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ρ-Almost periodic ultradistributions in $\mathbb{R}^n$

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Abstract. In this paper, we introduce and analyze ρ-almost periodic type ultradistributions in $\mathbb{R}^n$ with values in complex Banach spaces. We investigate the basic properties of ρ-almost periodic type ultradistributions and provide some structural results about them.

1. Introduction and preliminaries

The class of almost periodic functions was introduced by the Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (see the research monographs [3], [8], [9], [11], [15], [16] and [22] for further information concerning almost periodic functions and their applications). Suppose that $(X, \|\cdot\|)$ is a complex Banach space and $F : \mathbb{R}^n \to X$ is a continuous function ($n \in \mathbb{N}$). Then we say that the function $F(\cdot)$ is almost periodic if for each $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$ with

$$\left\|F(\mathbf{t} + \tau) - F(\mathbf{t})\right\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n;$$

here, $|\cdot - \cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$ and τ is usually called an ϵ-almost period of $F(\cdot)$. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic and a continuous function $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges uniformly to $F(\cdot)$.

The notion of a bounded distribution and the notion an almost periodic distribution were introduced in the pioneering papers by L. Schwartz (see, e.g., [26]), where the author analyzed the scalar-valued case. The bounded and almost periodic distributions with values in general Banach spaces were introduced by I. Cioranescu in [6] (1990); see also [1]-[2]. Further on, the class of scalar-valued almost periodic ultradistributions was introduced by I. Cioranescu [7] (1992) and the class of vector-valued almost periodic ultradistributions was introduced in [18] (2018).

In this paper, we consider various classes of ρ-almost periodic type ultradistributions in $\mathbb{R}^n$; for more details about one-dimensional almost periodic ultradistributions, we also refer the reader to the list of references quoted in [15]. In such a way, we continue the research study of ρ-almost periodic type distributions

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in $\mathbb{R}^n$ (see [21] for more details in this direction) as well as the research studies [19] and [20] (a joint work with S. Pilipović, D. Velinov and V. Fedorov), where we have analyzed some classes of one-dimensional c-almost periodic (ultra-)distributions.

Concerning our topic, we would like to recall that the scalar-valued almost periodic ultradistributions in $\mathbb{R}^n$ was analyzed by M. C. Gómez-Collado in [10] within the theory of ω-ultradistributions (2000); furthermore, all structural results established in [10] holds in the vector-valued setting. The main result of this research article, [10, Theorem 4.2], holds for the almost periodic ultradistributions of (M_p)-class and the almost periodic ultradistributions of $\{M_p\}$-class, provided that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$ as well as the conditions (M.1), (M.2) and (M.3) clarified below. Our main results are formulated within the Komatsu theory of ultradistributions, in the concrete situation in which the sequence (M_p) does not necessarily satisfy (M.3) but a slightly weaker condition (M.3').

We need the following notion (cf. [17, Definition 2.1.1] and [17, Section 2.1] for more details about Bohr (I', ρ)-almost periodic type functions):

Definition 1.1. Suppose that $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \to X$ is a continuous function, ρ is a binary relation on X and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr (I', ρ)-almost periodic if and only if for every $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that, for every $\mathbf{t} \in I$, there exists an element $y_{\mathbf{t}} \in \rho(F(\mathbf{t}))$ such that

$$\left\| F(\mathbf{t} + \tau) - y_{\mathbf{t}} \right\| \leq \epsilon.$$

(ii) $F(\cdot;\cdot)$ is (I', ρ)-uniformly recurrent if and only if there exists a sequence (τ_k) in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and that, for every $\mathbf{t} \in I$, there exists an element $y_{\mathbf{t}} \in \rho(F(\mathbf{t}))$ such that

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I} \left\| F(\mathbf{t} + \tau_k) - y_{\mathbf{t}} \right\| = 0.$$

Denote by $AP_{I',\rho}(I : X)$ $[UR_{I',\rho}(I : X)]$ the collection of all Bohr (I', ρ)-almost periodic functions $[(I', \rho)$-uniformly recurrent functions]; if $I' = I$, then we omit the term "I'" from the notation and, if $\rho = \mathrm{I}$, which stands for the identity operator on X here and hereafter, then we omit the term "ρ" from the notation.

1.1. Vector-valued ultradistributions

Let (M_p) be a sequence of positive real numbers satisfying $M_0 = 1$ and the following conditions:

(M.1): $M_p^2 \leq M_{p+1}M_{p-1}$, $p \in \mathbb{N}$,

(M.2): $M_p \leq AH^p \sup_{0\leq i\leq p} M_i M_{p-i}$, $p \in \mathbb{N}$, for some A, $H > 1$,

(M.3'): $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$.

Any use of the condition

(M.3): $\sup_{p\in\mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty$,

which is slightly stronger than (M.3′), will be explicitly emphasized. If $s > 1$, then we know that the Gevrey sequence $(p!^s)$ satisfies the above conditions. The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by $\mathcal{D}^{(M_p)}(\mathbb{R}^n) := \mathrm{indlim}_{K\Subset\Subset\mathbb{R}^n} \mathcal{D}_K^{(M_p)}$, resp., $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n) := \mathrm{indlim}_{K\Subset\Subset\mathbb{R}^n} \mathcal{D}_K^{\{M_p\}}$, where $\mathcal{D}_K^{(M_p)} := \mathrm{projlim}_{h\to\infty} \mathcal{D}_K^{M_p,h}$, resp., $\mathcal{D}_K^{\{M_p\}} := \mathrm{indlim}_{h\to 0} \mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h} := \left\{ \phi \in C^\infty(\mathbb{R}^n) : \mathrm{supp}\phi \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty \right\}$$

and

$$\|\phi\|_{M_p,h,K} := \sup\left\{ \frac{h^{|\alpha|}|\phi^{(\alpha)}(\mathbf{t})|}{M_{|\alpha|}} : \mathbf{t} \in K,\ \alpha \in \mathbb{N}_0^n \right\}.$$

The asterisk $*$ is used to denote both, the Beurling case (M_p) or the Roumieu case $\{M_p\}$. The space consisted of all continuous linear functions from $\mathcal{D}^*(\mathbb{R}^n)$ into X, denoted by $\mathcal{D}'^*(\mathbb{R}^n : X)$, is said to be the space of n-dimensional X-valued ultradistributions of $*$-class.

We say that the operator of infinite differentiation $P(D) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha D^\alpha$ is an ultradifferential operator of class (M_p), resp., of class $\{M_p\}$, if there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_\alpha| \leq C l^{|\alpha|}/M_{|\alpha|}$, $\alpha \in \mathbb{N}_0^n$; see [12] for further information. We introduce the space $\mathcal{E}^*(\mathbb{R}^n : X)$ and the convolution of an n-dimensional X-valued ultradistribution of $*$-class and an n-dimensional scalar-valued ultradifferentiable function in the same way as on pages 671 and 685 in [14]. If $T \in \mathcal{D}'^*(\mathbb{R}^n : X)$ and $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$, then we define $(T * \varphi)(t) := \langle T, \varphi(\mathbf{t} - \cdot)\rangle$, $\mathbf{t} \in \mathbb{R}^n$; then we know that $T * \varphi \in \mathcal{E}^*(\mathbb{R}^n : X)$. Set also $\langle T_{\mathbf{t}}, \varphi\rangle := \langle T, \varphi(\cdot - \mathbf{t})\rangle$ for $\mathbf{t} \in \mathbb{R}^n$.

The tempered ultradistributions of Beurling, resp., Roumieu type, are defined by S. Pilipović [25] as duals of the corresponding test spaces

$$\mathcal{S}^{(M_p)}(\mathbb{R}^n) := \text{projlim}_{h \to \infty} \mathcal{S}^{M_p,h}(\mathbb{R}^n), \quad \text{resp.,} \quad \mathcal{S}^{\{M_p\}}(\mathbb{R}^n) := \text{indlim}_{h \to 0} \mathcal{S}^{M_p,h}(\mathbb{R}^n),$$

where

$$\mathcal{S}^{M_p,h}(\mathbb{R}^n) := \left\{ \phi \in C^\infty(\mathbb{R}^n) : \|\phi\|_{M_p,h} < \infty \right\} \quad (h > 0),$$
$$\|\phi\|_{M_p,h} := \sup\left\{ \frac{h^{|\alpha|+|\beta|}}{M_{|\alpha|}M_{|\beta|}} (1 + |\mathbf{t}|^2)^{|\beta|/2} |\phi^{(\alpha)}(\mathbf{t})| : \mathbf{t} \in \mathbb{R}^n,\ \alpha,\ \beta \in \mathbb{N}_0^n \right\}.$$

A continuous linear mapping $\mathcal{S}^{(M_p)}(\mathbb{R}^n) \to X$, resp., $\mathcal{S}^{\{M_p\}}(\mathbb{R}^n) \to X$, is said to be an n-dimensional X-valued tempered ultradistribution of Beurling, resp., Roumieu type.

For any $h > 0$, we define

$$\mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h\big) := \Bigg\{ f \in C^\infty(\mathbb{R}^n : X)\ ;\ f^{(\alpha)} \in L^1(\mathbb{R}^n : X) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ and }$$
$$\|f\|_{1,h} := \sup_{\alpha \in \mathbb{N}_0^n} \frac{h^{|\alpha|}\|f^{(\alpha)}\|_1}{M_{|\alpha|}} < \infty \Bigg\},$$

where $\|\cdot\|_1$ denotes the norm in $L^1(\mathbb{R}^n)$. Then $(\mathcal{D}_{L^1}(\mathbb{R}^n, (M_p), h), \|\cdot\|_{1,h})$ is a Banach space and the space of all n-dimensional X-valued bounded Beurling ultradistributions of class (M_p), resp., n-dimensional X-valued bounded Roumieu ultradistributions of class $\{M_p\}$, is defined as the space consisting of all linear continuous mappings from $\mathcal{D}_{L^1}(\mathbb{R}^n, (M_p))$, resp., $\mathcal{D}_{L^1}(\mathbb{R}^n, \{M_p\})$, into X, where

$$\mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p)\big) := \text{projlim}_{h \to +\infty} \mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h\big),$$

resp.,

$$\mathcal{D}_{L^1}\big(\mathbb{R}^n, \{M_p\}\big) := \text{indlim}_{h \to 0+} \mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h\big).$$

We can simply prove that the statement of [7, Lemma 1] continues to hold in the higher-dimensional setting, even if the condition (M.3) is neglected; see also [10, Proposition 2.4]. Furthermore, we can simply prove that the restriction of any n-dimensional X-valued bounded ultradistribution of $*$-class to the space $\mathcal{S}^*(\mathbb{R}^n)$ is an n-dimensional X-valued tempered ultradistribution of $*$-class.

For our further work, we will introduce the following spaces of vector-valued bounded ultradistributions of $*$-class. For any $h > 0$, we define

$$\mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h, s\big) := \Bigg\{ f \in C^\infty(\mathbb{R}^n : X) \ ; \ \sup_{s \in B(\cdot,1)} \Big| f^{(\alpha)}(s) \Big| \in L^1(\mathbb{R}^n : X) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ and}$$

$$\|f\|_{1,h,s} := \sup_{\alpha \in \mathbb{N}_0^n} \frac{h^{|\alpha|} \Big| \sup_{\mathbf{s} \in B(\cdot,1)} \big| f^{(\alpha)}(\mathbf{s}) \big| \Big|_1}{M_{|\alpha|}} = \sup_{\alpha \in \mathbb{N}_0^n} \frac{h^{|\alpha|} \int_{\mathbb{R}^n} \Big| \sup_{\mathbf{s} \in B(x,1)} \big| f^{(\alpha)}(\mathbf{s}) \big| \Big| \, dx}{M_{|\alpha|}} < \infty \Bigg\}.$$

Then $(\mathcal{D}_{L^1}(\mathbb{R}^n, (M_p), h, s), \| \cdot \|_{1,h,s})$ is a Banach space which is continuously embedded into $(\mathcal{D}_{L^1}(\mathbb{R}^n, (M_p), h), \| \cdot \|_{1,h})$ for all $h > 0$. The space of all n-dimensional X-valued strongly bounded Beurling ultradistributions of class (M_p), resp., n-dimensional X-valued strongly bounded Roumieu ultradistributions of class $\{M_p\}$, is defined as the space consisting of all linear continuous mappings from $\mathcal{D}_{L^1}(\mathbb{R}^n, (M_p), s)$, resp., $\mathcal{D}_{L^1}(\mathbb{R}^n, \{M_p\}, s)$, into X, where

$$\mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), s\big) := \text{projlim}_{h \to +\infty} \mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h, s\big),$$

resp.,

$$\mathcal{D}_{L^1}\big(\mathbb{R}^n, \{M_p\}, s\big) := \text{indlim}_{h \to 0+} \mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h, s\big).$$

As above, we assume that these spaces are equipped with the strong topologies; we denote them by $\mathcal{D}'_{L^1}(\mathbb{R}^n, (M_p), s : X)$ and $\mathcal{D}'_{L^1}(\mathbb{R}^n, \{M_p\}, s : X)$, respectively. It is completely without scope of this paper to further analyze the topological properties and the structural theorems for the spaces $\mathcal{D}_{L^1}(\mathbb{R}^n, *, s)$ and their duals here; cf. also [24]. We will only note that a very simple argumentation shows that $\mathcal{S}^*(\mathbb{R}^n)$ is a linear subspace of $\mathcal{D}_{L^1}(\mathbb{R}^n, *, s)$.

The space of all continuous linear mappings from $\mathcal{D}_{L^1}(\mathbb{R}^n, *)$ into X, equipped with the topology of unform convergence over bounded subsets of $\mathcal{D}_{L^1}(\mathbb{R}^n, *, s)$, will be denoted by $\mathcal{D}'_{L^1,s}(\mathbb{R}^n, * : X)$. This space is locally convex since the family of seminorms $(\sup_{\varphi \in B} \| \langle \cdot, \varphi \rangle \|)_{B \in \mathcal{B}d}$, where $\mathcal{B}d$ denotes the collection of all bounded subsets of $\mathcal{D}'_{L^1}(\mathbb{R}^n, *, s : X)$, satisfies the conditions 1. and 2. from [23, Lemma 22.4], as easily approved; cf. also [23, Lemma 22.5]. It is not clear whether the spaces $\mathcal{D}'_{L^1,s}(\mathbb{R}^n, * : X)$ and $\mathcal{D}'_{L^1}(\mathbb{R}^n, *)$ are topologically equivalent.

We refer the reader to [12]-[14] for more details about the theory of ultradistributions.

2. Multi-dimensional ρ-almost periodic type ultradistributions in $\mathbb{R}^n$

We start this section by introducing the following notion (cf. [21, Definition 2.1] for the distributional analogue):

Definition 2.1. Suppose that $I' \subseteq \mathbb{R}^n$, ρ is a binary relation on X and $T \in \mathcal{D}'^*(\mathbb{R}^n : X)$. Then we say that T is an (I', ρ)-almost periodic ultradistribution of $*$-class [(I', ρ)-uniformly recurrent ultradistribution of $*$-class], if $T * \varphi \in AP_{I',\rho}(\mathbb{R}^n : X)$ for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ [$T * \varphi \in UR_{I',\rho}(\mathbb{R}^n : X)$ for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$]. If $I' = I$ and $\rho = \mathrm{I}$, then we also say that T is an almost periodic ultradistribution of $*$-class [uniformly recurrent ultradistribution of $*$-class].

It is clear that the structural characterizations of Bohr (I', ρ)-almost periodic functions [(I', ρ)-uniformly recurrent functions] can be used to provide certain results about (I', ρ)-almost periodic ultradistributions of $*$-class [(I', ρ)-uniformly recurrent ultradistributions of $*$-class]. For example, using [17, Corollary 2.1.4], we can immediately clarify the following result:

Proposition 2.2. *Suppose that $I' \subseteq \mathbb{R}^n$ and $\rho : X \to X$. If T is an (I', ρ)-almost periodic ultradistribution of $*$-class [(I', ρ)-uniformly recurrent ultradistribution of $*$-class], then T is an $(I' - I', \mathrm{I})$-almost periodic ultradistribution of $*$-class [$(I' - I', \mathrm{I})$-uniformly recurrent ultradistribution of $*$-class].*

We will omit such results in the sequel. Now we will reconsider the result established in [21, Theorem 2.2]:

Theorem 2.3. *Suppose that $\rho = A \in L(X)$, $\emptyset \neq I' \subseteq \mathbb{R}^n$, there exist an integer $k \in \mathbb{N}$ and (I', A)-almost periodic ((I', A)-uniformly recurrent) functions $F_j : \mathbb{R}^n \to X$ $(0 \leq j \leq k)$ such that the function*

$$\mathbf{t} \mapsto F(\mathbf{t}) \equiv \big(F_0(\mathbf{t}), \cdots, F_k(\mathbf{t})\big), \quad t \in \mathbb{R}^n \tag{2.1}$$

is (I', A_{k+1})-almost periodic ((I', A_{k+1})-uniformly recurrent), where $A_{k+1} \in L(X^{k+1})$ is given by $A_{k+1}(x_0, x_1, ..., x_k) := (Ax_0, Ax_1, ..., Ax_k)$, $(x_0, x_1, ..., x_k) \in X^{k+1}$. Set

$$T := \sum_{j=0}^{k} \sum_{\alpha \in \mathbb{N}_0^n} a_{\alpha,j} F_j^{(\alpha)}$$

and suppose that there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_{\alpha,j}| \leq C l^{|\alpha|}/M_{|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$ and $0 \leq j \leq k$. Then T is an (I', ρ)-almost periodic ultradistribution of $*$-class [(I', ρ)-uniformly recurrent ultradistribution of $*$-class].

Proof. We will provide all details of proof in the case of consideration of (I', ρ)-almost periodic ultradistribution of Beurling class. First of all, [14, Theorem 7.7] implies that $T \in \mathcal{D}'^*(\mathbb{R}^n : X)$; further on, for each $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ and $\mathbf{t} \in \mathbb{R}^n$ we have:

$$\begin{aligned}(T * \varphi)(\mathbf{t}) = \langle T, \varphi(\mathbf{t} - \cdot)\rangle &= \sum_{j=0}^{k} \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \varphi^{(\alpha)}(\mathbf{t} - v) F_j(v)\, dv \\ &= \sum_{j=0}^{k} \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \varphi^{(\alpha)}(v) F_j(\mathbf{t} - v)\, dv.\end{aligned}$$

Let $\varepsilon > 0$ be given. Then there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that, for every $\mathbf{t} \in \mathbb{R}^n$, we have

$$\big\|F_j(\mathbf{t} + \tau) - AF_j(\mathbf{t})\|_Y \leq \epsilon, \quad 0 \leq j \leq k.$$

Supopose that $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$ and $\operatorname{supp} \varphi \subseteq K$. Then there exists $h > l$ such that

$$\int_{\mathbb{R}^n} \Big|\varphi^{(\alpha)}(v)\Big|\, dv = \int_K \Big|\varphi^{(\alpha)}(v)\Big|\, dv \leq m(K) \|\varphi\|_{M_p,h,K} \frac{M_{|\alpha|}}{h^{|\alpha|}}, \quad \alpha \in \mathbb{N}_0^n.$$

Therefore, for every $\mathbf{t} \in \mathbb{R}^n$, we have:

$$\begin{aligned}&\|(T * \varphi)(t + \tau) - A(T * \varphi)(\mathbf{t})\| \\ &\quad\leq \sum_{j=0}^{k} \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \Big|\varphi^{(\alpha)}(v)\Big| \cdot \big\|F_j(\mathbf{t} + \tau - v) - AF_j(\mathbf{t} - v)\big\|\, dv \\ &\quad\leq \varepsilon \sum_{j=0}^{k} \sum_{\alpha \in \mathbb{N}_0^n} \frac{l^{|\alpha|}}{M_{|\alpha|}} \int_{\mathbb{R}^n} \Big|\varphi^{(\alpha)}(v)\Big|\, dv \leq \sum_{j=0}^{k} \sum_{\alpha \in \mathbb{N}_0^n} \frac{l^{|\alpha|}}{M_{|\alpha|}} m(K) \|\varphi\|_{M_p,h,K} \frac{M_{|\alpha|}}{h^{|\alpha|}},\end{aligned}$$

which simply implies the required statement. □

As already mentioned in the introductory part, if the sequence (M_p) additionally satisfies (M.3), $A = \mathrm{I}$ and T is an almost periodic ultradistribution of $*$-class, then there exist two almost periodic functions $F : \mathbb{R}^n \to X$, $G : \mathbb{R}^n \to X$ and an ultradifferential operator $P(D)$ of $*$-class such that $T = P(D)F + G$; see the formulation of [10, Theorem 4.2] and the statements of [10, Corollary 2.6] and [24, Lemma 5], which are the main auxiliary results needed for the proof of this result. Unfortunately, it is not clear how one can prove an analogue of this result for c-almost periodic ultradistributions of $*$-class (i.e., ρ-almost periodic ultradistributions of $*$-class with $\rho = c\mathrm{I}$ for some $c \in \mathbb{C} \setminus \{0\}$); see also [19, p. 18] for more details given in the one-dimensional setting.

Suppose now that $I' = I$ and $\rho = \mathrm{I}$. Then the requirements of Theorem 2.3 imply that the following condition holds:

(BC) The set of all translations $\{T_{\mathbf{t}} : \mathbf{t} \in \mathbb{R}^n\}$ is relatively compact in $\mathcal{D}'_{L^1}(\mathbb{R}^n, *)$.

Furthermore, the validity of (BC) implies that T is an almost periodic ultradistribution of $*$-class; see the proofs of [7, Theorem 2] and [10, Theorem 4.2].

Define now

$$\mathcal{E}^*_{I',\rho,AP}(\mathbb{R}^n : X) := \mathcal{E}^*\big(\mathbb{R}^n : X\big) \cap AP_{I',\rho}\big(\mathbb{R}^n : X\big)$$

and

$$\mathcal{E}^*_{I',\rho,UR}(\mathbb{R}^n : X) := \mathcal{E}^*\big(\mathbb{R}^n : X\big) \cap UR_{I',\rho}\big(\mathbb{R}^n : X\big).$$

If $\varphi : \mathbb{R}^n \to \mathbb{C}$, then we define $\check{\varphi}(x) := \varphi(-x)$, $x \in \mathbb{R}^n$; furthermore, by $(e_1, ..., e_n)$ we denote the standard basis of $\mathbb{R}^n$. We continue by stating the following result:

Theorem 2.4. *Suppose that T is an n-dimensional X-valued bounded ultradistribution of $*$-class, $I' \subseteq \mathbb{R}^n$, ρ is a binary relation on X, $D(\rho)$ is a closed subset of Y and ρ is continuous in the following sense:*

(C_ρ) For each $\epsilon > 0$ there exists $\delta > 0$ such that, for every $y_1,\ y_2 \in Y$ with $\|y_1 - y_2\|_Y < \delta$, we have $\|z_1 - z_2\|_Y < \epsilon/3$ for every $z_1 \in \rho(y_1)$ and $z_2 \in \rho(y_2)$.

Then T is an (I', ρ)-almost periodic ultradistribution of $*$-class [(I', ρ)-uniformly recurrent ultradistribution of $*$-class] if and only if there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{E}^*_{I',\rho,AP}(\mathbb{R}^n : X)$ [$\mathcal{E}^*_{I',\rho,UR}(\mathbb{R}^n : X)$] such that $\lim_{k\to+\infty} \psi_k = T$ for the topology of $\mathcal{D}'_{L^1,s}(\mathbb{R}^n, * : X)$.

Proof. We will consider the (I', ρ)-almost periodic ultradistributions of $\{M_p\}$-class, only. Suppose that there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{E}^*_{I',\rho,AP}(\mathbb{R}^n : X)$ with the prescribed properties. First of all, we will prove that for each fixed test function $\varphi \in \mathcal{D}^{\{M_p\}}$ the set of all translations $\{\varphi(\cdot - \mathbf{t}) : \mathbf{t} \in \mathbb{R}^n\}$ is bounded in $\mathcal{D}_{L^1}(\mathbb{R}^n, \{M_p\}, s : X)$. We know that there exist a compact set $K \subseteq \mathbb{R}^n$ and two real numbers $h > 0$ and $c > 0$ such that $|\varphi^{(\alpha)}(x)| \le cM_{|\alpha|}/h^{|\alpha|}$ for all $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$. Let K_1 denotes the compact set in $\mathbb{R}^n$ given by $K_1 = K + B(0, 1)$. Then we have

$$\sup_{\mathbf{t}\in\mathbb{R}^n;\alpha\in\mathbb{N}_0^n} \frac{h^{|\alpha|}\int_{\mathbb{R}^n} \sup_{\mathbf{s}\in B(x,1)}\big|\varphi^{(\alpha)}(\mathbf{s}-\mathbf{t})\big|\,dx}{M_{|\alpha|}} \le \sup_{\mathbf{t}\in\mathbb{R}^n;\alpha\in\mathbb{N}_0^n} \frac{h^{|\alpha|}\int_{\mathbb{R}^n} \sup_{\mathbf{s}\in B(x-\mathbf{t},1)}\big|\varphi^{(\alpha)}(\mathbf{s})\big|\,dx}{M_{|\alpha|}}$$

$$= \sup_{\mathbf{t}\in\mathbb{R}^n;\alpha\in\mathbb{N}_0^n} \frac{h^{|\alpha|}\int_{\mathbf{t}+K_1} cM_{|\alpha|}/h^{|\alpha|}\,dx}{M_{|\alpha|}} = cm\big(K_1\big).$$

Keeping in mind this fact, the required conclusion almost immediately from the argumentation contained in the first part of proof of [4, Proposition 7], since we have assumed that $D(\rho)$ is a closed subset of Y and ρ satisfies (C_ρ); cf. also [17, Theorem 2.1.12(v)]. Assume now that T is an (I', ρ)-almost periodic ultradistribution of $*$-class. Let $(\delta_k)_{k\in\mathbb{N}}$ be a sequence of infinitely ultradifferentiable functions of $\{M_p\}$-class such that $\operatorname{supp}\delta_k \subseteq [-(1/k), 1/k]^n$ and $\int_{\mathbb{R}^n} \delta_k(\mathbf{t})\,d\mathbf{t} = 1$ for all $k \in \mathbb{N}$. Set $\psi_k := T * \delta_k$ for all $k \in \mathbb{N}$. Then $(\psi_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{E}^*_{I',\rho,AP}(\mathbb{R}^n : X)$ and we only need to prove that $\lim_{k\to+\infty} \psi_k = T$ for the topology of $\mathcal{D}'_{L^1,s}(\mathbb{R}^n, \{M_p\} : X)$. Let $B \in \mathcal{B}d$ be fixed. Then there exists $h > 0$ such that B is contained and bounded in $\mathcal{D}_{L^1}\big(\mathbb{R}^n, (M_p), h_0H, s\big)$ for all $h_0 \in (0, h]$, where H denotes the constant from (M.2). We may assume without loss of generality that there exists $c > 0$ such that $\|\langle T, \varphi\rangle\| \le c\|\varphi\|_{1,h}$, $\varphi \in \mathcal{D}_{L^1}(\mathbb{R}^n, (M_p), h)$. Now we will estimate the term $\|[\check{\psi}_k * \varphi - \varphi]^{(\alpha)}\|_{L^1(\mathbb{R}^n)}$ ($k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$). The mean value theorem implies that for each $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ there exists a constant $c_{x,y} \in (0,1)$ such that, for every $k \geq 1+\sqrt{n}$, we have

$$\begin{aligned}
&\Big\|\big[\check{\psi}_k * \varphi - \varphi\big]^{(\alpha)}\Big\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\check{\psi}_k(y)\Big[\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)\Big]\,dy\right|dx\\
&\leq \int_{\mathbb{R}^n}\check{\psi}_k(y)\int_{\mathbb{R}^n}\Big|\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)\Big|\,dx\,dy \leq \sum_{i=1}^n\int_{\mathbb{R}^n}|y_i|\check{\psi}_k(y)\int_{\mathbb{R}^n}\Big|\varphi^{(\alpha+e_i)}\big(x-y+c_{x,y}y\big)\Big|\,dx\,dy\\
&\leq \sum_{i=1}^n\int_{\mathbb{R}^n}|y_i|\check{\psi}_k(y)\int_{\mathbb{R}^n}\sup_{\mathbf{s}\in[x-y,x]}\Big|\varphi^{(\alpha+e_i)}(\mathbf{s})\Big|\,dx\,dy \leq \frac{n}{k}\sum_{i=1}^n\int_{\mathbb{R}^n}\sup_{\mathbf{s}\in[x-y,x]}\Big|\varphi^{(\alpha+e_i)}(\mathbf{s})\Big|\,dx\\
&\leq \frac{n}{k}\int_{\mathbb{R}^n}\sup_{\mathbf{s}\in B(x,1)}\Big|\varphi^{(\alpha+e_i)}(\mathbf{s})\Big|\,dx,
\end{aligned}$$

where we have also used the Fubini theorem. Hence, we have the following:

$$\begin{aligned}
&\sup_{\varphi\in B}\Big\|\big\langle T*\psi_k - T,\varphi\big\rangle\Big\| = \sup_{\varphi\in B}\Big\|\big\langle T,\check{\psi}_k*\varphi-\varphi\big\rangle\Big\| \leq cd\sup_{\varphi\in B}\sup_{\alpha\in\mathbb{N}_0^n}\frac{h^{|\alpha|}\Big\|\big[\check{\psi}_k*\varphi-\varphi\big]^{(\alpha)}\Big\|_{L_1(\mathbb{R}^n)}}{M_{|\alpha|}}\\
&\quad\leq \frac{cnd}{k}\sup_{\varphi\in B}\sup_{\alpha\in\mathbb{N}_0^n}\frac{h^{|\alpha|}\sum_{i=1}^n\int_{\mathbb{R}^n}\sup_{\mathbf{s}\in B(x,1)}\Big|\varphi^{(\alpha+e_i)}(\mathbf{s})\Big|\,dx}{M_{|\alpha|}}\\
&\quad\leq \frac{nc}{kh}AM_1\sup_{\alpha\in\mathbb{N}_0^n}\frac{(hH)^{|\alpha|+1}\sum_{i=1}^n\int_{\mathbb{R}^n}\sup_{\mathbf{s}\in B(x,1)}\Big|\varphi^{(\alpha+e_i)}(\mathbf{s})\Big|\,dx}{M_{|\alpha|+1}}\\
&\quad\leq \frac{cn^2d}{kh}AM_1\sup_{\varphi\in B}\|\varphi\|_{1,hH,s},
\end{aligned}$$

which simply completes the proof. □

Remark 2.5. The argumentation contained in the second part of the proof of [4, Proposition 7] (cf. also [18, Lemma 1], where we have made the same mistake, and [5, Proposition 10]) is a little bit misleading since the equality

$$\int_{\mathbb{R}^n}\Big|\varphi^{(\alpha+e_i)}\big(x-y+c_{x,y}y\big)\Big|\,dx = \int_{\mathbb{R}^n}\Big|\varphi^{(\alpha+e_i)}(x)\Big|\,dx \tag{2.2}$$

is not true because the value of $c_{x,y}$ strongly depends on $x,\ y \in \mathbb{R}^n$ and it is not a constant so that the change of variable $x \mapsto x-y+c_{x,y}y$ cannot be done without further information on the Jacobian of $c_{x,y}$ for fixed $y \in \mathbb{R}^n$. Moreover, the inequality

$$\int_{\mathbb{R}^n}\Big|\varphi^{(\alpha)}(x-y)-\varphi^{(\alpha)}(x)\Big|\,dx \leq \sum_{i=1}^n|y_i|\int_{\mathbb{R}^n}\Big|\varphi^{(\alpha+e_i)}(x)\Big|\,dx \tag{2.3}$$

can be wrong, as the following counterexample shows: Suppose that $b > a > 0$, $y > 0$, $n = 1$, $\alpha = 0$ and $f(x) = x^2$, $x \geq 0$. Then

$$\begin{aligned}
\int_a^b|f(x+y)-f(y)|\,dx &= \int_a^b\big[y^2+2xy\big]\,dx = y\int_a^b\big[y+2x\big]\,dx\\
&= (b^3-a^3)/3 + y\int_a^b 2x\,dx = \big(b^3-a^3\big)/3 + y\int_a^b\big|f'(x)\big|\,dx > y\int_a^b\big|f'(x)\big|\,dx.
\end{aligned}$$

For the test functions, we can consider the sequence of smooth functions which sufficiently good approximates the function $f(\cdot)$ in the Sobolev space $W^{1,1}((a,b+y))$.

Supppose now that there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathcal{E}^*_{I',\rho,AP}(\mathbb{R}^n : X)$ such that $\lim_{k\to+\infty} \psi_k = T$ for the topology of $\mathcal{D}'_{L^1}(\mathbb{R}^n, * : X)$. Then T is an (I', ρ)-almost periodic ultradistribution of $*$-class [(I', ρ)-uniformly recurrent ultradistribution of $*$-class]; especially, in the case that $I' = I$ and $\rho = \mathrm{I}$, then T is an almost periodic ultradistribution of $*$-class [uniformly recurrent ultradistribution of $*$-class] and it can be approximated by trigonometric polynomials in the space of bounded X-valued ultradistributions of $*$-class, which has been used in [7] and [18] for the definition of an almost periodic ultradistribution of $*$-class. If the last condition holds, then we have $T * \varphi \in AP(\mathbb{R}^n : X)$, resp. $T * \varphi \in UR(\mathbb{R}^n : X)$, for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$; see the proof of implication (i) $\Leftrightarrow$ (iii) in [7, Theorem 2]. We can similarly reformulate the statements of [18, Theorem 1] and [18, Theorem 2] for almost periodic ultradistributions of $*$-class in $\mathbb{R}^n$. The statement of [19, Theorem 2.2] can be simply reformulated for c-almost periodic ultradistributions of $*$-class in $\mathbb{R}^n$ by replacing the space $\mathcal{D}'_{L^1,s}(\mathbb{R}^n, * : X)$ in its formulation with the space $\mathcal{D}'_{L^1}(\mathbb{R}^n, * : X)$; the same modification has to be done in the one-dimensional setting.

We will omit here all details concerning the existence of Bohr-Fourier coefficients of almost periodic ultradistributions of $*$-class; cf. [10] and the final part of [18, Section 2] for further information in this direction.

3. Conclusions and final remarks

In this paper, we have introduced and analyzed ρ-almost periodic type ultradistributions in $\mathbb{R}^n$ with values in complex Banach spaces. We have presented several structural results and useful remarks about the introduced classes of ρ-almost periodic type ultradistributions.

If $\emptyset \neq \mathbb{A} \subseteq C^\infty(\mathbb{R}^n : X)$, let us denote by $B'^*_{\mathbb{A}}(\mathbb{R}^n : X)$ the space of all vector-valued ultradistributions $T \in \mathcal{D}'^*(\mathbb{R}^n : X)$ such that $T * \varphi \in \mathbb{A}$ for all $\varphi \in \mathcal{D}^*(\mathbb{R}^n)$. The interested reader may try to reconsider the statements established in [18, Section 3] in the higher-dimensional setting. We close the paper with the observation that it could be also interesting to introduce and analyze the Colombeau ρ-almost periodic generalized functions in $\mathbb{R}^n$ and the Fourier ρ-almost periodic hyperfunctions in $\mathbb{R}^n$. The corresponding classes of almost automorphic generalized functions, with $\rho = \mathrm{I}$, can be also investigated.

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