,p_1,p_2)\Big\}\\ &= \max\Big\{\mathcal{S}(p_1,p_1,p_2),b\mathcal{S}(p_1,p_1,p_2)\Big\}.\end{aligned}$$

The following two cases arise. If $\max\Big\{\mathcal{S}(p_1,p_1,p_2),b\mathcal{S}(p_1,p_1,p_2)\Big\} = b\mathcal{S}(p_1,p_1,p_2)$, then from equation (10), we obtain

$$\mathcal{S}(p_1,p_1,p_2) \leq b\delta\mathcal{S}(p_1,p_1,p_2),$$

which is a contradiction, since $\delta < 1/b$ and $b \geq 1$.

Again, if $\max\Big\{\mathcal{S}(p_1,p_1,p_2),b\mathcal{S}(p_1,p_1,p_2)\Big\} = \mathcal{S}(p_1,p_1,p_2)$, then from equation (10), we obtain

$$\mathcal{S}(p_1,p_1,p_2) \leq \delta\mathcal{S}(p_1,p_1,p_2),$$

which is again a contradiction, since $\delta < 1/b$ and $b \geq 1$. Thus in both the cases, we get a contradiction. Hence, we conclude that $\mathcal{S}(p_1,p_1,p_2) = 0$ and so by condition (CS_b2), $p_1 = p_2$. By similar fashion, we can show that $q_1 = q_2$. Consequently, (p_1,q_1) is the unique coupled fixed point of F. □

Remark 3.5. *Theorem 3.4 extends and generalizes the results of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

If we take

$$\max\Big\{\mathcal{S}(p,p,u),\mathcal{S}(F(p,q),F(p,q),p),\mathcal{S}(F(p,q),F(p,q),u),\mathcal{S}(F(u,v),F(u,v),u)\Big\} = \mathcal{S}(p,p,u)$$

in Theorem 3.4, then we have the following result.

Corollary 3.6. *Let* $(\Xi, \mathcal{S})$ *be a complete cone* S_b*-metric space with* $b \geq 1$ *and* $\mathbb{P}$ *be a normal cone with normal constant* K*. Suppose that the mapping* $F\colon \Xi \times \Xi \to \Xi$ *satisfies the following contractive condition:*

$$\mathcal{S}(F(p,q), F(p,q), F(u,v)) \leq \delta\, \mathcal{S}(p,p,u), \tag{11}$$

for all $p,q,u,v \in \Xi$ *and* $\delta \in (0, 1/b)$ *is a constant. Then* F *has a unique coupled fixed point.*

Theorem 3.7. *Let* $(\Xi, \mathcal{S})$ *be a complete cone* S_b*-metric space with* $b \geq 1$ *and* $\mathbb{P}$ *be a normal cone with normal constant* K*. Suppose that the mapping* $F\colon \Xi \times \Xi \to \Xi$ *satisfies the following contractive condition:*

$$\mathcal{S}(F(p,q), F(p,q), F(u,v)) \leq \mathcal{A}(p,q,u,v), \tag{12}$$

where

$$\begin{aligned}\mathcal{A}(p,q,u,v) \;&=\; c_1\mathcal{S}(p,p,u) + c_2\mathcal{S}(q,q,v) + c_3\mathcal{S}(F(p,q),F(p,q),p)\\ &\quad +c_4\mathcal{S}(F(p,q),F(p,q),u) + c_5\mathcal{S}(F(u,v),F(u,v),u)\\ &\quad +c_6\mathcal{S}(F(u,v),F(u,v),p),\end{aligned}$$

for all $p,q,u,v \in \Xi$ *and* $c_1, c_2, c_3, c_4, c_5, c_6$ *are nonnegative constants such that* $c_1 + c_2 + c_3 b + c_5 b + c_6 b(2b + b^2) < 1$*. Then* F *has a unique coupled fixed point.*

Proof. Let $x_0, y_0 \in \Xi$ be arbitrary points and set

$$p_1 = F(p_0,q_0),\ q_1 = F(q_0,p_0), \ldots,\ p_{n+1} = F(p_n,q_n),\ q_{n+1} = F(q_n,p_n).$$

Assume that $p_n \neq p_{n+1}$. Then from equation (12) and using Lemma 2.26 and condition (CS_b3), we obtain

$$\begin{aligned}\mathcal{S}(p_n,p_n,p_{n+1}) \;&=\; \mathcal{S}(F(p_{n-1},q_{n-1}),F(p_{n-1},q_{n-1}),F(p_n,q_n))\\ &\leq\; \mathcal{A}(p_{n-1},q_{n-1},p_n,q_n),\end{aligned} \tag{13}$$

where

$$\begin{aligned}\mathcal{A}(p_{n-1},q_{n-1},p_n,q_n) \;&=\; c_1\mathcal{S}(p_{n-1},p_{n-1},p_n) + c_2\mathcal{S}(q_{n-1},q_{n-1},q_n)\\ &\quad +c_3\mathcal{S}(F(p_{n-1},q_{n-1}),F(p_{n-1},q_{n-1}),p_{n-1})\\ &\quad +c_4\mathcal{S}(F(p_{n-1},q_{n-1}),F(p_{n-1},q_{n-1}),p_n)\\ &\quad +c_5\mathcal{S}(F(p_n,q_n),F(p_n,q_n),p_n)\\ &\quad +c_6\mathcal{S}(F(p_n,q_n),F(p_n,q_n),p_{n-1})\\ &=\; c_1\mathcal{S}(p_{n-1},p_{n-1},p_n) + c_2\mathcal{S}(q_{n-1},q_{n-1},q_n)\\ &\quad +c_3\mathcal{S}(p_n,p_n,p_{n-1}) + c_4\mathcal{S}(p_n,p_n,p_n)\\ &\quad +c_5\mathcal{S}(p_{n+1},p_{n+1},p_n) + c_6\mathcal{S}(p_{n+1},p_{n+1},p_{n-1})\\ &=\; c_1\mathcal{S}(p_{n-1},p_{n-1},p_n) + c_2\mathcal{S}(q_{n-1},q_{n-1},q_n)\\ &\quad +c_3 b\mathcal{S}(p_{n-1},p_{n-1},p_n) + c_5 b\mathcal{S}(p_n,p_n,p_{n+1})\\ &\quad +c_6 b\mathcal{S}(p_{n-1},p_{n-1},p_{n+1})\\ &=\; (c_1 + c_3 b)\mathcal{S}(p_{n-1},p_{n-1},p_n) + c_5 b\mathcal{S}(p_n,p_n,p_{n+1})\\ &\quad +c_6 b[b(2\mathcal{S}(p_{n-1},p_{n-1},p_n) + b\mathcal{S}(p_n,p_n,p_{n+1}))]\\ &\quad +c_2\mathcal{S}(q_{n-1},q_{n-1},q_n)\\ &=\; (c_1 + c_3 b + 2c_6 b^2)\mathcal{S}(p_{n-1},p_{n-1},p_n)\\ &\quad +(c_5 b + c_6 b^3)\mathcal{S}(p_n,p_n,p_{n+1}) + c_2\mathcal{S}(q_{n-1},q_{n-1},q_n).\end{aligned}$$

Using this in equation (13), we obtain

$$\begin{aligned}\mathcal{S}(p_n,p_n,p_{n+1}) \;&\leq\; (c_1 + c_3 b + 2c_6 b^2)\mathcal{S}(p_{n-1},p_{n-1},p_n) + (c_5 b + c_6 b^3)\,\times\\ &\quad\; \mathcal{S}(p_n,p_n,p_{n+1}) + c_2\mathcal{S}(q_{n-1},q_{n-1},q_n).\end{aligned} \tag{14}$$

By similar fashion, we can have

$$\begin{aligned}\mathcal{S}(q_n,q_n,q_{n+1}) &\leq (c_1+c_3b+2c_6b^2)\mathcal{S}(q_{n-1},q_{n-1},q_n)+(c_5b+c_6b^3)\times\\ &\quad \mathcal{S}(q_n,q_n,q_{n+1})+c_2\mathcal{S}(p_{n-1},p_{n-1},p_n). \end{aligned}\tag{15}$$

From equations (14) and (15), we obtain

$$\begin{aligned}\mathcal{S}(p_n,p_n,p_{n+1})+\mathcal{S}(q_n,q_n,q_{n+1}) &\leq (c_1+c_3b+2c_6b^2)[\mathcal{S}(p_{n-1},p_{n-1},p_n)+\mathcal{S}(q_{n-1},q_{n-1},q_n)]\\ &\quad +(c_5b+c_6b^3)[\mathcal{S}(p_n,p_n,p_{n+1})+\mathcal{S}(q_n,q_n,q_{n+1})]\\ &\quad +c_2[\mathcal{S}(p_{n-1},p_{n-1},p_n)+\mathcal{S}(q_{n-1},q_{n-1},q_n)]. \end{aligned}\tag{16}$$

Let

$$\xi_n=\mathcal{S}(p_n,p_n,p_{n+1})+\mathcal{S}(q_n,q_n,q_{n+1}).$$

Then from equation (16), we obtain

$$\xi_n\leq (c_1+c_3b+2c_6b^2)\xi_{n-1}+(c_5b+c_6b^3)\xi_n+c_2\xi_{n-1}.$$

This implies that

$$\begin{aligned}\xi_n &\leq \Big(\frac{c_1+c_2+c_3b+2c_6b^2}{1-c_5b-c_6b^3}\Big)\xi_{n-1}\\ &= M\,\xi_{n-1}, \end{aligned}\tag{17}$$

for all $n\in\mathbb{N}$, where $M=\Big(\frac{c_1+c_2+c_3b+2c_6b^2}{1-c_5b-c_6b^3}\Big)<1$, since by assumption $c_1+c_2+c_3b+c_5b+c_6b(2b+b^2)<1$.

Continuing in the same way, we obtain

$$\xi_n\leq M\xi_{n-1}\leq M^2\xi_{n-2}\leq\cdots\leq M^n\xi_0,\tag{18}$$

for all $n\in\mathbb{N}$.

If $\xi_0=0$, then (p_0,q_0) is a coupled fixed point F. So, we assume that $\xi_0>0$.

Then for any $n,m\in\mathbb{N}$ with $m>n$, using Lemma 2.26 and condition (CS_b3) , we obtain as in Theorem 3.4 that

$$\begin{aligned}\mathcal{S}(p_n,p_n,p_m)\leq 2b\Big\{&\mathcal{S}(p_n,p_n,p_{n+1})+b^2\mathcal{S}(p_{n+1},p_{n+1},p_{n+2})\\ &+b^4\mathcal{S}(p_{n+2},p_{n+2},p_{n+3})+\dots\\ &+b^{2(m-n-1)}\mathcal{S}(p_{m-1},p_{m-1},p_m)\Big\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathcal{S}(q_n,q_n,q_m)\leq 2b\Big\{&\mathcal{S}(q_n,q_n,q_{n+1})+b^2\mathcal{S}(q_{n+1},q_{n+1},q_{n+2})\\ &+b^4\mathcal{S}(q_{n+2},q_{n+2},q_{n+3})+\dots\\ &+b^{2(m-n-1)}\mathcal{S}(q_{m-1},q_{m-1},q_m)\Big\}.\end{aligned}$$

Therefore, we have

$$\mathcal{S}(p_n,p_n,p_m)+\mathcal{S}(q_n,q_n,q_m)\leq\Big(\frac{2bM^n}{1-b^2M}\Big)\xi_0.$$

This implies that

$$\|\mathcal{S}(p_n,p_n,p_m)+\mathcal{S}(q_n,q_n,q_m)\|\leq\Big(\frac{2bM^nK}{1-b^2M}\Big)\|\xi_0\|\to 0 \text{ as } n,m\to\infty,$$

since $0 < M < 1$. Again, the above inequality implies that

$$\mathcal{S}(p_n, p_n, p_m) \to 0 \text{ and } \mathcal{S}(q_n, q_n, q_m) \to 0 \text{ as } n, m \to \infty.$$

This shows that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in Ξ. Since by hypothesis $(\Xi, \mathcal{S})$ is complete S_b-metric space, so their exist $s_1, t_1 \in \Xi$ such that $p_n \to s_1$ and $q_n \to t_1$ as $n \to \infty$. Now, we show that (s_1, t_1) is a coupled fixed point of F.

For this, using inequality (12), Lemma 2.26, conditions (CS_b2) and (CS_b3), we have

$$\begin{aligned}\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) &\leq 2b\mathcal{S}(F(s_1,t_1),F(s_1,t_1),p_{n+1}) + b\mathcal{S}(s_1,s_1,p_{n+1}) \\ &= 2b\mathcal{S}(F(s_1,t_1),F(s_1,t_1),F(p_n,q_n)) + b\mathcal{S}(s_1,s_1,p_{n+1}) \\ &\leq 2b\mathcal{A}(s_1,t_1,p_n,q_n) + b\mathcal{S}(s_1,s_1,p_{n+1}), \end{aligned} \tag{19}$$

where

$$\begin{aligned}\mathcal{A}(s_1,t_1,p_n,q_n) &= c_1\mathcal{S}(s_1,s_1,p_n) + c_2\mathcal{S}(t_1,t_1,q_n) + c_3\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) \\ &\quad +c_4\mathcal{S}(F(s_1,t_1),F(s_1,t_1),p_n) + c_5\mathcal{S}(F(p_n,q_n),F(p_n,q_n),p_n) \\ &\quad +c_6\mathcal{S}(F(p_n,q_n),F(p_n,q_n),s_1) \\ &= c_1\mathcal{S}(s_1,s_1,p_n) + c_2\mathcal{S}(t_1,t_1,q_n) + c_3\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) \\ &\quad +c_4\mathcal{S}(F(s_1,t_1),F(s_1,t_1),p_n) + c_5\mathcal{S}(p_{n+1},p_{n+1},p_n) \\ &\quad +c_6\mathcal{S}(p_{n+1},p_{n+1},s_1), \end{aligned}$$

taking the limit as $n \to \infty$, we get

$$\mathcal{A}(s_1,t_1,p_n,q_n) = (c_3 + c_4)\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1).$$

Using this in equation (19), we get

$$\begin{aligned}\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) &\leq 2b(c_3+c_4)\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) \\ &\quad +b\mathcal{S}(s_1,s_1,p_{n+1}). \end{aligned} \tag{20}$$

Taking the limit as $n \to \infty$ in equation (20) and using condition (CS_b2), we get

$$\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) \leq 2b(c_3+c_4)\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1),$$

which is a contradiction, since $2b(c_3 + c_4) < 1$. Hence, we conclude that $\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) = 0$ and so $F(s_1,t_1) = s_1$. Likewise, we can prove that $F(t_1,s_1) = t_1$. Thus, (s_1,t_1) is a coupled fixed point of F.

Now, we show that the uniqueness of coupled fixed point. For this, assume that (s_2,t_2) is another coupled fixed point of F such that $(s_1,t_1) \neq (s_2,t_2)$. Now, from equation (12) and using Lemma 2.26, conditions (CS_b2) and (CS_b3), we have

$$\begin{aligned}\mathcal{S}(s_1,s_1,s_2) &= \mathcal{S}(F(s_1,t_1),F(s_1,t_1),F(s_2,t_2)) \\ &\leq \mathcal{A}(s_1,t_1,s_2,t_2), \end{aligned} \tag{21}$$

where

$$\begin{aligned}\mathcal{A}(s_1,t_1,s_2,t_2) &= c_1\mathcal{S}(s_1,s_1,s_2) + c_2\mathcal{S}(t_1,t_1,t_2) + c_3\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_1) \\ &\quad +c_4\mathcal{S}(F(s_1,t_1),F(s_1,t_1),s_2) + c_5\mathcal{S}(F(s_2,t_2),F(s_2,t_2),s_2) \\ &\quad +c_6\mathcal{S}(F(s_2,t_2),F(s_2,t_2),s_1) \\ &= c_1\mathcal{S}(s_1,s_1,s_2) + c_2\mathcal{S}(t_1,t_1,t_2) + c_3\mathcal{S}(s_1,s_1,s_1) \\ &\quad +c_4\mathcal{S}(s_1,s_1,s_2) + c_5\mathcal{S}(s_2,s_2,s_2) + c_6\mathcal{S}(s_2,s_2,s_1) \\ &= (c_1+c_4)\mathcal{S}(s_1,s_1,s_2) + c_6 b\mathcal{S}(s_1,s_1,s_2) + c_2\mathcal{S}(t_1,t_1,t_2) \\ &= (c_1+c_4+c_6 b)\mathcal{S}(s_1,s_1,s_2) + c_2\mathcal{S}(t_1,t_1,t_2). \end{aligned}$$

Using this in equation (21), we obtain

$$\mathcal{S}(s_1, s_1, s_2) \leq (c_1 + c_4 + c_6 b)\mathcal{S}(s_1, s_1, s_2) + c_2\mathcal{S}(t_1, t_1, t_2). \tag{22}$$

By similar fashion, one can show that

$$\mathcal{S}(t_1, t_1, t_2) \leq (c_1 + c_4 + c_6 b)\mathcal{S}(t_1, t_1, t_2) + c_2\mathcal{S}(s_1, s_1, s_2). \tag{23}$$

From equations (22) and (23), we obtain

$$\begin{aligned} \mathcal{S}(s_1, s_1, s_2) + \mathcal{S}(t_1, t_1, t_2) &\leq (c_1 + c_4 + c_6 b)[\mathcal{S}(s_1, s_1, s_2) + \mathcal{S}(t_1, t_1, t_2)] \\ &\quad + c_2[\mathcal{S}(s_1, s_1, s_2) + \mathcal{S}(t_1, t_1, t_2)] \\ &= (c_1 + c_2 + c_4 + c_6 b)[\mathcal{S}(s_1, s_1, s_2) + \mathcal{S}(t_1, t_1, t_2)], \end{aligned}$$

which is a contradiction, since $c_1 + c_2 + c_4 + c_6 b < 1$. Hence, we conclude that $\mathcal{S}(s_1, s_1, s_2) + \mathcal{S}(t_1, t_1, t_2) = 0$. Thus, $\mathcal{S}(s_1, s_1, s_2) = 0$ and $\mathcal{S}(t_1, t_1, t_2) = 0$ and so $s_1 = s_2$ and $t_1 = t_2$. Hence (s_1, t_1) is the unique coupled fixed point of F. This completes the proof. □

Remark 3.8. *Theorem 3.7 also extends and generalizes the results of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

If we take $c_1 = k$, $c_2 = l$ and $c_3 = c_4 = c_5 = c_6 = 0$ in Theorem 3.7, then we have the following result.

Corollary 3.9. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p, q), F(p, q), F(u, v)) \leq k\,\mathcal{S}(p, p, u) + l\,\mathcal{S}(q, q, v), \tag{24}$$

for all $p, q, u, v \in \Xi$, where k, l are nonnegative constants such that $k + l < 1$. Then F has a unique coupled fixed point.

Remark 3.10. *Corollary 3.9 extends Theorem 2.2 of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

If we take $k = l = m$, where $m \in [0, \frac{1}{2})$ in Corollary 3.9, then we have the following result.

Corollary 3.11. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p, q), F(p, q), F(u, v)) \leq m\,[\mathcal{S}(p, p, u) + \mathcal{S}(q, q, v)], \tag{25}$$

for all $p, q, u, v \in \Xi$, where $m \in [0, \frac{1}{2})$ is a constant. Then F has a unique coupled fixed point.

Remark 3.12. *Corollary 3.11 extends Corollary 2.3 of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

If we take $c_1 = c_2 = c_4 = c_6 = 0$, $c_3 = k$ and $c_5 = l$ in Theorem 3.7, then we have the following result.

Corollary 3.13. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p, q), F(p, q), F(u, v)) \leq k\,\mathcal{S}(F(p, q), F(p, q), p) + l\,\mathcal{S}(F(u, v), F(u, v), u), \tag{26}$$

for all $p, q, u, v \in \Xi$, where k, l are nonnegative constants such that $k + l \in (0, \frac{1}{b})$. Then F has a unique coupled fixed point.

Remark 3.14. *Corollary 3.13 extends Theorem 2.5 of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

If we take $c_1 = c_2 = c_3 = c_5 = 0$, $c_4 = k$ and $c_6 = l$ in Theorem 3.7, then we have the following result.

Corollary 3.15. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p,q),F(p,q),F(u,v)) \leq k\,\mathcal{S}(F(p,q),F(p,q),u) + l\,\mathcal{S}(F(u,v),F(u,v),p), \tag{27}$$

for all $p,q,u,v \in \Xi$, where k,l are nonnegative constants such that $k + l \in [0, \frac{1}{b^2(2+b)})$. Then F has a unique coupled fixed point.

Remark 3.16. *Corollary 3.15 extends Theorem 2.6 of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

Remark 3.17. *Our results also generalize the results of Singh and Singh [26] for more general contractive conditions.*

If we set $k = l = n$, where $n \in [0, \frac{1}{2b})$ in Corollary 3.13, then we have the following result.

Corollary 3.18. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p,q),F(p,q),F(u,v)) \leq n\,[\mathcal{S}(F(p,q),F(p,q),p) + \mathcal{S}(F(u,v),F(u,v),u)], \tag{28}$$

for all $p,q,u,v \in \Xi$, where $n \in [0, \frac{1}{2b})$ is a constant. Then F has a unique coupled fixed point.

Remark 3.19. *Corollary 3.18 extends Corollary 2.7 of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

If we set $k = l = h$, where $h \in [0, \frac{1}{2b^2(2+b)})$ in Corollary 3.15, then we have the following result.

Corollary 3.20. *Let $(\Xi, \mathcal{S})$ be a complete cone S_b-metric space with $b \geq 1$ and $\mathbb{P}$ be a normal cone with normal constant K. Suppose that the mapping $F\colon \Xi \times \Xi \to \Xi$ satisfies the following contractive condition:*

$$\mathcal{S}(F(p,q),F(p,q),F(u,v)) \leq h\,[\mathcal{S}(F(p,q),F(p,q),u) + \mathcal{S}(F(u,v),F(u,v),p)], \tag{29}$$

for all $p,q,u,v \in \Xi$, where $h \in [0, \frac{1}{2b^2(2+b)})$ is a constant. Then F has a unique coupled fixed point.

Remark 3.21. *Corollary 3.20 extends Corollary 2.8 of Sabetghadam et al. [18] from cone metric space to the setting of cone S_b-metric space.*

Now, we illustrate an example in support of the result.

Example 3.22. *Let $\mathbb{E} = \mathbb{R}^2$, the Euclidean plane, $\mathbb{P} = \{(u,v) \in \mathbb{R}^2 : u,v \geq 0\}$ a normal cone in $\mathbb{E}$. Let $\Xi = \mathbb{R}$ and the function $\mathcal{S}\colon \Xi^3 \to \mathbb{E}$ defined by $\mathcal{S}(u,v,z) = \big(|u-z| + |v-z|, |u-z| + |v-z|\big)$ for all $u,v,z \in \Xi$. Then $(\Xi, \mathcal{S})$ is a cone S_b-metric space with coefficient $b = 1$ and $(\Xi, \mathcal{S})$ is a complete cone S_b-metric space. Now, we consider the mapping $F\colon \Xi \times \Xi \to \Xi$ by $F(u,v) = \frac{u+v}{5}$ for all $u,v \in \Xi$. Let $p,q,u,v \in \Xi$. Then, we have*

$$\begin{aligned}
\mathcal{S}(F(p,q),F(p,q),F(u,v)) &= \mathcal{S}\Big(\frac{p+q}{5}, \frac{p+q}{5}, \frac{u+v}{5}\Big) \\
&= \Big(\Big|\frac{p+q}{5} - \frac{u+v}{5}\Big| + \Big|\frac{p+q}{5} - \frac{u+v}{5}\Big|, \\
&\quad \Big|\frac{p+q}{5} - \frac{u+v}{5}\Big| + \Big|\frac{p+q}{5} - \frac{u+v}{5}\Big|\Big) \\
&= \frac{2}{5}\Big(|p+q-u-v|, |p+q-u-v|\Big) \\
&\leq \frac{2}{5}\Big((|p-u|, |p-u|) + (|q-v|, |q-v|)\Big).
\end{aligned}$$

$$\mathcal{S}(p,p,u) = \Big(|p-u|+|p-u|, |p-u|+|p-u|\Big) = 2(|p-u|, |p-u|).$$

$$\mathcal{S}(q,q,v) = \Big(|q-v|+|q-v|, |q-v|+|q-v|\Big) = 2(|q-v|, |q-v|).$$

And

$$\mathcal{S}(p,p,u) + \mathcal{S}(q,q,v) = 2[(|p-u|, |p-u|) + (|q-v|, |q-v|)].$$

Hence

$$\begin{aligned}\mathcal{S}(F(p,q),F(p,q),F(u,v)) &\leq \frac{2}{5}\Big((|p-u|, |p-u|) + (|q-v|, |q-v|)\Big) \\ &= \frac{1}{5}\Big[2\Big((|p-u|, |p-u|) + (|q-v|, |q-v|)\Big)\Big] \\ &= m\Big(\mathcal{S}(p,p,u) + \mathcal{S}(q,q,v)\Big),\end{aligned}$$

where $m = \frac{1}{5} \in [0, \frac{1}{2})$. *Thus all the conditions of Corollary 3.11 are satisfied. Hence by Corollary 3.11, F has a unique coupled fixed point. In this case,* $(0,0)$ *is the unique coupled fixed point.*

4. Conclusion

In this paper, we prove some coupled fixed point theorems via contractive type conditions in the framework of complete cone S_b-metric spaces. Furthermore, we provide some consequences as corollaries of the established results. Also, an illustrative example is given in support of the established result. The results obtained in this paper extend, unify and generalize several results from the existing literature (see, for example, [18], [26] and many others).

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