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Generalized ρ-almost periodic type sequences in locally convex spaces and applications

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Abstract. In this paper, we consider various classes of Weyl ρ-almost periodic type sequences, Doss ρ-almost periodic type sequences and Besicovitch ρ-almost periodic type sequences with values in locally convex spaces. We also investigate the existence and uniqueness of generalized ρ-almost periodic type solutions for some classes of the abstract (fractional) difference inclusions in locally convex spaces.

1. Introduction and preliminaries

Suppose that $(Y, \|\cdot\|)$ is a complex Banach space. Then we say that a sequence $x : \mathbb{Z}^n \to Y$ is (Bohr) almost periodic if, for every $\epsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{Z}^n$ there exists $\tau \in \mathbb{Z}^n \cap B(\mathbf{t}_0, l)$, where $B(\mathbf{t}_0, l) = \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$, such that

$$\big\|F(\mathbf{t}+\tau) - F(\mathbf{t})\big\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{Z}^n.$$

Any almost periodic Y-valued sequence is bounded and its range is relatively compact in Y. It is also well known that a sequence $(x_k)_{k\in\mathbb{Z}^n}$ in Y is almost periodic if and only if there exists an almost periodic function $F : \mathbb{R}^n \to Y$ such that $x_k = F(k)$ for all $k \in \mathbb{Z}^n$; for more details about almost periodic functions, almost periodic sequences and their applications, we refer to research monographs [7], [12], [17], [19], [20], [23] and [25].

The class of Stepanov almost periodic sequences, which has been introduced by J. Andres and D. Pennequin in the one-dimensional setting ([3]), reduces to the class of almost periodic sequences; a similar statement holds in the multi-dimensional setting. This is no longer true for the class of equi-Weyl almost periodic sequences, which provides a proper extension of the class of almost periodic sequences. This class of generalized almost periodic sequences has been introduced by A. Iwanik in [16], who considered the class of equi-Weyl almost periodic sequences with values in compact metric spaces.

Suppose that $1 \leq p < +\infty$. The class of (equi-)Weyl-p-almost periodic sequences and the class of Doss-p-almost periodic sequences have recently been introduced and analyzed in our joint research article with W.-S. Du and D. Velinov [8]. A sequence $x : \mathbb{Z}^n \to Y$ is said to be:

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(i) equi-Weyl-p-almost periodic if, for every $\epsilon > 0$, there exist $s \in \mathbb{N}$ and $L > 0$ such that, for every $\mathbf{t}_0 \in \mathbb{Z}^n$, the cube $I' \equiv \mathbf{t}_0 + [0, L]^n$ contains a point $\tau \in I' \cap \mathbb{Z}^n$ which satisfies

$$\sup_{\mathbf{t}_0 \in \mathbb{Z}^n} s^{-n/p} \left[\sum_{j \in (\mathbf{t}_0 + [0,s]^n) \cap \mathbb{Z}^n} \left\| x_{j+\tau} - x_j \right\|^p \right]^{1/p} < \epsilon; \tag{1.1}$$

(ii) Weyl-p-almost periodic if, for every $\epsilon > 0$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \mathbb{Z}^n$, the cube $I' \equiv \mathbf{t}_0 + [0, L]^n$ contains a point $\tau \in I' \cap \mathbb{Z}^n$ which satisfies that there exists an integer $s_\tau \in \mathbb{N}$ such that (1.1) holds for all integers $s \geq s_\tau$;

(iii) Doss-p-almost periodic if, for every $\epsilon > 0$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \mathbb{Z}^n$, the cube $I' \equiv \mathbf{t}_0 + [0, L]^n$ contains a point $\tau \in I' \cap \mathbb{Z}^n$ which satisfies

$$\limsup_{s \to +\infty} s^{-n/p} \left[\sum_{j \in [-s,s]^n \cap \mathbb{Z}^n} \left\| x_{j+\tau} - x_j \right\|^p \right]^{1/p} < \epsilon;$$

(iv) Besicovitch-p-almost periodic if for every $\epsilon > 0$ there exists a trigonometric polynomial $P(\cdot)$ such that

$$\limsup_{s \to +\infty} s^{-n/p} \left[\sum_{j \in [-s,s]^n \cap \mathbb{Z}^n} \left\| x_j - P(j) \right\|^p \right]^{1/p} < \epsilon.$$

The class of one-dimensional Besicovitch almost periodic sequences has been introduced by A. Bellow, V. Losert [24] and further analyzed by V. Bergelson et al. in [5] and [8] (cf. also the research article [15] by T. Downarowicz and A. Iwanik, where the authors have considered the notion of quasi-uniform convergence in compact dynamical systems).

On the other hand, the abstract difference equations in normed spaces have been considered in the research monograph [10] by M. I. Gil, where it has been assumed that all operator coefficients are bounded linear operators. Further on, the theory of discrete fractional calculus and the theory of fractional difference equations are rapidly growing fields of theoretical and applied mathematics, which are incredibly important in different fields like aerodynamics, rheology, interval-valued systems and discrete-time recurrent neural networks (cf. the monographs [1] by S. Abbas et al., and [11] by C. Goodrich and A. C. Peterson for further information in this direction). In our recent research monograph [21], we have considered the existence and uniqueness of almost periodic type solutions for various classes of the abstract fractional-differential-difference equations in Banach spaces.

Almost periodic sequences with values in locally convex spaces and abstract fractional difference equations in locally convex spaces have not received considerable attention of authors by now; cf. [22] for further information in this direction. In this paper, we continue our analysis from the above-mentioned research article by investigating some classes of Weyl ρ-almost periodic type sequences, Doss ρ-almost periodic type sequences and Besicovitch ρ-almost periodic type sequences with values in locally convex spaces. We also investigate the existence and uniqueness of generalized ρ-almost periodic type solutions for some classes of the abstract (fractional) difference inclusions in locally convex spaces.

The paper is very simply organized. After explaining the notation and preliminaries, in Section 2 we consider several new classes of generalized ρ-almost periodic type sequences in locally convex spaces. Some applications to abstract fractional difference equations in locally convex spaces are given in Section 3. With the exception of the fifth application to the abstract semilinear difference inclusions given in Section 3, the remaining part of paper is written as a scientific report.

Notation and preliminaries. If Y is a Hausdorff locally convex space over the field of complex numbers, then the abbreviation $\circledast_Y$ stands for the fundamental system of seminorms which defines the topology of Y; I denotes the identity operator on Y. If Y is sequentially complete, then we simply write that Y is an SCLCS. If X is also a Hausdorff locally convex space over the field $\mathbb{C}$, then $L(X,Y)$ denotes the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. For more details about the integration of functions with values in SCLCSs, the multivalued linear operators (MLOs) in SCLCSs and the solution operator families subgenerated by MLOs in SCLCSs, we refer the reader to [18]. We will use the same notion and notation as in this monograph.

If X is a topological space, then by a trigonometric polynomial $P : \Lambda \times X \to Y$ we mean any linear combination of functions like

$$e^{i[\lambda_1 t_1+\lambda_2 t_2+\cdots+\lambda_n t_n]}c(x),$$

where λ_i are real numbers $(1 \leq i \leq n)$ and $c : X \to Y$ is a continuous mapping. Further on, if Y is a locally convex space and $\mathcal{W}$ is a base of balanced neighbourhoods of zero in Y, then a continuous function $f : \mathbb{R}^n \to Y$ is said to be (Bohr) almost periodic if, for every $W \in \mathcal{W}$, there exists a number $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists a point $\tau \in B(\mathbf{t}_0, l)$ such that $f(t+\tau) - f(t) \in W$ for all $\mathbf{t} \in \mathbb{R}^n$. In this case, the range of $f(\cdot)$ is totally bounded in Y and $f(\cdot)$ is uniformly continuous; the space of all almost periodic functions, denoted by $AP(\mathbb{R}^n : Y)$, is translation invariant, closed under uniform convergence and closed under reflexions at zero. We have recently proved that $AP(\mathbb{R}^n : Y)$ is a vector space with the usual operations if Y is a general locally convex space; the proof of this fact will appear somewhere else (cf. also the pioneering papers [13] and [14] by G. M. N'Guérékata). Concerning the class of ρ-almost periodic functions with values in locally convex spaces and the class of Weyl ρ-almost periodic functions with values in locally convex spaces, we refer the reader to the recent research articles [9] and [2].

The Gamma function will be denoted by $\Gamma(\cdot)$ and the principal branch will be always used to take the powers; define $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $0^\zeta := 0$ $(\zeta > 0,\ t > 0)$. Given a number $s \in \mathbb{R}$, we set $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. If $m \in \mathbb{N}$, then we define $\mathbb{N}_m := \{1, ..., m\}$.

If $\alpha > 0$ and $v \in \mathbb{N}_0$, then the Cesàro sequence $(k^\alpha(v))_{v\in\mathbb{N}_0}$ is defined by

$$k^\alpha(v) := \frac{\Gamma(v+\alpha)}{\Gamma(\alpha)v!}.$$

Then we know that, for every $\alpha > 0$ and $\beta > 0$, we have $k^\alpha *_0 k^\beta \equiv k^{\alpha+\beta}$, where $*_0$ is the finite convolution product given by $(k^\alpha *_0 k^\beta)(v) := \sum_{j=0}^{v} k^\alpha(v-j)k^\beta(j)$, $v \in \mathbb{N}_0$, and $|k^\alpha(v) - g_\alpha(v)| = O(g_\alpha(v)|1/v|)$, $v \in \mathbb{N}$.

The Weyl fractional derivative $[\Delta_W^\alpha u](\cdot)$ of arbitrary order $\alpha > 0$ is defined as follows. Suppose that $m = \lceil\alpha\rceil$ and $u : \mathbb{Z} \to Y$ satisfies $\sum_{v=-\infty}^{\infty} p(u(v)) \cdot (1+|v|)^{m-\alpha-1} < +\infty$ for all $p \in \circledast_Y$. Then $[\Delta_W^\alpha u](\cdot)$ is defined by

$$\Big[\Delta_W^\alpha u\Big](v) := \Big[\Delta^m\Big(\Delta_W^{-(m-\alpha)}u\Big)\Big](v), \quad v \in \mathbb{Z},$$

where

$$\Big(\Delta_W^{-(m-\alpha)}u\Big)(v) := \sum_{l=-\infty}^{v} k^{m-\alpha}(v-l)u(l), \quad v \in \mathbb{Z}$$

and

$$\Delta^m u_v := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}u_{v+j}, \quad v \in \mathbb{Z}.$$

2. Generalized ρ-almost periodic type sequences in locally convex spaces

In this section, we will consider Weyl, Besicovitch and Doss classes of generalized ρ-almost periodic type sequences with values in locally convex spaces. Unless stated otherwise, we will always assume henceforth that Y is a (Hausdorff) locally convex space and X is an arbitrary non-empty set.

Suppose that $\Lambda = \Lambda_1 \times \Lambda_2 \times ... \times \Lambda_n$, where for each $j \in \mathbb{N}_n$ there exists an integer $a \in \mathbb{Z}$ such that $\Lambda_j = \mathbb{Z}$, $\Lambda_j = \{..., a-2, a-1, a\}$ or $\Lambda_j = \{a, a+1, a+2, ...\}$. Set $\Lambda'' := \{\mathbf{a} \in \mathbb{Z}^n : \mathbf{a} + \Lambda \subseteq \Lambda\}$. If $l \in \mathbb{N}$, then we introduce the set P_l consisting of all closed subrectangles of Λ which contains exactly $(l+1)^n$ points with all integer coordinates. Suppose also that a function $\mathbb{F}_l : \{l\} \times P_l \to [0, \infty)$ is given for each integer $l \in \mathbb{N}$.

We start this section by introducing the following extension of [21, Definition 2.1.13], where we have considered the case in which Y is a complex Banach space and $1 \leq p < +\infty$:

Definition 2.1. Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $p > 0$, $\emptyset \neq \Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. Then we say that $F(\cdot;\cdot)$ is:

(i) equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exist $l \in \mathbb{N}$ and $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that, for every $J \in P_l$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that

$$\sup_{x \in B} \big[\mathbb{F}_l(l, J)\big]^p \sum_{j \in J} \Big[\kappa\big(F(j+\tau;x) - z_{j,x}\big)\Big]^p < \epsilon; \tag{2.1}$$

(ii) Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that there exists an integer $l_\tau \in \mathbb{N}$ such that, for every $l \geq l_\tau$, $J \in P_l$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2.1) holds.

Any equi-Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic sequence is Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic and any Weyl-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic sequence is Doss-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic in the following sense:

Definition 2.2. Suppose that $F : \Lambda \times X \to Y$ is a given sequence, $p > 0$, $\emptyset \neq \Lambda' \subseteq \Lambda''$ and ρ is a binary relation on Y. Then we say that $F(\cdot;\cdot)$ is Doss-$(\mathcal{B}, \Lambda', \mathbb{F}., p, \rho)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exists $L > 0$ such that, for every $\mathbf{t}_0 \in \Lambda'$, there exists a point $\tau \in \Lambda' \cap B(\mathbf{t}_0, L)$ which satisfies that there exists an increasing sequence (l_k) of positive integers such that, for every $k \in \mathbb{N}$, $J \in P_{l_k}$, $j \in J$ and $x \in B$, there exists $z_{j,x} \in \rho(F(j;x))$ such that (2.1) holds with the number l replaced by the number l_k therein.

As in the Banach space setting, the situation in which the following condition holds:

(FV) There exists a function $\mathbb{F} : (0, \infty) \to (0, \infty)$ such that $\mathbb{F}(l, J) = \mathbb{F}(l)$ for all $l \in \mathbb{N}$ and $J \in P_l$

will be the most important for us. Now we will state the following slight extension of [21, Proposition 2.1.16]; the proof is similar to the proof of the above-mentioned result and therefore omitted:

Proposition 2.3. *Suppose that $F : \Lambda \times X \to Y$ is a given sequence, X is a pseudometric space, $p > 0$, $\Lambda' = \Lambda''$ and $\rho : Y \to Y$ is a continuous function. If* (FV) *holds and $F(\cdot;\cdot)$ is equi-Weyl-$(\mathcal{B}, \mathbb{F}, p, \rho)$-almost periodic, then for each bounded set $B \in \mathcal{B}$ the set $\{F(\mathbf{t};x) : \mathbf{t} \in \Lambda; x \in B\}$ is bounded as well.*

In [21, Theorem 2.1.18], we have stated an important result concerning the extensions of (equi-)Weyl ρ-almost periodic type sequences and Doss ρ-almost periodic type sequences. This result also holds in the case that X is an arbitrary non-empty set, Y is a locally convex space and $p > 0$; moreover, the result stated in [21, Corollary 2.1.20] holds if $1 \leq p < +\infty$ and Y is a locally convex space (see [9] for more details).

The statement of [21, Proposition 2.1.24] can be extended in the following way, with a general exponent $p > 0$ (see [2] for the notion of a Bohr almost periodic function $H : \Lambda \to Y$):

Proposition 2.4. *Suppose that $F : \Lambda \to Y$ is a given sequence, $p > 0$, $\Lambda' = \Lambda''$, $\rho = \mathrm{I}$ and $F(l, J) \equiv l^{-n/p}$ for all $l \in \mathbb{N}$ and $J \in P_l$. If for each $\epsilon > 0$ and $\kappa \in \circledast_Y$ there exist a Bohr almost periodic function $H : \Lambda \to Y$ and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have*

$$l^{-n} \sum_{j \in J} \Big[\kappa\big(F(j) - H(j)\big)\Big]^p \leq \epsilon, \tag{2.2}$$

then $F(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}., p, \rho)$-almost periodic.

Unfortunately, in the present situation, we do not know whether the well-known result of A. Iwanik [16, Lemma 1] and the statement of [21, Proposition 2.1.22] can be extended to the equi-Weyl-p-almost periodic functions with values in locally convex spaces.

Now we will introduce the class of Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic sequences in locally convex spaces; cf. [21, Definition 2.1.27] for the case in which Y is a complex Banach space and $1 \leq p < +\infty$:

Definition 2.5. Suppose that X is a topological space, $F : \Lambda \times X \to Y$ is a given sequence, $\mathbb{F} : (0, \infty) \to [0, \infty)$ and $p > 0$. Then we say that $F(\cdot; \cdot)$ is Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic if, for every $\epsilon > 0$, $\kappa \in \circledast_Y$ and $B \in \mathcal{B}$, there exists a trigonometric polynomial $P(\cdot; \cdot)$ such that

$$\limsup_{l \to +\infty} \left\{ \bigl[\mathbb{F}(l)\bigr]^p \cdot \sup_{x \in B} \sum_{j \in [-l,l]^n \cap \Lambda} \Bigl[\kappa\bigl(F(j; x) - P(j; x)\bigr)\Bigr]^p \right\} < \epsilon.$$

If $\mathbb{F}(l) \equiv l^{-n/p}$, then we omit the term "$\mathbb{F}$" from the notation.

The set of all Besicovitch-$(\mathcal{B}, \mathbb{F}, p)$-almost periodic sequences is a vector space with the usual operations. The assertions of [21, Theorem 2.1.28, Corollary 2.1.29] continue to hold in locally spaces with the obvious terminological changes; in particular, any Besicovitch-p-almost periodic sequence $F : \mathbb{Z}^n \to Y$ is Besicovitch p-bounded in the sense that

$$\limsup_{l \to +\infty} \left\{ \frac{1}{l^n} \cdot \sup_{x \in B} \sum_{\mathbf{t} \in [-l,l]^n \cap \Lambda} \Bigl[\kappa\bigl(F(\mathbf{t}; x)\bigr)\Bigr]^p \right\} < +\infty \quad (p > 0),$$

and the mean value

$$M(F) := \lim_{T \to +\infty} \frac{1}{T^n} \sum_{\mathbf{t} \in (\mathbf{s} + T[0,1]^n) \cap \mathbb{Z}^n} F(\mathbf{t})$$

exists if $1 \leq p < +\infty$ and Y is an SCLCS, uniformly in $\mathbf{s} \in \mathbb{Z}^n$.

For simplicity, we will not consider here the metrically generalized ρ-almost periodic sequences in locally convex spaces; cf. [21, Subsection 2.1.4] for the Banach space setting. The structural results established in [2, Proposition 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.9] and the structural results established in [2, Subsection 2.1], where we have investigated the connections between the completions of locally convex spaces and Weyl ρ-almost periodicity, can be clarified for the Weyl ρ-almost peirodic type sequences; details can be left to interested readers.

3. Applications to abstract fractional difference equations in locally convex spaces

In this section, we will present some applications of the introduced notion to the abstract fractional difference inclusions in locally convex spaces. For the sake of brevity, we will always asssume that the underlying locally convex spaces are sequentially complete.

We will divide the remainder of this section into five parts.

1. In [4, Section 3], D. Araya, R. Castro and C. Lizama have investigated the almost automorphic solutions of the first-order linear difference equation

$$u(k+1) = Au(k) + f(k), \quad k \in \mathbb{Z}, \tag{3.1}$$

where Y is a complex Banach space, $A \in L(Y)$ and $(f_k \equiv f(k))_{k \in \mathbb{Z}}$ is an almost automorphic sequence. All results concerning the existence and uniqueness of almost periodic (automorphic) type solutions of (3.1) established in [4] and [21] continue to hold if Y is a general SCLCS and $A = \lambda \mathrm{I}$, where $|\lambda| \neq 1$. In this case, the assertions of [21, Theorem 2.1.40, Theorem 2.1.41] remain true in SCLCSs.

Concerning the statement of [21, Theorem 2.1.42], we will first clarify the following extension of [21, Theorem 2.1.45] in SCLCSs:

Theorem 3.1. *([22]) Suppose that (f_k) is a (polynomially) bounded sequence, $\mathcal{A}$ is a closed* MLO *in Y, $C \in L(Y)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $\mathcal{A}^{-1}C \in L(Y)$, there exists $r > 1$ such that for each $x \in Y$ the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$, $|\lambda| < r$ is continuous and for each seminorm $p \in \circledast_Y$ there exist $c > 0$ and $q \in \circledast_Y$ such that $p((z - \mathcal{A})^{-1}Cx) \leq cq(x)$, $x \in Y$, $|z| < r$. Then the abstract difference inclusion*

$$u(k+1) \in \mathcal{A}u(k) + Cf(k), \quad k \in \mathbb{Z}, \tag{3.2}$$

has a (polynomially) bounded solution $u(\cdot)$, given by

$$u(k) = -\sum_{m=k}^{\infty} \mathcal{A}^{k-m-1}Cf(m) = -\sum_{v=1}^{\infty} \mathcal{A}^{-v}Cf(k-1+v), \quad k \in \mathbb{Z}. \tag{3.3}$$

Suppose now, as in the formulation of [21, Theorem 2.1.42], that $\mathbb{F} : (0, \infty) \to (0, \infty)$, $1 \leq p < +\infty$, $\rho = T \in L(X)$, (FV) holds and the sequence $f(\cdot)$ is equi-Weyl-$(\mathbb{F}, p, T)$-almost periodic [polynomially bounded Weyl-$(\mathbb{F}, p, T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F}, p, T)$-almost periodic]. Then the solution $u(\cdot)$ of (3.2), given by (3.3), is (equi-)Weyl-$(\mathbb{F}, p, T)$-almost periodic [polynomially bounded Weyl-$(\mathbb{F}, p, T)$-almost periodic; polynomially bounded Doss-$(\mathbb{F}, p, T)$-almost periodic].

2. In [21], we have proved several structural results about the existence and uniqueness of generalized almost periodic (automorphic) solutions of the abstract fractional difference inclusion

$$\Delta_W^{\alpha} u(k) \in \mathcal{A}u(k+1) + f(k), \quad k \in \mathbb{Z}, \tag{3.4}$$

where $\mathcal{A}$ is a closed multivalued linear operator on a complex Banach space Y and $0 < \alpha \leq 1$. A sequence $(u(k))_{k\in\mathbb{Z}}$ is said to be a strong solution to (3.4) if for each seminorm $p \in \circledast_Y$ one has $\sum_{k\in\mathbb{Z}} p(u(k)) \cdot g_\alpha(|k|) < +\infty$, $u(k) \in D(\mathcal{A})$ for all $k \in \mathbb{Z}$ and (3.4) holds. If $\mathcal{A}$ is the integral generator of a discrete (α, C)-resolvent family $(S_\alpha(v))_{v\in\mathbb{N}_0} \subseteq L(Y)$, then a sequence $(u(k))_{k\in\mathbb{Z}}$ is said to be a mild solution to (3.4) if

$$u(k) := \sum_{j=-\infty}^{k-1} S_\alpha(k-1-j)f(j), \quad k \in \mathbb{Z} \tag{3.5}$$

and the above series is absolutely convergent, with the meaning clear; cf. [22] for the notion.

Suppose now that $\mathcal{A}$ is the integral generator of an exponentially stable C-regularized semigroup $(T(t))_{t\geq 0}$ on Y; here it is worth recalling that the differential operators with constant coefficients generate exponentially stable C-regularized semigroups in E_l-type spaces and their projective limits under very mild conditions (cf. [9] for more details). If this is the case, then there exists a unique strongly continuous operator family $(R_\alpha(t))_{t\geq 0} \subseteq L(Y)$ such that

$$\int_0^{+\infty} e^{-\lambda t} R_\alpha(t)x\, dt = \big(\lambda^\alpha - \mathcal{A}\big)^{-1}x, \quad \Re\lambda > 0,\ x \in Y,$$

and $\mathcal{A}$ is the integral generator of an exponentially equicontinuous (g_α, g_α)-regularized C-resolvent family $(R_\alpha(t))_{t\geq 0}$. After that, we define the Poisson transform of $(R_\alpha(t))_{t\geq 0}$ by

$$S_\alpha(v)x := \int_0^{+\infty} e^{-t}\frac{t^v}{v!} R_\alpha(t)x\, dt, \quad v \in \mathbb{N}_0,\ x \in Y.$$

Then $(S_\alpha(v))_{v\in\mathbb{N}_0} \subseteq L(Y)$ is a discrete (α, C)-resolvent family $(S_\alpha(v))_{v\in\mathbb{N}_0}$ with the integral generator $\mathcal{A}$ and for each seminorm $\kappa \in \circledast_Y$ there exist a real number $d > 0$ and a seminorm $\kappa_1 \in \circledast_Y$ such that $\sum_{v=0}^{+\infty} \kappa(S_\alpha(v)x) \leq d\kappa_1(x)$ for all $x \in Y$. Suppose now that the inhomogeneity $f(\cdot)$ is equi-Weyl-$(\mathbb{F}, 1, T)$-almost periodic [bounded Weyl-$(\mathbb{F}, 1, T)$-almost periodic; bounded Doss-$(\mathbb{F}, 1, T)$-almost periodic; bounded Besicovitch-$(\mathbb{F}, 1)$-almost periodic]. Then a mild solution $u(\cdot)$ of (3.4), given by (3.5), has the same property (cf. [21, pp. 71–72] for more details).

3. Suppose that $f : \mathbb{Z}^n \to Y$, $\lambda_1,\ \lambda_2, ...,\ \lambda_n$ are complex numbers and

$$\max\big(|\lambda_1|, |\lambda_2|, ..., |\lambda_n|\big) < 1.$$

For each tuple $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$, we define

$$\begin{aligned} u\big(k_1, k_2, ..., k_n\big) &:= \sum_{l_1 \leq k_1, l_2 \leq k_2, ..., l_n \leq k_n} \lambda_1^{k_1-l_1}\lambda_2^{k_2-l_2} \cdot ....\lambda_n^{k_n-l_n} f\big(l_1-1, l_2-1, ..., l_n-1\big) \\ &= \sum_{v_1 \geq 0, v_2 \geq 0, ..., v_n \geq 0} \lambda_1^{v_1}\lambda_2^{v_2} \cdot .... \cdot \lambda_n^{v_n} f\big(k_1-v_1-1, k_2-v_2-1, ..., k_n-v_n-1\big). \end{aligned}$$

Using the same argumentation as in the Banach space setting, we can prove that, if (FV) holds and the sequence $f(\cdot)$ is equi-Weyl-$(\Lambda', \mathbb{F}, p, T)$-almost periodic [polynomially bounded Weyl-$(\Lambda', \mathbb{F}, p, T)$-almost periodic; polynomially bounded Doss-$(\Lambda', \mathbb{F}, p, T)$-almost periodic], then the sequence $u(\cdot)$ enjoys the same property as $f(\cdot)$; a similar statement can be deduced for the class of generalized Besicovitch-p-almost periodic sequences ($\rho = T \in L(X)$). It is very simple to find the form of function $F : \mathbb{Z}^n \to X$ such that

$$u\big(k_1+1, k_2+1, ..., k_n+1\big) = \lambda_1\lambda_2...\lambda_n \cdot u\big(k_1, k_2, ..., k_n\big) + F\big(k_1, k_2, ..., k_n\big),$$

for all $(k_1, k_2, ..., k_n) \in \mathbb{Z}^n$; cf. [21, pp. 81–82] for more details.

4. In [8], we have recently analyzed the existence and uniqueness of generalized almost periodic type solutions to the abstract impulsive Volterra integro-differential equations in Banach spaces. The statement of [8, Lemma, p. 16] continues to hold in SCLCSs; concerning the assertion of [8, Theorem 8], we would like to make the following comment: Let us assume that X is an SCLCS and let us replace the condition (ew-M1), resp, (w-M1), in the formulation of this result with the following condition:

(ew-M1-T-$\circledast$) For every $\epsilon > 0$ and $\kappa \in \circledast_X$, there exist $s \in \mathbb{N}$ and $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$ which satisfies that there exists an integer $q_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all $i \in \mathbb{N}$ and

$$\sup_{|J|=s}\left[\frac{1}{s}\sum_{j\in J}\Big[\kappa\big(y_{j+q_\tau} - Ty_j\big)\Big]^p\right]^{1/p} < \epsilon, \tag{3.6}$$

where the supremum is taken over all segments $J \subseteq \mathbb{N}$ of length s and $\rho = T \in L(X)$.

(w-M1-T-$\circledast$) For every $\epsilon > 0$ and $\kappa \in \circledast_X$, there exists $L > 0$ such that every interval $I' \subseteq [0, \infty)$ of length L contains a point $\tau \in I'$ which satisfies that there exist an integer $q_\tau \in \mathbb{N}$ and an integer $s_\tau \in \mathbb{N}$ such that $|t_{i+q_\tau} - t_i - \tau| < \epsilon$ for all integers $i \in \mathbb{N}$ and (3.6) holds for all integers $s \geq s_\tau$, with $\rho = T \in L(X)$.

Then the function $G_2 : [0, \infty) \to X$, appearing in the formulation of [8, Theorem 8], will be (equi-)Weyl-(p, T)-almost periodic; all other conclusions stated in the formulation of the above-mentioned result remain the same in SCLCSs.

5. Suppose that Y is an SCLCS and $1 \leq p < +\infty$. Then by $B-e-W^p_{ap}(\mathbb{Z} : Y)$ we denote the collection of all sequences $F : \mathbb{Z} \to Y$ such that for each $\epsilon > 0$ and $\kappa \in \circledast_Y$ there exist a Bohr almost periodic function $H : \mathbb{Z} \to Y$ and an integer $l \in \mathbb{N}$ such that, for every $J \in P_l$, we have (2.2). Since Bohr almost periodic sequences form vector space with usual operations, it readily follows that $\mathcal{Y} := l_\infty(\mathbb{Z} : Y) \cap B-e-W^p_{ap}(\mathbb{Z} : Y)$ is vector space with usual operations. If we endow $\mathcal{Y}$ with the fundamental system of seminorms $(\kappa_\infty)_{\kappa\in\circledast_Y}$, where $\kappa_\infty(F) := \sup_{k\in\mathbb{Z}} \kappa(F(k))$ for all $F \in \mathcal{Y}$ and $\kappa \in \circledast_Y$, then $\mathcal{Y}$ is an SCLCS, as easily approved.

Suppose, further, that $\mathcal{A}$ is a closed MLO in Y, $C \in L(Y)$, $C\mathcal{A} \subseteq \mathcal{A}C$, $\mathcal{A}^{-1}C \in L(Y)$, there exists $r > 1$ such that for each $x \in Y$ the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}Cx$, $|\lambda| < r$ is continuous and for each seminorm $p \in \circledast_Y$ there exists $c > 0$ such that $p((z - \mathcal{A})^{-1}Cx) \leq cp(x)$, $x \in Y$, $|z| < r$. Then we know that there exist a finite real number $M > 0$ and a number $a \in (0, 1)$ such that

$$p\big(\mathcal{A}^{-v}Cx\big) \leq Ma^v p(x), \quad v \in \mathbb{N},\ x \in X; \tag{3.7}$$

cf. the proof of [22, Theorem 3.1]. Consider now the abstract semilinear difference inclusion

$$u(k+1) \in \mathcal{A}u(k) + Cf(k, u(k)), \quad k \in \mathbb{Z}. \tag{3.8}$$

Taking into account Theorem 3.1, it seems reasonable to say that a sequence $u : \mathbb{Z} \to Y$ is a mild solution of (3.8) if

$$u(k) = -\sum_{v=1}^{\infty} \mathcal{A}^{-v} Cf\big(k-1+v, u(k-1+v)\big), \quad k \in \mathbb{Z},$$

where we assume that the last series converges absolutely. Concerning the function $f : \mathbb{Z} \times Y \to Y$, we assume that the following conditions hold:

(i) For each $\kappa \in \circledast_Y$, there exists a finite real number $c_\kappa \in (0, (1-a)/(Ma))$ such that $\kappa(f(k,x)-f(k,y)) \leq c_\kappa \kappa(x-y)$ for all $k \in \mathbb{Z}$ and $x,\ y \in Y$.

(ii) For each bounded set $B \subseteq Y$, the set $\{f(k,x) : k \in \mathbb{Z},\ x \in B\}$ is bounded as well.

(iii) $f : \mathbb{Z} \times Y \to Y$ is Bohr almost periodic on bounded subsets of Y, i.e., for each $\epsilon > 0$, $\kappa \in \circledast_Y$ and for each bounded set $B \subseteq Y$, we have that there exists $l > 0$ such that for each $t_0 \in \mathbb{Z}$ there exists $\tau \in B(t_0, l) \cap \mathbb{Z}$ such that

$$\kappa\big(f(k+\tau, u(k+\tau)) - f(k, u(k))\big) \leq \epsilon, \quad k \in \mathbb{Z},\ x \in B. \tag{3.9}$$

Suppose now that a Bohr almost periodic sequence $H : \mathbb{Z} \to Y$ is given. Then the range of $H(\cdot)$, denoted by B, is bounded in Y and the function $f_B : \mathbb{Z} \to l_\infty(B : Y)$ given by $[f_B(k)](x) := f(k,x)$, $k \in \mathbb{Z}$, $x \in B$ is Bohr almost periodic. Keeping in mind this fact, we can prove that, for every $\epsilon > 0$ and $\kappa \in \circledast_Y$, there exists $l > 0$ such that for each $t_0 \in \mathbb{Z}$ there exists $\tau \in B(t_0, l) \cap \mathbb{Z}$ such that (3.9) holds and $\kappa(H(k+\tau) - H(k)) \leq \epsilon$ for all $k \in \mathbb{Z}$. Using this fact and decomposition

$$\begin{aligned}
&\kappa\big(f(k+\tau, H(k+\tau)) - f(k, H(k))\big) \\
&\quad \leq \kappa\big(f(k+\tau, H(k+\tau)) - f(k, H(k+\tau))\big) + \kappa\big(f(k, H(k+\tau)) - f(k, H(k))\big) \\
&\quad \leq \sup_{x \in B} \kappa\big(f(k+\tau, x) - f(k,x)\big) + c_\kappa \kappa\big(H(k+\tau) - H(k)\big), \quad k \in \mathbb{Z},
\end{aligned}$$

it follows that the function $k \mapsto f(k, H(k))$, $k \in \mathbb{Z}$ is Bohr almost periodic. Using now the assumptions (i) and (ii), it follows that for each $u \in \mathcal{Y}$ we have $f(\cdot, u(\cdot)) \in \mathcal{Y}$. Furthermore, the sequence

$$k \mapsto \sum_{v=1}^{\infty} \mathcal{A}^{-v} CH(k-1+v), \quad k \in \mathbb{Z}$$

is Bohr almost periodic and we can repeat verbatim the argumentation contained in the proof of [21, Theorem 2.1.42] to show that for each $u \in \mathcal{Y}$ the sequence

$$k \mapsto \sum_{v=1}^{\infty} \mathcal{A}^{-v} Cu(k-1+v), \quad k \in \mathbb{Z}$$

also belongs to $\mathcal{Y}$. By the foregoing, the mapping $\Pi : \mathcal{Y} \to \mathcal{Y}$, given by

$$\big[\Pi(u)\big](k) := -\sum_{v=1}^{\infty} \mathcal{A}^{-v} Cf\big(k-1+v, u(k-1+v)\big), \quad k \in \mathbb{Z}$$

is well-defined. Moreover, for each seminorm $\kappa \in \circledast_Y$ and $u_1,\ u_2 \in \mathcal{Y}$ we have

$$\kappa_\infty\Big(\Pi\big(u_1\big) - \Pi\big(u_2\big)\Big) \leq Mc_\kappa \frac{a}{1-a} \kappa_\infty\big(u_1 - u_2\big);$$

cf. also (3.7). Applying the well-known fixed point theorem of A. Deleanu and G. Marinescu [6, Theorem 1, p. 92], we get that there exists a unique solution of the abstract semilinear difference inclusion (3.8) which belongs to the space $\mathcal{Y}$.

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