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Some fixed point results for discontinuous mappings via the degree of nondensifiability

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Abstract. In this paper we introduce the of concept generalized DND-condensing mapping which, as the name suggests, is a generalization of the so called DND-condensing mappings, and prove that some fixed point results for certain discontinuous mappings hold for this new class of mappings. To be more precise, by using the degree of nondensifiability (DND), we show the existence of fixed points (resp. r-fixed points) for mappings which are both generalized DND-condensing and half-continuous (resp. r-continuous). In other words, we generalize some already proved fixed point results for half-continuous and r-continuous mappings which, themselves, generalize a version of the well known Schauder fixed point theorem. As a consequence of our main result, we prove the existence of an approximated fixed point sequence for a half-continuous self mapping defined on a bounded, closed and convex subset of a Banach space and satisfying a weaker condition than the required for generalized DND-condensing ones.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space. For given $x \in X$ and $r > 0$, we denote by $B(x, r) = \{y \in X : \|x - y\| \leq r\}$ the closed ball centered at x and with radius r. As usual, the canonical pairing of X and X^* is written as $\langle x, x^* \rangle$ for all $x \in X$ and $x^* \in X^*$. One version of the celebrated Schauder fixed point theorem (see, for instance, [2, Theorem I.2.1]) states that every continuous mapping $f : C \longrightarrow C$, where C is a compact and convex subset of X, has a fixed point. Several papers have already extended the Schauder fixed point theorem to noncontinuous mappings; see, for instance, [5, 6, 8, 9, 19, 27] and references therein.

Next, we give some definitions and results from [5] that we will use later. The first of them is the concept of half-continuous mapping, see [5, Definition 2.1].

Definition 1.1. *Let $C \subseteq X$ be non-empty. A mapping $f : C \longrightarrow X$ is said to be half-continuous if for each $x \in C$ with $x \neq f(x)$ there is $p^* \in X^*$ and $\varepsilon > 0$ such that for all $x' \in B(x, \varepsilon) \cap C$ with $x' \neq f(x')$, we have $\langle f(x') - x', p^* \rangle > 0$.*

In our framework, a continuous mapping is, in particular, half-continuous. Indeed, as X is a Hausdorff locally convex topological space, X^* separates points of X, see, for instance, [1, Corollary 5.82] . Thus, the

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assertion follows from [27, Proposition 3.2]. However, as shown by several examples in [5], the class of continuous mappings is strictly contained in the class of half-continuous ones. As Bich pointed out in [5], roughly speaking a mapping f is half-continuous if every x which is not a fixed point of f can be separated from its image $f(x)$, the separation being robust to a small perturbation of x. For half-continuous mappings, we have the following generalization of the Schauder fixed point theorem, proved in [5, Theorem 3.1].

Theorem 1.2. *Let $C \subset X$ be non-empty, compact and convex and $f : C \longrightarrow C$ a half-continuous mapping. Then, f has a fixed point.*

The most common notion of approximated fixed point is the following, see [5, Definition 4.1] or [4, Definition 1.1].

Definition 1.3. *Let $C \subseteq X$ be non-empty and $r > 0$. We say that a mapping $f : C \longrightarrow X$ admits an r-fixed point if for every $\varepsilon > 0$ there is $x_\varepsilon \in C$ such that $\|f(x_\varepsilon) - x_\varepsilon\| \leq r + \varepsilon$.*

Of course, if f has a fixed point, it admits an r-fixed point for each $r > 0$. Furthermore, if $f : C \longrightarrow C$ and C is non-empty and bounded, then f admits an r-fixed point for each $r \geq 2\text{Diam}(C)$, $\text{Diam}(C)$ being the diameter of C. The following concept will be used later, see [5, Definition 4.2].

Definition 1.4. *Let $C \subseteq X$ be non-empty and $r > 0$. A mapping $f : C \longrightarrow X$ is said to be r-continuous if, for each $x \in C$ and $\varepsilon > 0$, there is $\delta > 0$ such that for all $y \in B(x, \delta) \cap C$, we have $\|f(x) - f(y)\| < r + \varepsilon$.*

For r-continuous mappings, we have the following result, see [5, Theorem 4.4].

Theorem 1.5. *Let $C \subset X$ be non-empty, compact and convex and $f : C \longrightarrow C$ a r-continuous mapping for some $r > 0$. Then, f admits an r-fixed point.*

It is important to stress that the above result improves the result of [6] and unifies the two results in [6, 19]. The following concept is well known, see, for instance, [2, Definition VII.1.1].

Definition 1.6. *Let $C \subseteq X$ be non-empty and $f : C \longrightarrow X$. We will say that f has an approximated fixed point sequence (a.f.p.s.) if there is a sequence $(\tilde{x}_n)_{n\geq 1} \subset C$ such that*

$$\lim_n \|f(\tilde{x}_n) - \tilde{x}_n\| = 0.$$

In such case, $(\tilde{x}_n)_{n\geq 1} \subset C$ is said to be an a.f.p.s. for f.

The following relationships between the above concepts, assuming the set C is closed, are clear, and, in general, the reciprocals do not hold.

$$\begin{array}{ccc} f \text{ has a fixed point} & \Rightarrow & f \text{ has an a.f.p.s.} \\ \Downarrow & & \\ f \text{ admits an } r\text{-fixed point, for some } r > 0 & \Leftarrow & f \text{ has an a.f.p.s.} \end{array}$$

On the other hand, in this paper, we prove that the compactness condition of Theorems 1.2 and 1.5 can be replaced by a weaker one based in certain contractiveness condition on the mapping f. Such contractiveness condition is given in terms of the so called degree of nondensifiability (DND), explained in detail in Section 2. To be more precise, in Definition 3.1 we generalize the concept of DND-condensing mapping (see Definition 2.9), by introducing the concept of generalized DND-condensing mapping and prove in our main result, Theorem 3.3, that Theorem 1.2 (resp. 1.5) remains true for half-continuous (resp. r-continuous) mappings that are also generalized DND-condensing mappings, assuming that C is non-empty, bounded, convex and closed instead of compact. By several examples, we show that a generalized DND-condensing mapping is a *real* generalization of the DND-condensing mappings, and that our main result works in cases where Theorems 1.2 or 1.5 not.

As a consequence of our main result, in Corollary 3.6, we show the existence of an a.f.p.s. for half-continuous self mappings defined on a non-empty, bounded, convex and closed subset of a Banach space obeying a weaker condition given in the definition of generalized DND-condensing mappings.

2. The degree of nondensifiability and DND-condensing mappings

Before recalling the definition of degree of nondensifiability, it is convenient to recall the definition of α-dense curves and densifiables sets, introduced in 1997 by Mora and Cherruault [22]. In what follows, for a given metric space (M, d), $\mathcal{B}(M)$ denotes the class of the non-empty and bounded subsets of M. Also, as in Section 1, X stands for a Banach space.

Definition 2.1. *Let (M, d) a metric space, $B \in \mathcal{B}(M)$ and $\alpha \geq 0$. A continuous mapping $\gamma : [0, 1] \longrightarrow (M, d)$ is said to be an α-dense curve in B if the following conditions hold:*

(i) $\gamma([0, 1]) \subset B$.

(ii) For each $x \in B$ there is $y \in \gamma([0, 1])$ such that $d(x, y) \leq \alpha$.

If for each $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Let us note that, for each $B \in \mathcal{B}(M)$, there always exists an α-dense curve in B for each $\alpha \geq \mathrm{Diam}(B)$, the diameter of B. Indeed, fix any $x_0 \in B$ and $\alpha \geq \mathrm{Diam}(B)$, the mapping $\gamma(t) := x_0$ for all $t \in [0, 1]$ is, clearly, an α-dense curve in B for all $\alpha \geq \mathrm{Diam}(B)$. Also, the α-dense curves are a generalization of the so called *space-filling curves*, see [25]. In fact, for $M := \mathbb{R}^n$ and $B \in \mathcal{B}(M)$ with non-empty Jordan content, a space-filling curves in B is a continuous and onto mapping $g : [0, 1] \longrightarrow B$, that is, a 0-dense curve in B. Also, we have the following result, proved in [20, 24].

Proposition 2.2. *Let $B \in \mathcal{B}(M)$ be arc-wise connected. Then, B is densifiable if and only if it is totally bounded.*

Remark 2.3. *According to [26], we mean that $B \in \mathcal{B}(M)$ is totally bounded if and only if, for each $\varepsilon > 0$, there is a finite set $F \subset M$ such that for each $x \in B$ there is $y \in F$ with $d(x, y) \leq \varepsilon$. In the case that M is complete, totally bounded means precompact or relatively compact (i.e., its closure is compact).*

For a detailed exposition of the concepts of α-dense curves and densifiable sets, we refer to [7, 20–24] and references therein

Now, we can give the following definition (see [18, 23]).

Definition 2.4. *For a given $B \in \mathcal{B}(M)$, the degree of nondensifiability, in short DND, of B is defined as*

$$\phi(B) := \inf\left\{\alpha \geq 0 : \Gamma_{B,\alpha} \neq \emptyset\right\},$$

where $\Gamma_{B,\alpha}$ stands for the class of α-dense curves in B.

As we have shown above, for a given $B \in \mathcal{B}(M)$ there is always an α-dense curve in B, for any $\alpha \geq \mathrm{Diam}(B)$. Therefore, $\phi(B) \in [0, \mathrm{Diam}(B)]$. Also, it is worth to say that the DND is a *quantitative* version of the Hahn-Mazurkiewicz theorem, see for instance, [25, 28]. We recall that such theorem states that $B \in \mathcal{B}(M)$ is the continuous image of $[0, 1]$ if and only if it is compact, connected and locally connected. So, roughly speaking, the DND measures, in the specified sense, the distance from a given $B \in \mathcal{B}(M)$ to the class of its Peano Continua.

Example 2.5. *Let U_X be the closed unit ball of a Banach space X. Then, according to [23], we have*

$$\phi(U_X) = \begin{cases} 1, & X \text{ has infinite dimension} \\ 0, & \text{otherwise} \end{cases}.$$

Some basic properties of the DND are given in the following result, see [18].

Proposition 2.6. *The DND ϕ satisfies the following properties:*

(M-1) Let $B \in \mathcal{B}(M)$ be arc-wise connected. Then, B is precompact if, and only if, then $\phi(B) = 0$.

(M-2) Invariant under closure: $\phi(B) = \phi(\bar{B})$, for each $B \in \mathcal{B}(M)$.

Additionally, if $M := X$ is a Banach space, then the following conditions are also satisfied:

(B-1) Semi-homogeneity: $\phi(cB) = |c|\phi(B)$, for each $c \in \mathbb{R}$ and $B \in \mathcal{B}(X)$.

(B-2) Invariant under translations: $\phi(x_0 + B) = \phi(B)$, for each $x_0 \in X$ and $B \in \mathcal{B}(X)$.

(B-3) For each $B_1, B_2 \in \mathcal{B}(X)$,

$$\phi\Big(\mathrm{Conv}(B_1 \cup B_2)\Big) \leq \max\Big\{\phi\Big(\mathrm{Conv}(B_1)\Big), \phi\Big(\mathrm{Conv}(B_2)\Big)\Big\} \leq \max\Big\{\phi(B_1), \phi(B_2)\Big\}.$$

In particular, $\phi(\mathrm{Conv}(B_1)) \leq \phi\Big(B_1)$.

Despite the above properties, and it was proved in [18], the DND is a so called measure of noncompactness (see [2, 3]). However, there are some relationships between the DND and some measures of noncompactness. For instance, by recalling that the Hausdorff measure of noncompactness is defined as

$$\chi(B) := \inf\Big\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radii } \leq \varepsilon\Big\},$$

for each $B \in \mathcal{B}(X)$, X being a Banach, in [18, Theorem 2.5] it was proved the following result.

Proposition 2.7. *For each arc-wise connected $B \in \mathcal{B}(X)$, the following intequalities hold*

$$\chi(B) \leq \phi(B) \leq 2\chi(B),$$

and are the best possible.

To highlight the differences between the DND ϕ and the Hausdorff measure of noncompactness χ, we give the following example.

Example 2.8. *Let X be a Banach space and $x, y \in X$ with $x \neq y$. Define $B := \{x, y\}$. Then, noticing the properties of the Hausdorff measure of noncompactness χ we have*

$$\chi(B) = \chi\Big(\mathrm{Conv}(B)\Big) = 0.$$

Let γ any α-dense curve in B, for some $\alpha > 0$. As $\gamma([0,1])$ is arc-wise connected and $x \neq y$, must be $\gamma([0,1]) = x$ or $\gamma([0,1]) = y$. So, $\phi(B) = \|x - y\| > 0$. Also, as the mapping $\gamma(t) := (1-t)x + ty$, for all $t \in [0,1]$ satisfies that $\gamma([0,1]) = \mathrm{Conv}(B)$, we find that

$$\phi\Big(\mathrm{Conv}(B)\Big) = 0 < \|x - y\| = \phi(B).$$

In the above examples, we have the same results replacing χ by any other measure of noncompactness which is invariant under the passage to the convex hull.

On the other hand, the DND has been used to prove several fixed point results, see [10, 15–17] and references therein. Such fixed point results based in the DND (and, in particular, Theorem 2.10 below) work in some cases where the measures of noncompactness not. The following concept will be crucial for our goals (see [16, 17]).

Definition 2.9. *Let $C \in \mathcal{B}(X)$ and $f : C \longrightarrow C$ continuous. We will say that f is DND-condensing if*

$$\phi\Big(f(B)\Big) < \phi(B) \quad \textit{whenever } \phi(B) > 0,$$

for each $B \subset C$ non-empty and convex.

We have the following fixed point result for DND-condensing mappings (again, [16, 17]).

Theorem 2.10. *Let $C \in \mathcal{B}(X)$ convex and closed, and $f : C \longrightarrow C$ a DND-condensing mapping. Then, f has a fixed point.*

As a consequence of the above theorem we have the following result (see also [11, Theorem 3.1])

Corollary 2.11. *Let $C \in \mathcal{B}(X)$ convex and closed. Assume that $f : C \longrightarrow C$ is a continuous mapping such that*

$$\phi\Big(f(B)\Big) \leq \phi(B), \quad \textit{for each non-empty and convex } B \subset C.$$

Then, f has an a.f.p.s.

Proof. Let $(\lambda_n)_{n\geq 1} \subset (0, 1)$ with $\lim_n \lambda_n = 0$, and $x_0 \in C$. For each integer $n \geq 1$ define $f_n : C \longrightarrow C$ as

$$f_n(x) := \lambda_n x_0 + (1 - \lambda_n) f(x), \quad \text{for all } x \in C.$$

Then, given any non-empty and convex $B \subset C$ with $\phi(B) > 0$, noticing Proposition 2.6, we have

$$\phi\Big(f_n(B)\Big) = (1 - \lambda_n)\phi\Big(f(B)\Big) \leq (1 - \lambda_n)\phi(B) < \phi(B).$$

So, for each $n \geq 1$, f_n is a DND-condensing mapping and by Theorem 2.10 has a fixed point, put $\tilde{x}_n$. Now, it is easy to show that the sequence $(\tilde{x}_n)_{n\geq 1} \subset C$ is an a.f.p.s. for f. □

3. The main result

We start this Section with the following definition:

Definition 3.1. *Let $C \in \mathcal{B}(X)$ be closed and convex and $f : C \longrightarrow C$ a mapping. We will say that f is a generalized DND-condensing mapping if*

$$\phi\Big(\mathrm{Conv}(f(B))\Big) < \phi(B) \quad \textit{whenever } \phi(B) > 0,$$

for each non-empty and convex $B \subset C$.

Let us note that a DND-condensing mapping is, in particular, a generalized DND-condensing mapping. Indeed, if $C \in \mathcal{B}(X)$ is closed and convex and $f : C \longrightarrow C$ is DND-condensing, then for each $B \subset C$ non-empty and convex with $\phi(B) > 0$, noticing Proposition 2.6, we have

$$\phi\Big(\mathrm{Conv}(f(B))\Big) \leq \phi\Big(f(B)\Big) < \phi(B).$$

However, as we show in the following example, a DND-condensing mapping is not necessarily generalized DND-condensing.

Example 3.2. *Let ℓ_∞ be the Banach space of the real bounded sequences, endowed its usual supremum norm, and U_{ℓ_∞} its closed unit ball. Denote by $\mathbf{0}$ the null of ℓ_∞ and by $\mathbf{1} := (1, 1, \ldots, 1, \ldots,) \in \ell_\infty$. Define $f : U_{\ell_\infty} \longrightarrow U_{\ell_\infty}$ as*

$$f(x) := \begin{cases} \mathbf{1}, & 0 \leq \|x\| \leq \frac{1}{2} \\ \mathbf{0}, & \frac{1}{2} < \|x\| \leq 1 \end{cases}, \quad \text{for each } x \in U_{\ell_\infty}$$

Clearly, f is not continuous and therefore it can not be a DND-condensing mapping. Moreover, from Examples 2.5 and 2.8, we have

$$\phi\Big(f(U_{\ell_\infty})\Big) = \phi(\{\mathbf{1}, \mathbf{0}\}) = 1 = \phi(U_{\ell_\infty}),$$

and so, the contractiveness condition of Definition 2.9 of DND-condensing mapping is not satisfied. Also, f has not a fixed point and $f(B) \subset \{\mathbf{1}, \mathbf{0}\}$ is compact (in fact, finite) for each non-empty $B \subset U_{\ell_\infty}$. Therefore, given a non-empty and convex $B \subset U_{\ell_\infty}$ with $\phi(B) > 0$, $\mathrm{Conv}(f(B))$ is compact. So, from properties (M-1) and (M-2) of Proposition 2.6 we have

$$\phi\Big(\mathrm{Conv}(f(B))\Big) = 0 < \phi(B).$$

Therefore, f is a generalized DND-condensing mapping.

Now, we can state and prove our main result.

Theorem 3.3. *Let $C \in \mathcal{B}(X)$ be closed and convex, and $f : C \longrightarrow C$ a generalized DND-condensing mapping. Then, we have:*

(1) If f is half-continuous then it has a fixed point.

(2) If f is r-continuous, for some $r > 0$, then it admits an r-fixed point.

Proof. Fixed any $x_0 \in C$ define

$$\Sigma := \Big\{B \subset C : B = \overline{\mathrm{Conv}}(B), x_0 \in B, f(B) \subset B\Big\}.$$

As $C \in \Sigma$, $\Sigma \neq \emptyset$. Let also

$$\mathcal{F} := \bigcap_{B \in \Sigma} B, \quad \mathcal{G} := \overline{\mathrm{Conv}}\Big(\mathrm{Conv}(f(\mathcal{F})) \cup \{x_0\}\Big).$$

Let us note that $\mathcal{F} \neq \emptyset$ (as $x_0 \in \mathcal{F}$) and it is non-empty, closed and convex. Also, it is clear that that $f(\mathcal{F}) \subset \mathcal{F}$.

Since $x_0 \in \mathcal{F}$, $\mathcal{F}$ is closed and convex and $\mathrm{Conv}(f(\mathcal{F})) \subset \mathcal{F}$, we find that $\mathcal{G} \subset \mathcal{F}$. This implies $f(\mathcal{G}) \subset f(\mathcal{F}) \subset \mathcal{G}$ and as $x_0 \in \mathcal{G}$ and $\mathcal{G} = \overline{\mathrm{Conv}}(\mathcal{G})$ we infer that $\mathcal{G} \in \Sigma$. Therefore, $\mathcal{F} \subset \mathcal{G}$ and thus $\mathcal{F} = \mathcal{G}$.

We claim that $\mathcal{F}$ is compact. Otherwise, as $\mathcal{F}$ is closed it is not precompact. So, noticing Proposition 2.6, we have

$$\phi(\mathcal{F}) = \phi(\mathcal{G}) \leq \max\Big\{\phi\Big(\mathrm{Conv}(f(\mathcal{F}))\Big), \phi(\{x_0\})\Big\} = \phi\Big(\mathrm{Conv}(f(\mathcal{F}))\Big) < \phi(\mathcal{F}),$$

which is contradictory. So, $\mathcal{F}$ is convex and compact and the statements (1) and (2) follow directly form Theorems 1.2 and 1.5, respectively.

□

At this point, we present some examples to illustrate the above result. Let us note that the non-empty, closed and convex domain C of the shown mappings in the below examples is not compact. Therefore, in the above examples the existence of fixed points or r-fixed points can not be derived from Theorem 1.2 or 1.5.

Example 3.4. *In the Banach space c_0 of the real sequences converging to zero, endowed with its usual norm, define the closed and convex set*

$$C := \Big\{x := (x_n)_{n\geq 1} \in c_0 : 0 \leq x_n \leq 1, \text{ for all } n \geq 1\Big\}.$$

Consider $f : C \longrightarrow C$ given by

$$f(x) := \begin{cases} (1,0,\ldots,0,\ldots), & \frac{1}{2} \leq x_1 \leq 1 \\ (1, \dfrac{1}{3}x_2, \ldots, \dfrac{1}{3}x_n, \ldots), & \text{otherwise} \end{cases}, \quad \text{for all } x := (x_n)_{n\geq 1} \in C.$$

One can check straightforwardly that f is not continuous. Indeed, define the sequence

$$x^{(m)} := (\frac{m}{1+2m}, 1, 0, \ldots, 0, \ldots) \in C \quad \text{for each } m \geq 1,$$

which converges to $x_0 := (1/2, 1, 0, \ldots, 0, \ldots)$. But

$$f(x^{(m)}) = (1, \frac{1}{3}, 0, \ldots, 0, \ldots) \not\longrightarrow f(x_0) = (1, 0, \ldots, 0, \ldots).$$

So, f is not continuous. Also, $\tilde{x} := (1, 0, \ldots, 0, \ldots)$ is the unique fixed point of f. We will show in the lines below that f is half-continuous.

Let $x \in C$ with $f(x) \neq x$. If $x_1 = 1$, take $i > 1$ such that $x_i \in (0, 1]$ and $\varepsilon > 0$ obeying $x_i - \varepsilon > 0$. Then, for $p^* = (0, \ldots, -1, 0, \ldots) \in \ell_1$ (the dual of c_0), with the -1 in the i-th position, for each $x' = (x'_n)_{n\geq 1} \in B(x, \varepsilon) \cap C$ we have

$$\langle f(x') - x', p^* \rangle = \begin{cases} x'_i > x_i - \varepsilon > 0, & x'_1 = 1 \\ \dfrac{2}{3}x'_i > \dfrac{2}{3}(x_i - \varepsilon) > 0, & 0 \leq x'_1 < 1 \end{cases}.$$

If $0 \leq x_1 < 1$, by taking $0 < \varepsilon < 1 - x_1$ and $p^* = (1, 0, \ldots, 0, \ldots) \in \ell_1$ we get

$$\langle f(x') - x', p^* \rangle = 1 - x'_1 \geq 1 - x_1 - \varepsilon > 0, \quad \text{for all } x' = (x'_n)_{n\geq 1} \in B(x, \varepsilon) \cap C.$$

Next, we will prove that f is a generalized DND-condensing mapping. Given any non-empty and convex $B \subset C$ with $\phi(B) > 0$, let us denote

$$B_1 := \{x \in B : \frac{1}{2} \leq x_1 \leq 1\}, \quad \text{and } B_2 := B \setminus B_1.$$

So, we can write $B := B_1 \cup B_2$. If $B_1 = \emptyset$, then $B = B_2$ and it is immediate to check that

$$\phi\Big(\mathrm{Conv}(f(B))\Big) = \phi\Big(\mathrm{Conv}(f(B_2))\Big) \leq \phi\Big(f(B_2)\Big) = \frac{1}{3}\phi(B_2) = \frac{1}{3}\phi(B) < \phi(B). \tag{1}$$

If $B_2 = \emptyset$, then $B = B_1$ and we have

$$\phi\Big(\mathrm{Conv}(f(B))\Big) = \phi\Big(\mathrm{Conv}(f(B_1))\Big) = \phi(\{\tilde{x}\}) = 0 < \phi(B). \tag{2}$$

Now, assume $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$. In this case, noticing Proposition 2.6 and inequalities (1) and (2) we have

$$\begin{aligned} &\phi\Big(\mathrm{Conv}(f(B))\Big) = \phi\Big(\mathrm{Conv}(f(B_1) \cup f(B_2))\Big) \leq \\ &\max\Big\{\mathrm{Conv}\Big(\phi(B_1)\Big) \cup \mathrm{Conv}\Big(\phi(B_2)\Big)\Big\} = \frac{1}{3}\phi(B_2). \end{aligned} \tag{3}$$

By Proposition 2.7 and noticing the properties of the Hausdorff measure of noncompactness, f we have

$$\frac{1}{3}\phi(B_2) \leq \frac{2}{3}\chi(B_2) \leq \frac{2}{3}\chi(B) \leq \frac{2}{3}\phi(B) < \phi(B). \tag{4}$$

So, joining (3) and (4) we infer that f is a generalized DND-condensing mapping.

Example 3.5. *Let $C([0,1],\mathbb{R})$ be the Banach space of all continuous functions $x : [0,1] \longrightarrow \mathbb{R}$, endowed with its usual norm, and C its closed unit ball. Fixed $x_0 \in C$, $x_0 \not\equiv \mathbf{0}$ (the identically null function) and $0 < r < \frac{1}{2}$ define $f : C \longrightarrow C$ as*

$$f\Big(x(t)\Big) := \begin{cases} rx_0(t), & x \not\equiv \mathbf{0} \\ \\ rx(t), & \text{otherwise} \end{cases}, \quad \text{for all } x \in C.$$

It is clear that f is not continuous and has not any fixed point. Also, for each $x \in C$ and $\varepsilon > 0$, for a given $\delta > 0$ it is easy to show that

$$\|f(x) - f(y)\| \leq 2r < 2r + \varepsilon, \quad \text{for all } y \in B(x,\delta) \cap C.$$

So, f is $2r$-continuous. Now, given $B \subset C$ non-empty and convex let γ be an α-dense curve in B for some $\alpha > 0$. As $\gamma([0,1])$ is compact, for a given $\epsilon > 0$ there exists a finite set $F := \{x_1,\ldots,x_n\} \subset \gamma([0,1])$ such that $\gamma([0,1]) \subset F + \epsilon C$. So, we find

$$B \subset \gamma([0,1]) + \alpha C \subset F + (\alpha + \epsilon)C.$$

Noticing the above inclusion, it is easy to prove that $f(B) \subset f(F) + (r\alpha + \epsilon)C$ and consequently

$$\mathrm{Conv}\Big(f(B)\Big) \subset \mathrm{Conv}\Big(f(F)\Big) + (r\alpha + \epsilon)C. \tag{5}$$

Taking into account that $\mathrm{Conv}(f(F))$ is convex and compact, by the Hahn-Mazurkiewicz theorem (see [25, 28]) there is a continuous mapping $\omega : [0,1] \longrightarrow C([0,1],\mathbb{R})$ such that $\omega([0,1]) = \mathrm{Conv}(f(F))$. From (5), ω is an $(r\alpha+\epsilon)$-dense curve in C. By letting $\epsilon \to 0$ and $\alpha \to \phi(B)$, we have

$$\phi\Big(\mathrm{Conv}(f(B))\Big) \leq r\phi(B) < \phi(B).$$

Therefore, f is a generalized DND-condensing mapping.

On the other hand, and as consequence of Theorem 3.3, we can prove, under suitable conditions, the existence of an a.f.p.s. for certain class of mappings, which satisfy a weaker contractiveness condition than that of the generalized DND-condensing mappings. Moreover, we prove that such weaker contractiveness condition is sufficient to prove the existence of a r-fixed point for a r-continuous mappings.

Corollary 3.6. *Let $C \in \mathcal{B}(X)$ closed and convex, and $f : C \longrightarrow C$ such that for each non-empty and convex $B \subset C$ we have*

$$\phi\Big(\mathrm{Conv}(f(B))\Big) \leq \phi(B). \tag{6}$$

Then, we have:

(1) If there is $x_0 \in C$ satisfying that for each $x \in C$ with $f(x) \neq x$ there are $p^* \in X^*$ and $\varepsilon > 0$ such that for each $0 < \lambda < 1$

$$\lambda\langle x_0 - x', p^*\rangle + (1-\lambda)\langle f(x') - x', p^*\rangle > 0 \quad \text{for all } x' \in B(x,\varepsilon) \text{ with } f(x') \neq x',$$

then f has an a.f.p.s.

(2) If f is r-continuous, for some $r > 0$, it admits an r-fixed point.

Proof. Let $(\lambda_n)_{n\geq 1} \subset (0,1)$ a sequence with $\lim_n \lambda_n = 0$ and define, for each $n \geq 1$, $f_n : C \longrightarrow C$ as

$$f_n(x) := \lambda_n x_0 + (1-\lambda_n) f(x), \quad \text{for all } x \in C.$$

where $x_0 \in C$ is given in (1). Given any $x \in C$ with $f(x) \neq x$ and $p^* \in X^*$, $\varepsilon > 0$ obeying the conditions in (1) we have

$$\langle f_n(x') - x', p^* \rangle = \langle \lambda_n x_0 + (1-\lambda_n) f(x') - \lambda_n x' - (1-\lambda_n) x', p^* \rangle =$$
$$\lambda_n \langle x_0 - x', p^* \rangle + (1-\lambda_n) \langle f(x') - x', p^* \rangle > 0,$$

for all $x' \in B(x,\varepsilon)$ with $f(x') \neq x'$. So, f_n is half-continuous for each $n \geq 1$. Now, given $B \subset C$ non-empty and convex noticing Proposition 2.6 and (6), we find

$$\phi\Big(\mathrm{Conv}(f_n(B))\Big) = \phi\Big(\lambda_n x_0 + (1-\lambda_n)\mathrm{Conv}(f(B))\Big) =$$
$$(1-\lambda_n)\phi\Big(\mathrm{Conv}(f(B))\Big) < \phi(B).$$

So, for each $n \geq 1$, f_n is a generalized DND-condensing mapping and consequently, by Theorem 3.3, f_n has a fixed point, put $\tilde{x}_n \in C$. Then, we have

$$\lim_n \|\tilde{x}_n - f(\tilde{x}_n)\| = \lim_n \|f_n(\tilde{x}_n) - f(\tilde{x}_n)\| = \lim_n \lambda_n \|x_0 - f(\tilde{x}_n)\| = 0$$

Therefore $(\tilde{x}_n)_{n\geq 1} \subset C$ is an a.f.p.s. for f.

Now, assume condition (2) holds and let $(\varepsilon_n)_{n\geq 1} \subset (0,1)$ be such that $\lim_n \varepsilon_n = 0$. It is easy to show that f_n is $(1-\lambda_n)r$-continuous for each $n \geq 1$. As we have proved above, f_n is also a generalize DND-condensing mapping. Then, by Theorem 3.3, for each $n \geq 1$, f_n admits a $(1-\lambda_n)r$-fixed point. This means that, for each $n \geq 1$, there is $\xi_n \in C$ such that

$$\|\xi_n - f_n(\xi_n)\| \leq (1-\lambda_n) r + \varepsilon_n. \tag{7}$$

Then, from (7), we have

$$\begin{aligned} &\|\xi_n - f(\xi_n)\| \leq \|\xi_n - f_n(\xi_n)\| + \|f_n(\xi_n) - f(\xi_n)\| \leq \\ &(1-\lambda_n) r + \varepsilon_n + \lambda_n \|x_0 - f(x_n)\| \leq (1-\lambda_n) r + \varepsilon_n + \lambda_n \mathrm{Diam}(C). \end{aligned} \tag{8}$$

So, for a given $\varepsilon > 0$, taking large enough n_ε such that

$$(1-\lambda_{n_\varepsilon}) r + \varepsilon_{n_\varepsilon} + \lambda_{n_\varepsilon} \mathrm{Diam}(C) < r + \varepsilon, \tag{9}$$

from (8) and (9) we conclude that

$$\|\xi_{n_\varepsilon} - f(\xi_{n_\varepsilon})\| < r + \varepsilon.$$

So, ξ_{n_ε} is an r-fixed point for f and the proof is now complete. □

Remark 3.7. *Looking at the above proof, if f is continuous then f_n is continuous too. So, by the considerations of Section 1, f_n is, in particular, half-continuous. Therefore, if f is continuous, the assumption given in (1) of the above corollary is redundant.*

4. Final remarks and conclusions

In the present paper we have introduced a generalization of the so called DND-condensing mappings. Some fixed points results for certain discontinuous mappings based on others existing fixed point results and this new class of mappings have been proved. Through examples, we have shown that our main result generalizes the previously proved fixed-point results mentioned above. As a consequence of our main result, we have proved under suitable conditions the existence of an a.f.p.s. for mappings obeying a weaker contractiveness condition than the required in the definition of generalized DND-condensing mappings.

On the other hand, it could be interesting to extend our results out of the frame work of the Banach spaces, as we have already done with the DND-condensing mappings in [15, 17]. Likewise, DND-condensing mappings have been used to prove the existence of attractor sets [16] or solutions of certain integral equations [13, 14, 16, 17]. So, in future works we could try to apply the presented results in this paper to prove similar existence results to the mentioned ones.

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