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Common solutions to a system of variational inequality problems coupled with compostion of multivalued and single-valued mappings in Banach spaces and its application to nonconvex constrained optimization problem

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Abstract. In this paper, we develop a new fixed points technique for characterizing fixed point set of composition involving multivalued and single-valued mappings in Banach spaces without commuting assumption. We use our new result as tool to obtain the strong convergence of the sequence generated by a modified Krasnoselskii-Mann algorithm for finding a common solutions to a system of variational inequalities and common fixed point problems in q-uniformly smooth and p-uniformly convex Banach spaces with $q > 1, p > 1$. Finally, our theorems are applied to nonconvex constrained optimization problem.

1. Introduction

Let E be a Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0, +\infty) \to [0, \infty)$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}. \tag{1}$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\}, \forall\, x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x),\ \ x \neq 0.$$

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Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.

The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth.* Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p) \ or\ W_p^m \text{ is}$$

$$\begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases} \tag{2}$$

Let J_q denote the generalized duality mapping from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}.$$

J_2 is called the normalized duality mapping and is denoted by J. It is known that E is smooth if and only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E. Following Browder [1], we say that a Banach space has a weakly continuous normalized duality map if J is a single-valued and is weak-to-weak* sequentially continous, i.e., if $(x_n) \subset E$, $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup J(x)$ in E^*. Weak continuity of duality map J plays an important role in the fixed point theory for nonlinear operators. Recall that a Banach space E satisfies Opial property (see, e.g., [4]) if $\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|$ whenever $x_n \rightharpoonup x$, $x \neq y$. A Banach space E that has a weakly continuous normalized duality map satisfies Opial's property.

Let $C \subseteq E$ be a nonempty set. An operator $A : C \to E$ is said to be accretive if there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq 0, \ \ \forall x, y \in C.$$

An operator $A : C \to E$ is said to be α-inverse strongly accretive if, for some $\alpha > 0$,

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \alpha\|Ax - Ay\|^q, \ \ \forall x, y \in C.$$

Let C be a nonemty subset of q-uniformly smooth Banach space E and $A : C \to E$ be a nonlinear operator. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Ax^*, J_q(x - x^*)\rangle \geq 0, \ \ \forall x \in C. \tag{3}$$

The set of solutions of the variational inequality in Banach space is denoted by $VI_q(C, A)$. If $q = 2$, then $VI_q(C, A)$ is reduced to $VI(C, A)$, where $VI(C, A)$ is the set of solutions of the generalized variational inequality in Banach spaces proposed by Aoyama et al., [5] in 2005. For solving $VI(C,A)$, Aoyama et al. [5] introduced an iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C(I-\lambda_n A)x_n \ n \geq 0, \tag{4}$$

$\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ are two real number sequences. Aoyama et al. [5] proved the following weak convergence theorem to a solution of $VI(C,A)$.

Theorem 1.1. *[5] Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse strongly accretive operator with $VI(C,A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in \left[a, \frac{\alpha}{K^2}\right]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to a solution of variational inequality $VI(C,A)$, where K is the 2-uniformly smoothness constant of E.*

Recently, many authors studied the following convex feasibility problem (for short, CFP):

$$\textit{finding} \quad \textit{an } x^* \in \bigcap_{i=1}^{m} K_i \tag{5}$$

where $m \geq 1$ is an integer and each K_i is a nonempty closed convex subset of H. There is a considerable investigation on the CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [19, 21], computer tomography and radiation therapy treatment planning [20]. Recently, iterative methods for single-valued nonexpansive mappings have been applied to solve fixed points problems and variational inequality problems in Hilbert spaces, see, e.g.,[23, 24] and the references therein. Recently, Censor, Gibali and Reich [16], proved the following weak convergence theorem for solving system of variationnal inequality problem in a real Hilbert space.

Theorem 1.2. *[16] Let H be a Hilbert space. For each $1 \leq i \leq N$, let an operator $h_i : H \to H$ and a nonempty, closed and convex subset $C_i \subset H$ be given. Assume that $\bigcap_{i=1}^N C_i \neq \emptyset$ and $\Psi = \bigcap_{i=1}^N VI(C, A_i) \neq \emptyset$ and that for each $1 \leq i \leq N$, h_i is α_i-ism. Set $\alpha := \min\{\alpha_i\}$ and take $\lambda \in (0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \sum_{i=1}^{N} \omega_{n,i}(P_{C_i}(I-\lambda h_i)x_n), \ \ n \geq 0, \tag{6}$$

where $\sum_{i=1}^{N} \omega_{n,i} = 1$. Then the sequence $\{x_n\}$converges weakly to a point $z \in \Psi$.

On the other hand, many problems arising in different areas of mathematics, such as Game Theory, Control theory, Dynamic systems theory, Signal and image processing, Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations and Differential inclusions, Optimization theory equations, can be modeled by the equation

$$x \in Tx, \tag{7}$$

where T is a multivalued mapping. The solution set of this equation coincides with the fixed point set of T. The Pompeiu *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A,B) = \max\left\{ \sup_{a\in A} d(a,B), \sup_{b \in B} d(b,A) \right\}$$

for all $A, B \in CB(K)$ (see, Berinde [2]). A multivalued mapping $T : D(T) \subseteq E \to CB(E)$ is called *β-Lipschitzian* if there exists $\beta > 0$ such that

$$H(Tx,Ty) \leq \beta\|x-y\| \ \ \forall x, y \in D(T). \tag{8}$$

When $\beta \in (0,1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $\beta = 1$. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $Fix(T) := \{x \in D(T) : x \in Tx\}$. A multivalued map T is called quasi-nonexpansive if

$$H(Tx, Tp) \leq \|x - p\|$$

holds for all $x \in D(T)$ and $p \in Fix(T)$.

Remark 1.3. *It is easy to see that the class of mulivalued quasi-nonexpansive mappings properly includes that of multivalued nonexpansive maps with fixed points.*

During the last decades, the study of common fixed point problems involving multivalued and singlevalued mappings has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, [1, 5]). Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in Game Theory and Market Economy and in other areas of mathematics, such as in Non-Smooth Differential Equations and Differential Inclusions, Optimization theory. We describe briefly the connection of fixed point theory for multivalued mappings with these applications.

1.1. Optimization problems with constraints

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $A : H \to H$ be a single-valued mapping. Consider the following optimization problem:

$$(P) \quad \begin{cases} \min f(x) \\ \\ \quad Ax0. \end{cases}$$

It is known that the multivalued map, ∂f the subdifferential of f, is maximal monotone, where for $x, w \in H$,

$$w \in \partial f(x) \Leftrightarrow f(y) - f(x) \geq \langle y - x, w \rangle, \ \ \forall \ y \in H$$

$$\Leftrightarrow x \in \operatorname{argmin}(f - \langle \cdot, w \rangle).$$

It is easily seen that, for $x \in H$ with x is a solution of (P) if and only if

$$x \in Fix(T_1) \cap Fix(T_2),$$

with $T_1 := I - \partial f$ and $T_2 := I - A$, where I is the identity map of H. Therefore, x is a solution of (P) if and only if x is a solution of common fixed point problem involving multivalued and single-valued mappings. Recently, N. Tahat et al. [25], proved the following theorem for common fixed points problem involving single-valued and multivalued maps in G-metric spaces.

Theorem 1.4. *[25] Let (X, G) be a G-metric space. Set $g : X \to X$ and $T : X \to CB(X)$. Assume that there exists a function a $\alpha[0, +\infty) \to [0, 1)$ satisfying $\limsup_{r \to t} \alpha(r) < 1$ for every $t \geq 0$ such that*

$$H_G(Tx, Ty, Tz) \leq \alpha(G(gx, gy, gz))G(gx, gy, gz),$$

for all $x, y, z \in X$. If for any $x \in X, Tx \subseteq g(X)$ and $g(X)$ is a G-complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ implies $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then

1. *g and T have a unique point of coincidence.*
2. *If in addition g and T are weakly compatible, then g and T have a unique common fixed point.*

Remark 1.5. *Most existing results for solving common fixed points problem require that the operators of underlying operators must be commuting and also, the intersection of the fixed point sets $Fix(T_1) \cap Fix(T_2)$ must be nonempty.*

Motivated and inspired by the recent research work going on in this field, we consider the problem of finding a common solutions of systems of variational inequality and fixed point problems of composition involving multivalued and single-valued maps in Banach space. Thus the CSVIFP is formulated as finding a point x^* with the property

Problem 1.6.

$$x^* \in \Big(\bigcap_{i=1}^{m} VI_q(C, A_i)\Big) \;\; and \; x^* \in Fix(T_1 \circ T_2),$$

$A_i : C \to E$ is α_i-inverse strongly accretive for $i = 1, 2, ..., m, T_1 : C \to C$ be a single-valued mapping and $T_2 : C \to CB(C)$ be a multivalued mapping. Note that, the set $Fix(T_2 \circ T_1)$ is in general larger than the set $Fix(T_1) \bigcap Fix(T_2)$, see for example [3]. It is well known that Krasnoselskii-Mann iteration of nonexpansive mappings find application in many areas of mathematics and know to be weakly convergent in the infinite dimensional setting. In this paper, we introduce and study an explicit iterative scheme by a modified Krasnoselskii-Mann algorithm for approximating a solution to Problem 1.6 in q-uniformly smooth and p-uniformly convex Banach spaces with $p > 1, q > 1$. There is no compactness assumption. The results obtained in this paper are significant improvement on important recent results.

2. Preliminaries

Let C be a nonempty subsets of a smooth real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_C x + t(x - Q_C x)) = Q_C x$$

for each $x \in E$ and $t \geq 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_C x = x$ for each $x \in C$.

Lemma 2.1. *[17] Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z)\rangle \leq 0$$

for all $z \in C$ and $y \in D$.

It is noted that Lemma 2.1 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

Remark 2.2. *If K is a nonempty, closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.*

Definition 2.3. *Let E be real Banach space and $T : D(T) \subset E \to 2^E$ be a multivalued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.*

Lemma 2.4. *[26] Let E be a real Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $I - T$ is demiclosed*

Definition 2.5. *Let E be a smooth real Banach space and $T : D(T) \subset E \to E$, then T is said to be firmly nonexpansive if for all $x, y \in D(T)$, we have*

$$\|Tx - Ty\|^p \leq \langle x - y, J_p(Tx - Ty)\rangle. \tag{9}$$

If $p = 2$, we have the important special version of 9 :

$$\|Tx - Ty\|^2 \leq \langle x - y, J(Tx - Ty)\rangle.$$

Definition 2.6. *(Chidume [15]). Let E be a real Banach space, which is reflexive, smooth, and strictly convex. Define the following function $\phi_p : E \times E \to \mathbb{R}$ defined by :*

$$\phi_p(x, y) = \|x\|^p - p\langle x, J_p y\rangle + (p-1)\|y\|^p. \tag{10}$$

Lemma 2.7. *[14] For $p > 1$, let E be a p-uniformly convex smooth real Banach space E. Then, there exists $d_p > 0$ such that :*

$$d_q\|x - y\|^p \leq \phi_p(x, y), \quad \forall\, x, y \in E.$$

Lemma 2.8 (Song and Cho [13]). *Let K be a nonempty subset of a real Banach space and $T : K \to P(K)$ be a multi-valued map. Then the following are equivalent:*
(i) $x^* \in Fix(T)$;
(ii) $P_T(x^*) = \{x^*\}$;
(iii) $x^* \in Fix(P_T)$.
Moreover, $Fix(T) = Fix(P_T)$.

The resolvent operator has the following properties:

Lemma 2.9. *[3] For any $r > 0$,*
(i) A is accretive if and only if the resolvent J_r^A of A is single-valued and firmly nonexpansive;
(ii) A is m-accretive if and only if J_r^A of A is single-valued and firmly nonexpansive and its domain is the entire E;
(iii) $0 \in A(x^*)$ if and only if $x^* \in Fix(J_r^A)$, where $Fix(J_r^A)$ denotes the fixed-point set of J_r^A.

Lemma 2.10 (Xu, [12]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that (a) $\sum_{n=0}^{\infty}\alpha_n = \infty$, (b) $\limsup_{n\to\infty}\sigma_n \leq 0$ or $\sum_{n=0}^{\infty}|\sigma_n\alpha_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

*

Lemma 2.11 ([11]). *Assume that a Banach space E has a weakly continous duality mapping J_φ with jauge φ.*

$$\varphi(\|x + y\|) \leq \varphi(\|x\| + \langle y, J_\varphi(x + y)\rangle \tag{11}$$

for all x,y $\in E$ where $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma$ $t \geq 0$. In particular, for the normilized duality mapping, we have the important special version of 11:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle.$$

Lemma 2.12 (Chang et al. [8]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$, a closed ball with center 0 and radius $r > 0$. For any given sequence $\{u_1, u_2, ..., u_m\} \subset B(0)_r$ and for $i = 1, 2, ..., m$, any positive real numbers $\{\lambda_1, \lambda_2, ..., \lambda_m\}$ with $\sum_{k=1}^{m}\lambda_k = 1$, then there exists a continuous, strictly increasing and convex function*

$$g : [0, 2r] \to \mathbb{R}^+,\ g(0) = 0,$$

such that for any integer i, j with $i < j$,

$$\|\sum_{k=1}^{m}\lambda_k u_k\|^2 \leq \sum_{k=1}^{m}\lambda_k\|u_k\|^2 - \lambda_i\lambda_j g(\|u_i - u_j\|).$$

Lemma 2.13. *[18] Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \leq t_{\tau(n)+1}.$$

Lemma 2.14. *[10] Let C be a nonempty closed convex subset of q-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $A : C \to E$ be a mapping. Then $VI_q(C, A) = Fix(Q_C(I - \lambda A))$, for all $\lambda > 0$.*

Lemma 2.15. *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E. Let $\lambda > 0$ and let A be an α-inverse strongly accretive operator of C into E. If $0 < \lambda < \left(\frac{q\alpha}{C_q}\right)^{\frac{1}{q-1}}$, where C_q is the q-uniformly smooth constant of E. Then $Q_C(I - \lambda A)$ is a nonexpansive mapping.*

Proof. Let $x, y \in C$, we have

$$\begin{aligned}
\|Q_C(I-\lambda A)x - Q_C(I-\lambda A)y\|^q &= \|(I-\lambda A)x - (I-\lambda A)y\|^q \\
&\leq \|x-y\|^q - q\lambda\langle Ax - Ay, J_q(x-y)\rangle + C_q\lambda^q\|Ax - Ay\|^q \\
&\leq \|x-y\|^q - q\lambda\alpha\|Ax - Ay\|^q + C_q\lambda^q\|Ax - Ay\|^q \\
&\leq \|x-y\|^q - \lambda(q\alpha - C_q\lambda^{q-1})\|Ax - Ay\|^q \\
&\leq \|x-y\|^q.
\end{aligned}$$

Then $Q_C(I - \lambda A)$ is a nonexpansive mapping. □

3. Main Results

We now state and prove the following result.

Lemma 3.1. *For $p > 1$, let E be a p-uniformly convex smooth real Banach space and C be a nonempty closed convex subset of E. Let $T_1 : C \to C$ be a firmly nonexpansive mapping and $T_2 : C \to CB(C)$ be a multivalued quasi-nonexpansive mapping such that $T_2 p = \{p\} \, \forall \, p \in Fix(T_2)$ and $Fix(T_2) \cap Fix(T_1) \neq \emptyset$. Then, $Fix(T_2) \cap Fix(T_1) = Fix(T_2 \circ T_1)$ and $T_2 \circ T_1$ is a multivalued quasi-nonexpansive mapping on C.*

Proof. Clearly, we have $Fix(T_2) \cap Fix(T_1) \subseteq Fix(T_2 \circ T_1)$. Let $p \in Fix(T_2) \cap Fix(T_1)$ and $q \in Fix(T_2 \circ T_1)$. By using properties of T_2, we have

$$\begin{aligned}
\|q-p\|^p &\leq H(T_2 \circ T_1 q, T_2 p)^p \\
&\leq \|T_1 q - p\|^p.
\end{aligned} \tag{12}$$

Using the fact that T_1 is firmly nonexpansive, we have

$$\|T_1 q - p\|^p \leq \langle q - p, J_p(T_1 q - p)\rangle. \tag{13}$$

Furthermore, using properties of function ϕ_p, we have

$$\phi_p(q-p, T_1 q - p) = \|q-p\|^p - p\langle q-p, J_p(T_1 q - p)\rangle + (p-1)\|T_1 q - p\|^p.$$

Hence,

$$\langle q-p, J_p(T_1 q - p)\rangle = \frac{1}{p}\Big(\|q-p\|^p + (p-1)\|T_1 q - p\|^p - \phi_p(q-p, T_1 q - p)\Big). \tag{14}$$

Using 13 and 14, we obtain

$$\|T_1q - p\|^p \leq \|q - p\|^p - \phi_p(q - p, T_1q - p). \tag{15}$$

From 12, we have

$$\phi_p(q - p, T_1q - p) \leq 0.$$

By lemma 2.7, we have $\|T_1q - q\| = 0$ which implies that

$$q = T_1q.$$

Using the fact that $q \in Fix(T_2 \circ T_1)$, we get

$$q = T_1q \in T_2 \circ T_1q = T_2q.$$

Thus, $q \in Fix(T_2) \cap Fix(T_1)$. Hence, $Fix(T_2) \cap Fix(T_1) = Fix(T_2 \circ T_1)$.
Next, we show $T_1 \circ T_1$ is a quasi-nonexpansive mapping on C. Let $x \in C$ and $p \in Fix(T_2 \circ T_1)$. Then, $p \in Fix(T_2) \cap Fix(T_1)$ by step 1. We have,

$$\begin{aligned} H(T_2 \circ T_1x, T_2 \circ T_1p) &= H(T_2 \circ T_1x, T_2p) \\ &\leq \|T_1x - p\| \\ &\leq \|x - p\|. \end{aligned}$$

This completes the proof. □

We are now in a position to state and prove our main result.

Theorem 3.2. *For $p > 1, q > 1$, let E be a p-uniformly convex and q-uniformly smooth real Banach space having a weakly continuous duality map J_φ. Let C be a nonempty, closed convex cone of E and Q_C be a sunny nonexpansive retraction from E onto C. Let $A_i : C \to E$ is α_i-inverse strongly accretive for $i = 1, 2, ..., m$ and $\theta_i \in \Big[a, \Big(\frac{q\alpha_i}{C_q}\Big)^{\frac{1}{q-1}}\Big]$ for some $a > 0$. Let $T_1 : C \to C$ be a firmly nonexpansive mapping and $T_2 : C \to CB(C)$ be a multivalued quasi-nonexpansive mapping such that $\Gamma := \Big(VI_q(C, A_i)\Big) \bigcap Fix(T_2) \bigcap Fix(T_1) \neq \emptyset$ and $T_2p = \{p\}$, for all $p \in \Gamma$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in C, \text{ choosen arbitrarily,} \\ z_n = \beta_n x_n + (1 - \beta_n)v_n, \ v_n \in T_2 \circ T_1x_n, \\ y_n = \gamma_0 z_n + \displaystyle\sum_{i=1}^{m} \gamma_i Q_C(I - \theta_i A_i)z_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)y_n, \end{cases} \tag{16}$$

where $\sum_{i=0}^{m} \gamma_i = 1$, $\beta_n \in [c, d] \subset (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, 1)$. Assume that the above control sequences satisfy the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty}(1 - \lambda_n)\alpha_n = \infty$.
Assume that $I - T_2 \circ T_1$ is demiclosed at the origin. Then, the sequence $\{x_n\}$ generated by (16) converges strongly to $x^* \in \Gamma$, where $x^* = Q_\Gamma(0)$.

Proof. It is well known that if C is a closed and convex cone of a real Banach $E,$ we have $\lambda x \in C$ for all $\lambda \in (0,1)$ and $x \in C$. Therefore, the sequence $\{x_n\}$ generated by 16 is well defined. Fixing $p \in \Gamma.$ We prove that the sequence $\{x_n\}$ is bounded. Using (16) and Lemma 3.1, we have

$$\begin{aligned} \|z_n - p\| &= \|\beta_n x_n + (1-\beta_n)v_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|v_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)H(T_2 \circ T_1 x_n, T_2 \circ T_1 p) \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\|. \end{aligned}$$

Hence,

$$\|z_n - p\| \leq \|x_n - p\|. \tag{17}$$

From (16), Lemmas 2.14 and 2.15, it follows that

$$\begin{aligned} \|y_n - p\| &= \|\gamma_0 z_n + \sum_{i=1}^{m} \gamma_i Q_C(I - \theta_i A_i)z_n - p\| \\ &\leq \gamma_0\|z_n - p\| + \sum_{i=1}^{m} \gamma_i\|Q_C(I - \theta_i A_i)z_n - p\| \\ &\leq \|z_n - p\|. \end{aligned}$$

Therefore, we have

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{18}$$

Hence,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq [1 - (1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \max\{\|x_n - p\|,\ \|p\|\}. \end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|,\ \|p\|\}, \quad n \geq 1.$$

Using Lemmas 2.12 and 3.1, we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\beta_n x_n + (1-\beta_n)v_n - p\|^2 \\ &\leq (1-\beta_n)\|v_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|v_n - x_n\|) \\ &\leq (1-\beta_n)H(T_2 \circ T_1 x_n, T_2 \circ T_1 p)^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|) \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|). \end{aligned}$$

Hence,

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - v_n\|). \tag{19}$$

Therefore, by Lemma 2.11 and inequality 19, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\lambda_n x_n)+(1-\alpha_n)y_n-p\|^2\\
&= \|\alpha_n\lambda_n\big(x_n-p\big)+(1-\alpha_n)\big(y_n-p\big)-(1-\lambda_n)\alpha_n p\|^2\\
&\le \|\alpha_n\big(\lambda_n x_n-\lambda_n p\big)+(1-\alpha_n)\big(y_n-p\big)\|^2+2(1-\lambda_n)\alpha_n\langle p, J(p-x_{n+1})\rangle\\
&\le \alpha_n\lambda_n^2\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2+2(1-\lambda_n)\alpha_n\langle p, J(p-x_{n+1})\rangle\\
&\le \alpha_n\lambda_n\|x_n-p\|^2+(1-\alpha_n)\Big[\|x_n-p\|^2-\beta_n(1-\beta_n)g(\|x_n-v_n\|)]\\
&\quad +2(1-\lambda_n)\alpha_n\langle p, J(p-x_{n+1})\rangle\\
&\le [1-(1-\lambda_n)\alpha_n]\|x_n-p\|^2-(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n-v_n\|)\\
&\quad +2(1-\lambda_n)\alpha_n\langle p, J(p-x_{n+1})\rangle.
\end{aligned}$$

Therefore,

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n-v_n\|)\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2(1-\lambda_n)\alpha_n\langle p, J(p-x_{n+1})\rangle. \tag{20}$$

Since $\{x_n\}$ is bounded, then there exists a constant $B>0$ sucht that

$$(1-\lambda_n)\langle p, J(p-x_{n+1})\rangle\le B,\ \text{ for all, }\ n\ge 0.$$

Hence,

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n-v_n\|)\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n B. \tag{21}$$

Now, we prove that $\{x_n\}$ converges strongly to x^*.
We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n-p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n-p\|\}$ is convergent. Clearly, we have

$$\|x_n-p\|^2-\|x_{n+1}-p\|^2\to 0.$$

It then implies from 21 that

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n-v_n\|)=0. \tag{22}$$

Using the fact that $\beta_n\in[a,b]\subset(0,1)$ and property of g, we have

$$\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{23}$$

Hence,

$$\lim_{n\to\infty}d(x_n, T_2\circ T_1x_n)=0. \tag{24}$$

Now, observing that,

$$\begin{aligned}
\|z_n-x_n\| &= \|(1-\beta_n)x_n+\beta_n v_n-x_n\|\\
&= \|(1-\beta_n)x_n+\beta_n v_n-\beta_n x_n-(1-\beta_n)x_n\|\\
&\le \|v_n-x_n\|.
\end{aligned}$$

Therefore, from 23 we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{25}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^*-x_n)\rangle\le 0$. Since E is reflexive and $\{x_n\}_{n\ge0}$ is bounded there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in C and

$$\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^*-x_n)\rangle=\lim_{j\to+\infty}\langle x^*, J_\varphi(x^*-x_{n_j})\rangle.$$

From 24 and $I - T_2 \circ T_1$ is demiclosed, we obtain $a \in Fix(T_2 \circ T_1)$. From Lemma 2.12, the fact that $Q_C(I - \theta_i A_i)$ is nonexpansive and 18, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\gamma_0 z_n + \sum_{i=1}^{m} \gamma_i Q_C(I - \theta_i A_i) z_n - p\|^2 \\ &\leq \gamma_0 \|z_n - p\|^2 + \sum_{i=1}^{m} \gamma_i \|Q_C(I - \theta_i A_i) z_n - p\|^2 - \gamma_0 \gamma_i g(\|Q_C(I - \theta_i A_i) z_n - z_n\|) \\ &\leq \|x_n - p\|^2 - \gamma_0 \gamma_i g(\|Q_C(I - \theta_i A_i) z_n - z_n\|).\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1 - \alpha_n) y_n - p\|^2 \\ &= \|\alpha_n \lambda_n \big(x_n - p\big) + (1 - \alpha_n)\big(y_n - p\big) - (1 - \lambda_n)\alpha_n p\|^2 \\ &\leq \|\alpha_n \big(\lambda_n x_n - \lambda_n p\big) + (1 - \alpha_n)\big(y_n - p\big)\|^2 + 2(1 - \lambda_n)\alpha_n \langle p, J(p - x_{n+1})\rangle \\ &\leq \alpha_n \lambda_n^2 \|x_n - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 + 2(1 - \lambda_n)\alpha_n \langle p, J(p - x_{n+1})\rangle \\ &\leq \alpha_n \lambda_n \|x_n - p\|^2 + (1 - \alpha_n)\Big[\|x_n - p\|^2 - \gamma_0 \gamma_i g(\|Q_C(I - \theta_i A_i) z_n - z_n\|)\Big] \\ &\quad +2(1 - \lambda_n)\alpha_n \langle p, J(p - x_{n+1})\rangle \\ &\leq [1 - (1 - \lambda_n)\alpha_n]\|x_n - p\|^2 - (1 - \alpha_n)\gamma_0 \gamma_i g(\|Q_C(I - \theta_i A_i) z_n - z_n\|) \\ &\quad +2(1 - \lambda_n)\alpha_n \langle p, J(p - x_{n+1})\rangle.\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $D > 0$ sucht that

$$(1 - \alpha_n)\gamma_0 \gamma_i g(\|Q_C(I - \theta_i A_i) z_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n D. \tag{26}$$

Thus we have

$$\lim_{n \to \infty} g(\|Q_C(I - \theta_i A_i) z_n - z_n\|) = 0. \tag{27}$$

Using property of g, we have

$$\lim_{n \to \infty} \|Q_C(I - \theta_i A_i) z_n - z_n\| = 0. \tag{28}$$

From 28 and Lemma 2.4, we obtain $a \in \bigcap_{i=1}^{m} Fix(Q_C(I - \theta_i A_i))$. Using Lemma 2.14, we have $a \in \bigcap_{i=1}^{m} VI_q(C, A_i)$. Therefore, $a \in \Gamma$. On the other hand, by using $x^* = Q_\Gamma(0)$ and the assumption that the duality mapping J_φ is weakly continuous, we have,

$$\begin{aligned}\limsup_{n \to +\infty} \langle x^*, J_\varphi(x^* - x_n)\rangle &= \lim_{j \to +\infty} \langle x^*, J_\varphi(x^* - x_{n_j})\rangle \\ &= \langle x^*, J_\varphi(x^* - a)\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. In fact, since $\Phi(t) = \int_0^t \varphi(\sigma) d\sigma,\ \forall t \geq 0$, and φ is a gauge function, then for $1 \geq k \geq 0,\ \Phi(kt) \leq k\Phi(t)$. From 16 and Lemma 2.11, we get that

$$\begin{aligned}\Phi(\|x_{n+1} - x^*\|) &= \Phi(\|\alpha_n(\lambda_n x_n) + (1 - \alpha_n) y_n - x^*\|) \\ &\leq \Phi(\|\alpha_n \lambda_n (x_n - x^*) + (1 - \alpha_n)(y_n - x^*)\|) + (1 - \lambda_n)\alpha_n \langle x^*, J_\varphi(x^* - x_{n+1})\rangle \\ &\leq \Phi(\alpha_n \lambda_n \|x_n - x^*\| + \|(1 - \alpha_n)(y_n - x^*)\|) + (1 - \lambda_n)\alpha_n \langle x^*, J_\varphi(x^* - x_{n+1})\rangle \\ &\leq \Phi(\alpha_n \lambda_n \|x_n - x^*\| + (1 - \alpha_n)\|x_n - x^*\|) + (1 - \lambda_n)\alpha_n \langle x^*, J_\varphi(x^* - x_{n+1})\rangle \\ &\leq \Phi((1 - (1 - \lambda_n)\alpha_n)\|x_n - x^*\|) + (1 - \lambda_n)\alpha_n \langle x^*, J_\varphi(x^* - x_{n+1})\rangle \\ &\leq [1 - (1 - \lambda_n)\alpha_n]\Phi(\|x_n - x^*\|) + (1 - \lambda_n)\alpha_n \langle x^*, J_\varphi(x^* - x_{n+1})\rangle.\end{aligned}$$

From Lemma 2.10, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \ B_k \leq B_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. From 21, we have

$$(1 - \alpha_{\tau(n)})\beta_{\tau(n)}(1 - \beta_{\tau(n)})g(\|x_{\tau(n)} - v_{\tau(n)}\|) \leq 2\alpha_{\tau(n)}B \to 0 \text{ as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \|x_{\tau(n)} - v_{\tau(n)}\| = 0. \tag{29}$$

By same argument as in case I, we can show that $x_{\tau(n)}$ converges weakly in E and $\limsup_{n\to+\infty}\langle x^*, J_\varphi(x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \Phi(\|x_{\tau(n)+1} - x^*\|) - \Phi(\|x_{\tau(n)} - x^*\|) \leq \Big(1 - \lambda_{\tau(n)}\Big)\alpha_{\tau(n)}[-\Phi(\|x_{\tau(n)} - x^*\|) + \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle],$$

which implies that

$$\Phi(\|x_{\tau(n)} - x^*\|) \leq \langle x^*, J_\varphi(x^* - x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.13, we conclude that

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Therefore $\{x_n\}$ converges strongly to $x^* \in \Gamma$. □

Remark 3.3. *In our theorem, we assume that C is a cone. But, in some cases, for example, if C is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in C$ for all $\lambda \in (0,1)$ and $x \in C$. Therefore, in the case where E is a real Hilbert space or $E = l_p$, $1 < p < \infty$, our results can be used to approximated fixed ponts of multivalued quasi-nonexpansive mappings from the closed unit ball to itself.*

Remark 3.4. *Many already studied algorithms for solving variational inequality problem coupled with common fixed points problem in the literature can be considered as special cases of this paper.*

Now, using the similar arguments as in the proof of Theorem 3.2 and Lemma 2.8, we obtain the following result by replacing $T_2 \circ T_1$ by $P_{T_2} \circ T_1$ and removing the rigid restriction on Γ ($T_2 p = \{p\},\ \forall\, p \in \Gamma$) and assumption that E has a weakly continuous duality map J_φ.

Theorem 3.5. *Assume that $E = l_p$, $1 < p < \infty$ or E is a real Hilbert space. Let $\mathbb{B}$ be the closed unit ball of E and $Q_{\mathbb{B}}$ be a sunny nonexpansive retraction from E onto $\mathbb{B}$. Let $A_i : \mathbb{B} \to E$ is α_i-inverse strongly accretive for $i = 1, 2, ..., m$. Let $T_1 : \mathbb{B} \to \mathbb{B}$ be a firmly nonexpansive mapping and $T_2 : \mathbb{B} \to CB(\mathbb{B})$ be a multivalued mapping such that P_{T_2} is quasi-nonexpansive. Assume that $\Gamma := \Big(VI_p(\mathbb{B}, A_i)\Big) \bigcap Fix(T_2) \bigcap Fix(T_1) \neq \emptyset$ and Assume that $I - P_{T_2} \circ T_1$ is demiclosed at the origin. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in \mathbb{B}, \text{ choosen arbitrarily}, \\ z_n = \beta_n x_n + (1 - \beta_n)v_n,\ v_n \in I - P_{T_2} \circ T_1 x_n, \\ y_n = \gamma_0 z_n + \displaystyle\sum_{i=1}^{m} \gamma_i Q_{\mathbb{B}}(I - \theta_i A_i)z_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n)y_n, \end{cases} \tag{30}$$

where $\sum_{i=0}^{m} \lambda_i = 1$, $\beta_n \in [c,d] \subset (0,1)$, $\theta_i > 0$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,1)$. *Assume that the above control sequences satisfy the following conditions:*
(i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *generated by (30) converges strongly to* $x^* \in \Gamma$, *where* $x^* = Q_\Gamma(0)$.

4. Application to non-convex optimization problem

DC Programming and DCA were introduced by Pham Dinh Tao in 1985 in their preliminary form and extensively developed by Le Thi Hoai An and Pham Dinh Tao since 1994 to become now classical, see for example [22] and references therein. Their original key idea relies on the DC structure of the objective function in nonconvex programs which are explored and exploited in a deep and suitable way. A DC program is of the form :

$$\inf\{f(x) = g(x) - h(x) \quad x \in H\}. \tag{31}$$

The complexity of DC programs resides on the lack of practical global optimality conditions. So the following necessary local optimality conditions for 31 were developed:

$$\partial g(x^*) \cap \partial h(x^*) \neq \emptyset,$$

such a point x^* is called a critical point of $g - h$ or for 31. We consider the following problem :

Problem 4.1.

$$\textit{find } x^* \in C \ \textit{ such that } \partial g(x^*) \cap \partial h(x^*) \neq \emptyset. \tag{32}$$

We denote the set of solutions of Problem 4.1 by Ω_1.
Problem 4.1 has many applications such as multicommodity network, image restoration processing, discrete tomography, clustering and seems particularly well suited to model several nonconvex industrial problems (portfolio optimization, fuel mixture, molecular biology, phylogenetic analysis ...), see for example [22].

Problem 4.2. *We also consider the following common solutions of a system variational inequality problem :*

$$\textit{find } x^* \in C \ \textit{ such that } x^* \in \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big), \tag{33}$$

We denote the set of solutions of Problem 4.1 by Ω_2.

Theorem 4.3. *Let* H *be a real Hilbert space and* C *be be a nonempty, closed convex cone of* H. *Let* g, h *be two convex lower semicontinuous functions defined on* C. *Let* $A_i : C \to H$ *is* α_i*-inverse strongly monotone for* $i = 1, 2, ..., m$. *Assume that that* $\Omega_1 \cap \Omega_2 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined iteratively from arbitrary* $x_0 \in C$ *by:*

$$\begin{cases} x_0 \in C, \ \textit{choosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n) J_\mu^{\partial g} \circ J_\mu^{\partial h} x_n, \\ y_n = \gamma_0 z_n + \sum_{i=1}^{m} \gamma_i P_C(I - \theta_i A) z_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n) y_n, \end{cases} \tag{34}$$

where $\sum_{i=0}^{m} \lambda_i = 1$, $\beta_n \in [c,d] \subset (0,1), \theta_i \in \Big(0, 2\alpha_i\Big]$, $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,1)$. *Assume that the above control sequences satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$; *(ii)* $\lim_{n\to\infty} \lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *generated by 34 converges strongly to a critical point of* $f-g$ *which is a common solution of a system of variational inequality problems.*

Proof. Using properties of resolvent operators and Lemma 3.1, we have $J_r^{\partial g} \circ J_r^{\partial h}$ is nonexpansive on C and $\partial g^{-1}0 \cap \partial h^{-1}0 = Fix(J_r^{\partial g} \circ J_r^{\partial h}) = Fix(J_r^{\partial g}) \cap Fix(J_r^{\partial h})$. Therefore, it follows Theorem 3.2 that $\{x_n\}$ converges strongly to some point $x^* \in Fix(J_r^{\partial g}) \bigcap Fix(J_r^{\partial h}) \bigcap \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big) \iff 0 \in \partial g(x^*) \cap \partial h(x^*)$ and $x^* \in \Big(\bigcap_{i=1}^{m} VI(C, A_i)\Big)$, completing the proof. □

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