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A set of asymptotially almost automorphic functions

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Abstract. For a set $\mathbb{S}$ of real numbers we introduce the concept of $\mathbb{S}$-asymptotically almost automorphic functions valued in a Banach space. It generalizes in particular the one of $\mathbb{Z}$-asymptotically almost automorphic functions. This enables us to study the existence of asymptotically almost automorphic solutions of a differential equation with piecewise constant argument of generalized type.

1. Introduction

The almost periodic functions have been introduced by Bohr in 1925 and describe phenomenons that are similar to the periodic oscillations which can be observed in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, plasma physics, engineering. An important generalization of the almost periodicity is the concept of the almost automorphy introduced in the literature [3]-[6] by Bochner. In [16], the author presents the theory of almost automorphic functions and their applications to differential equations.

The study of differential equations with piecewise constant argument (EPCA) is an important subject because these equations have the structure of continuous dynamical systems in intervals of unit length. Therefore they combine the properties of both differential and difference equations. There have been many papers studying DEPCA, see for instance [17],[19]-[23] and the references therein.

Some papers deal with the existence of asympotically ω-periodic solutions (see for instance [11]), S-asymptotically ω-periodic solutions of DEPCA (see [12]). Other articles deal with the existence of almost automorphic solutions of EPCA (see [10],[18]). In this paper, we study the existence of asymptoticlly almost automorphic solutions of the following differential equation with piecewise constant argument of generalized (DEPCAG) type (see [1], [2], [8])

$$x'(t) = A(t)x(\varphi(t)) + f(t, x(\varphi(t))),\ t \in \mathbb{R} \tag{1}$$

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where φ is a step function, $A : \mathbb{R} \to \mathbb{R}^{q\times q}$ is continuous in $\mathbb{R} \setminus \mathbb{S}$ and $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is continuous. More precisely, there exists a strictly increasing sequence of real numbers $t_i, i \in \mathbb{Z}$, such that $t_i \to^{+}_{-} \infty$ as $i \to^{+}_{-} \infty$ and on each interval $[t_i, t_{i+1}[$, $\varphi(t)$ is constant:

$$\varphi(t) = g_n, \quad t_n \le t < t_{n+1}.$$

In order to give sufficient conditions of existence and uniqueness of asymptotically almost automorphic solutions of the equation (1), we introduce the concept of $\mathbb{S}$-asymptotically almost automorphic functions that generalizes the one of $\mathbb{Z}$-asymptotically almost automorphic ones, where $\mathbb{S}$ is a subset of $\mathbb{R}$. We refer to [7, 14] to have more information about this.
The set of the new results presented in this paper allow to study the existence of Asymptotically Almost Automorphy of the equation (1), while in [13], we studied the existence of Almost Automorphy of the equation(1). Therefore, the study of the existence of Asymptotically Almost Automorphy of the equation (1) is new.
The paper is organized as follows. In Section 2, we recall definitions and properties about almost automorphic functions and introduce the concept of $\mathbb{S}$-almost automorphic functions. In Section 3, we also study the existence and uniqueness of almost automorphic solutions of the equation (1) considering the concept of $\mathbb{S}$-almost automorphic functions and using the Banach fixed point Theorem.

2. Asymptotically almost automorphic functions with respect to a set

Let $\mathbb{S}$ denote a subset of $\mathbb{R}$. For every non zero real number r we consider the function $\varphi_r : \mathbb{R} \to \mathbb{R}$ such that for every $(t, s) \in \mathbb{R} \times \mathbb{S}$:

$$\varphi_r(t + s) = \varphi_r(t) + rs. \tag{2}$$

In particular for all $s \in \mathbb{S}$ we have:

$$\varphi_r(s) = rs + \varphi_r(0).$$

Definition 2.1. *A subset A of $\mathbb{R}$ is said to be r-stable if it is invariant under the homothety of ratio r and center 0.*

We give an example of such a set $\mathbb{S}$ and an associated function φ_r.

Example 2.2. *Let $\mathbb{S}$ be a discrete subgroup of $\mathbb{R}$, then $\mathbb{S} = \alpha\mathbb{Z}$ for some (non negative) real α, and $\mathbb{S}$ is obviously r-stable for all non zero integer r. Set $\varphi_r(t) = [rt/\alpha]\alpha + c$ where $[.]$ is the integer part function and c is a constant; then it is easily seen that (2) is satisfied.*

Proposition 2.3. *The function φ_r satisfies the following properties*

i) $\forall (t, s) \in \mathbb{R} \times \mathbb{S}, \varphi_r(t - s) = \varphi_r(t) - rs.$
i) $\forall (s_1, s_2, \cdots, s_p) \in \mathbb{S}^p, \forall (m_1, m_2, \cdots, m_p) \in \mathbb{Z}^p$:

$$\varphi_r(m_1 s_1 + \cdots + m_p s_p) = r(m_1 s_1 + \cdots + m_p s_p) + \varphi_r(0).$$

In all the sequel $\mathbb{X}$ denotes a real or complex Banach space.

Definition 2.4. *A function $f : \mathbb{R} \to \mathbb{X}$ is said to be $\mathbb{S}$-continuous if it is continuous in $\mathbb{R} \setminus \mathbb{S}$, which is refered as an $\mathbb{S}$-continuous function.*

The set of all $\mathbb{S}$-continuous functions $f : \mathbb{R} \to \mathbb{X}$ will be denoted by $\mathbb{S}C(\mathbb{R}, \mathbb{X})$ and the set of those that are bounded by $\mathbb{S}C_b(\mathbb{R}, \mathbb{X})$. Clearly $\mathbb{S}C_b(\mathbb{R}, \mathbb{X})$ is a closed subspace of the Banach space $C_b(\mathbb{R}, \mathbb{X})$ of bounded continuous functions and then it is also a Banach space.

Definition 2.5. *A bounded $\mathbb{S}$-continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic with respect to the set $\mathbb{S}$ if for every real sequence s' valued in $\mathbb{S}$, there are a subsequence s and a function $g : \mathbb{R} \to \mathbb{X}$ such that for all $t \in \mathbb{R}$:*

$$\lim_{n\to\infty} f(t+s_n) = g(t) \text{ and } \lim_{n\to\infty} g(t-s_n) = f(t). \tag{3}$$

Such a function f is called $\mathbb{S}$-almost automorphic and if the above limits are uniform, it is called $\mathbb{S}$-almost periodic.

The set of all $\mathbb{S}$-almost automorphic (resp. almost periodic) functions will be denoted by $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ (resp. $\mathbb{S}AP(\mathbb{R},\mathbb{X})$). Clearly $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ is a subspace of the Banach space $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$; we have the following:

Theorem 2.6. *The space $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ is a Banach space.*

Proposition 2.7. *Let $\mathbb{S}$ be r-stable and $\varphi_r \in \mathbb{S}C(\mathbb{R},\mathbb{X})$. If $f \in AA(\mathbb{R},\mathbb{X})$ (resp. $AP(\mathbb{R},\mathbb{X})$), then $f \circ \varphi_r \in \mathbb{S}AA(\mathbb{R},\mathbb{X})$ (resp. $\mathbb{S}AP(\mathbb{R},\mathbb{X})$). If $f \in \mathbb{S}AA(\mathbb{R},\mathbb{X})$ (resp. $\mathbb{S}AP(\mathbb{R},\mathbb{X})$) and $\mathbb{S} \cap \varphi_r(\mathbb{R}\setminus\mathbb{S}) = \emptyset$, the same conclusion holds.*

We associate to the subset $\mathbb{S}$ the following property:

(P1) There is a bounded set K_0 in $\mathbb{R}$ such that all real t can be written as $t = \alpha + \xi$ where $\alpha \in K_0$ and $\xi \in \mathbb{S}$.

(P2) For every $T > 0$, there exist $T_\gamma \in \mathbb{R}$ such that if $t > T_\gamma$ then $\varphi_r(t) > T$.

Example 2.8. *Let $\mathbb{S} = \alpha\mathbb{Z}$ for some (non negative) real α. Set $\varphi_r(t) = [rt/\alpha]\alpha + c$ where $[.]$ is the integer part function, $r \in \mathbb{N}^*$ and c is a constant; then it is easily seen that* **(P2)** *is satisfied. In fact, since $[rt/\alpha]\alpha + c \le rt + c < [rt/\alpha]\alpha + \alpha + c$ and $\lim_{t\to+\infty} rt + c = +\infty$, we deduce that $\lim_{t\to+\infty}[rt/\alpha]\alpha + \alpha + c = +\infty$. Hence, we have $\lim_{t\to+\infty}[rt/\alpha]\alpha + c = +\infty$ and $\varphi_r(t) = [rt/\alpha]\alpha + c$ satisfy* **(P2)**.

Proposition 2.9. *Let $\mathbb{S}$ satisfy* **(P1)** *and let f be an $\mathbb{S}$-almost automorphic (resp. $\mathbb{S}$-almost periodic) function. If f is uniformly continuous, then f is almost automorphic (resp. almost periodic).*

Remark 2.10. *We note that $\mathbb{S} = \mathbb{Z}$ satisfies the condition* **(P1)***: it suffices to take $K_0 = [0, 1[$, since for every real number x, $x - [x] \in [0, 1[$.*

Definition 2.11. *A continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$, if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that for each $t \in \mathbb{R}$ and $x \in \mathbb{X}$,*

$$\lim_{n\to\infty} f(t+s_n, x) = g(t,x) \quad \textit{and} \quad \lim_{n\to\infty} g(t-s_n, x) = f(t,x).$$

Then we have the following result.

Theorem 2.12. *[16, Theorem 2.2.5] If f is almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$ and if f is Lipschitzian in x uniformly in t, then g satisfies the same Lipschitz condition in x uniformly in t.*

Using the above theorem we obtain:

Theorem 2.13. *Let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$. Assume that f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}$. Let also $\phi : \mathbb{R} \to \mathbb{X}$ be almost automorphic. Then the function $F : \mathbb{R} \to \mathbb{X}$ defined by $F(t) = f(t, \phi(\varphi_r(t)))$ is $\mathbb{S}$-almost automorphic.*

Remark 2.14. *Let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ satisfy the conditions of the previous theorem. We have that the function $G : \mathbb{R} \to \mathbb{X}$ defined by $G(t) = g(t, \phi(\varphi_r(t)))$ is bounded.*

Definition 2.15. *A bounded continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be asymptotically almost automorphic if it can be decomposed as $f(t) = g(t) + h(t)$ where*

$$g(t) \in AA(\mathbb{R}, \mathbb{X}),\ h(t) \in C_0(\mathbb{R}, \mathbb{X}).$$

Here

$$C_0(\mathbb{R}, \mathbb{X}) := \{h : \mathbb{R} \to \mathbb{X} : h \text{ is continuous on } \mathbb{R} \text{ and } \lim_{t\to\infty} h(t) = 0\}.$$

Denote by $AAA(\mathbb{R}, \mathbb{X})$ the set of all such functions.

Lemma 2.16. *$AAA(\mathbb{R}, \mathbb{X})$ is a Banach space with the suprenum norm.*

Definition 2.17. *A bounded $\mathbb{S}$-continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be $\mathbb{S}$-asymptotically almost automorphic if it can be decomposed as $f(t) = g(t) + h(t)$ where*

$$g(t) \in \mathbb{S}AA(\mathbb{R}, \mathbb{X}),\ h(t) \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X}).$$

Here

$$\mathbb{S}C_0(\mathbb{R}, \mathbb{X}) := \{h : \mathbb{R} \to \mathbb{X} : h \text{ is continuous on } \mathbb{R} \setminus \mathbb{S} \text{ and } \lim_{t\to\infty} h(t) = 0\}.$$

Denote by $\mathbb{S}AAA(\mathbb{R}, \mathbb{X})$ the set of all such functions.

Proposition 2.18. *We assume also that there exists a sequence s' valued in $\mathbb{S}$ such thar $\lim_{n\to\infty} s'_n = +\infty$. Then the decomposition of a $\mathbb{S}$-Asymptotically almost automorphic function is unique.*

Proof. We assume that $f(t) = g_1(t) + h_1(t)$, $f(t) = g_2(t) + h_2(t)$ with $g_1,\ g_2 \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $h_1,\ h_2 \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$. Then $g_1(t) - g_2(t) + h_1(t) - h_2(t) = 0$ and $\lim_{t\to\infty} g_1(t) - g_2(t) = 0$. Consider the sequence s'_k such thar $\lim_{k\to\infty} s'_k = \infty$. Therefore there exist a subsequence s_n such that

$$\lim_{k\to+\infty} g_1(t + s_k) - g_2(t + s_k) = F(t)$$

and

$$\lim_{k\to+\infty} F(t - s_k) = g_1(t) - g_2(t)$$

for all $t \in \mathbb{R}$. We deduce that for all $t \in \mathbb{R}$, $F(t) = 0$. Therefore for all $t \in \mathbb{R}$, $g_1(t) - g_2(t) = 0$ and $h_1(t) - h_2(t) = 0$. $\square$

Lemma 2.19. *We assume also that there exists a sequence s' valued in $\mathbb{S}$ such thar $\lim_{n\to\infty} s'_n = +\infty$. If $f \in \mathbb{S}AAA(\mathbb{R}, \mathbb{X})$, that is, $f = g + h$ where $g \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$ then*

$$\overline{\{g(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}\}}.$$

Proof. Since $g \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} s_n = +\infty$ such that

$$\lim_{n\to+\infty} g(t + s_n) = p(t) \qquad (a)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to+\infty} p(t - s_n) = g(t) \qquad (b)$$

for each $t \in \mathbb{R}$.

Now for any fixed t_0, we have $\lim_{n\to+\infty} t_0 + s_n = +\infty$, and deduce that

$$\lim_{n\to+\infty} f(t_0 + s_n) = \lim_{n\to+\infty} g(t_0 + s_n) + h(t_0 + s_n) = p(t_0).$$

Consequently, $p(t_0) \in \overline{\{f(t) : t \in \mathbb{R}\}}$ and $\overline{\{p(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$. According (a) and (b), we have $\overline{\{g(t) : t \in \mathbb{R}\}} = \overline{\{p(t) : t \in \mathbb{R}\}}$. Therefore

$$\overline{\{g(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}\}}.$$

□

Remark 2.20. *If $f \in \mathbb{S}AAA(\mathbb{R}, \mathbb{X})$, that is $f = g + h$ where $g \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$, we define*

$$||f||_{\mathbb{S}AAA} := \sup_{t\in\mathbb{R}} ||g(t)|| + \sup_{t\in\mathbb{R}} ||h(t)||.$$

Lemma 2.21. *We assume also that there exists a sequence s' valued in $\mathbb{S}$ such thar $\lim_{n\to\infty} s'_n = +\infty$. $\mathbb{S}AAA(\mathbb{R}, \mathbb{X})$ is a Banach space with the norm $|| \cdot ||_{\mathbb{S}AAA}$.*

Proof. If $f \in \mathbb{S}AAA(\mathbb{R}, \mathbb{X})$, that is, $f = g + h$ where $g \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $h \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$, then $||g|| = \sup_{t\in\mathbb{R}}||g(t)|| \leq \sup_{t\in\mathbb{R}}||f(t)|| = ||f||$ considering the last lemma. Now

$$\begin{aligned}
||f|| \leq ||f||_{\mathbb{S}AAA(\mathbb{R},\mathbb{X})} \\
&= \sup_{t\in\mathbb{R}} ||g(t)|| + \sup_{t\in\mathbb{R}} ||h(t)|| \\
&= \sup_{t\in\mathbb{R}} ||g(t)|| + \sup_{t\in\mathbb{R}} ||f(t) - g(t)|| \\
&\leq 3 \sup_{t\in\mathbb{R}} ||f(t)|| \\
&\leq 3||f||.
\end{aligned}$$

Let f_n be a cauchy sequence in $\mathbb{S}AAA(\mathbb{R}, \mathbb{X})$. Then $f_n = g_n + h_n$ with $g_n \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $h_n \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$.
Since f_n is a cauchy sequence in $\mathbb{S}AAA(\mathbb{R}, \mathbb{X})$, then g_n is a cauchy sequence in $\mathbb{S}AA(\mathbb{R}, \mathbb{X})$.
Since g_n is a cauchy sequence in the Banach space $\mathbb{S}AA(\mathbb{R}, \mathbb{X})$, then there exists $g \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ such that $g_n \to g$ uniformly on $\mathbb{R}$.
h_n is also a Cauchy sequence of $\mathbb{S}$-continuous functions with respect to the norm sup. Therefore, there exists $h : \mathbb{R} \to \mathbb{X}$ such that $h_n \to h$ uniformly on $\mathbb{R}$. Since $||h(t)|| \leq ||h(t) - h_n(t)|| + ||h_n(t)||$, we deduce that

$$\lim_{t\to\infty} ||h(t)|| = 0.$$

For all $x_0 \in \mathbb{R} \setminus \mathbb{S}$, h_n is continuous in x_0. Since $h_n \to h$ uniformly on $\mathbb{R}$ and h_n is continuous in x_0, then h is continuous in x_0. Therefore $h : \mathbb{R} \to \mathbb{X}$ is continous on $\mathbb{R} \setminus \mathbb{S}$. We obtain so that $h \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$.
Therefore, the function f defined as $f := g + h \in \mathbb{S}AAA(\mathbb{R}, \mathbb{X})$ and $f_n \to f$. □

Lemma 2.22. *The following assertions hold:*
i) Let $\mathbb{S}$ satisfy **(P1)** *and $f = g + h$, $g \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $\mathbb{S}C_0(\mathbb{R}, \mathbb{X})$. We assume also that there exist a sequence s' valued in $\mathbb{S}$ such thar $\lim_{n\to\infty} s'_n = \infty$. If f is uniformly continuous on $\mathbb{R}$, then g is also uniformly continuous on $\mathbb{R}$.*
ii) Let $\mathbb{S}$ satisfy **(P1)**. *We assume also that there exist a sequence s' valued in $\mathbb{S}$ such that $\lim_{n\to\infty} s'_n = \infty$If f is uniformly continuous on $\mathbb{R}$ and $f \in \mathbb{S}AAA(\mathbb{R}, \mathbb{X})$. Then $f \in AAA(\mathbb{R}, \mathbb{X})$.*

Proof. i) There exist a sequence $s'_n \in \mathbb{S}$ with $\lim_{n\to\infty} s'_n = \infty$. Therefore, there exist a subsequence $\{s_n\} \subset \{s'_n\}$ such that

$$\lim_{n\to\infty} g(t+s_n) = u(t) \text{ and } \lim_{n\to\infty} u(t-s_n) = g(t).$$

We observe that for every $x,\ y \in \mathbb{R}$:

$$\begin{aligned} ||u(x)-u(y)|| &\le ||u(x)-g(x+s_n)|| + ||g(x+s_n)-g(y+s_n)|| + ||g(y+s_n)-u(y)|| \\ &\le ||u(x)-g(x+s_n)|| + ||f(x+s_n)-f(y+s_n)|| + ||g(y+s_n)-u(y)|| \\ &\quad + ||h(x+s_n)|| + ||h(y+s_n)|| \end{aligned}$$

Sinc f is uniformly continuous, and g is a $\mathbb{S}$ almost automophic function, then u is also uniformly continuous. We observe that for every $x,\ y \in \mathbb{R}$:

$$||g(x)-g(y)|| \le ||g(x)-u(x-s_n)|| + ||u(x-s_n)-u(y-s_n)||$$
$$+||u(y-s_n)-g(y).||$$

We deduce so that g is uniformly continuous.
ii) Since f is uniformly continuous, then g is uniformly continuous. Since g is uniformly continuous and a $\mathbb{S}$ almost-automorphic function, then g is an almost automorphic function. Since f and g are continuous then $h = f - g$ is continuous. Therefore $h \in C_0(\mathbb{R},\ \mathbb{X})$. □

Example 2.23. *The function $f(t) = \sin(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)})$ is an almost automorphic function but is not almost periodic (see [16]). For all real $\alpha \neq 0$, the function $g(t) = \sin(\frac{1}{2+\cos([\frac{t}{\alpha}]\alpha)+\cos(\sqrt{2}[\frac{t}{\alpha}]\alpha)})$ is a $\alpha\mathbb{Z}$-almost automorphic function.*

Proposition 2.24. *We assume that $\mathbb{S}$ be r-stable, that* **(P1)** *and* **(P2)** *are satisfied. We assume also that $\varphi_r \in \mathbb{S}C(\mathbb{R},\mathbb{X})$. If $f \in AAA(\mathbb{R},\mathbb{X})$, then $f \circ \varphi_r \in \mathbb{S}AAA(\mathbb{R},\mathbb{X})$.*

Proof. Let $f \in AAA(\mathbb{R},\mathbb{X})$. Therefore $f = g + h$ where $g \in AA(\mathbb{R},\mathbb{X})$ and $h \in C_0(\mathbb{R},\ \mathbb{X})$. According to the proposition 2.7, $g \circ \varphi_r \in \mathbb{S}AA(\mathbb{R},\mathbb{X})$. Since $h \in C_0(\mathbb{R},\ \mathbb{X})$, then we have

$$\forall \epsilon > 0,\ \exists\, T > 0, t > T \implies ||h(t)|| < \epsilon.$$

Let $\epsilon > 0$. According to **(P2)**, there exists $T_\gamma \in \mathbb{R}$ such thath if $t > T_\gamma$, then $\varphi_r(t) > T$. Therefore, if $t > T_\gamma$, then $||h(\varphi_r(t))|| < \epsilon$. We deduce so that $h \in \mathbb{S}C_0(\mathbb{R},\ \mathbb{X})$. □

Definition 2.25. *A bounded continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is said to vanish at infinity if $\lim_{t\to\infty} ||f(t,x)|| = 0$ uniformly on any bounded subset of $\mathbb{X}$. Denote by $C_0(\mathbb{R}\times\mathbb{X},\ \mathbb{X})$ the set of all these functions.*

Definition 2.26. *A bounded continuous function $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ is said to be asymptotically almost automorphic if it can be decomposed as $f = g + h$ where*

$$g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X}),\ h \in C_0(\mathbb{R}\times\mathbb{X},\mathbb{X}).$$

Denote by $AAA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ the set of all such functions.

Theorem 2.27. *Let $f : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ be asymptotically almost automorphic. Assume that f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}$. Let also $\phi : \mathbb{R} \to \mathbb{X}$ be asymptotically almost automorphic. Then the function $F : \mathbb{R} \to \mathbb{X}$ defined by $F(t) = f(t, \phi(\varphi_r(t))$ is $\mathbb{S}$-Asymptotically almost automorphic.*

Proof. Let $f = g + h$, $\phi = \phi_1 + \phi_2$ where $g \in AA(\mathbb{R} \times \mathbb{X}, \mathbb{X})$, $\phi_1 \in AA(\mathbb{R}, \mathbb{X})$, $h \in C_0(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ and $\phi_2 \in C_0(\mathbb{R}, \mathbb{X})$. We have

$$f(t, \phi(\varphi_r(t)) = g(t, \phi_1(\varphi_r(t)) + f(t, \phi(\varphi_r(t)) - g(t, \phi_1(\varphi_r(t))$$

$$= g(t, \phi_1(\varphi_r(t)) + g(t, \phi(\varphi_r(t)) - g(t, \phi_1(\varphi_r(t)) + h(t, \phi(\varphi_r(t)).$$

Since f is lipschitzian then g is also lipschitzian. The idea is the same idea that in i of the lemma 2.22. Since g is lipschitzian, then the function $G(t) = g(t, \phi_1(\varphi_r(t)))$ is a $\mathbb{S}$-almost automorphic function.
Since

$$||g(t, \phi(\varphi_r(t)) - g(t, \phi_1(\varphi_r(t))|| \leq K_g ||\phi_2(\varphi_r(t))||,$$

we deduce that $g(t, \phi(\varphi_r(t)) - g(t, \phi_1(\varphi_r(t)) \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$. Obviously $h(t, \phi(\varphi_r(t)) \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$. □

3. A differential equation with a general piecewise constant argument

We consider the differential equation (1) where φ is a step function, $A : \mathbb{R} \to \mathbb{R}^{q \times q}$ is continuous in $\mathbb{R} \setminus \mathbb{S}$ and $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is continuous. Thus, in the sequel $\mathbb{X} = \mathbb{R}^q$. Moreover, in addition to **(P1)**, we consider the two following conditions:

(P3) $\forall (t, s) \in \mathbb{R} \times \mathbb{S},\ \varphi(t + s) = \varphi(t) + s \quad \text{and} \quad \varphi(t) \leq t.$

(P4) $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is asymptotically almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$ and f satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}$.

We give a consequence of **(P1)** that will be useful for the sequel.

Proposition 3.1. *[13] Assume that* **(P1)** *is satisfied, then there exists a bounded set K_1 in $\mathbb{R}$ such that: $\forall t \in \mathbb{R}$, $t - \varphi(t) \in K_1$.*

Definition 3.2. *A solution of (1) is a function $x(t)$ defined on $\mathbb{R}$ for which the following conditions hold:*

(1) *$x(t)$ is continuous on $\mathbb{R}$.*
(2) *The derivative $x'(t)$ exists at each point $t \in \mathbb{R}$, with possible exception at the points $t_i, i \in \mathbb{Z}$, where one-sided derivatives exist.*
(3) *The equation (1) is satisfied on each interval $[t_i, t_{i+1}[$, $i \in \mathbb{Z}$.*

Theorem 3.3. *[13] Let f satisfy* **(P3)** *and* **(P4)**. *Then the solution of (1) satisfies*

$$x(t) = x(\varphi(t)) + \int_{\varphi(t)}^{t} A(s)x(\varphi(s))ds + \int_{\varphi(t)}^{t} f(s, x(\varphi(s))ds.$$

Lemma 3.4. *Assume that* **(P1), (P2), (P3)** *and* **(P4)** *are satisfied. We assume also that $A(t)$ is an $\mathbb{S}$-Asymptotically almost automorphic operator. Then*

$$(\wedge\phi)(t) = \phi(\varphi(t)) + \int_{\varphi(t)}^{t} A(s)\phi(\varphi(s))ds + \int_{\varphi(t)}^{t} f(s, \phi(\varphi(s)))ds$$

maps $\mathbb{S}AAA(\mathbb{X})$ into itself.

We set $M_1 = \sup(K_1)$, where K_1 is the bounded subset of $\mathbb{R}$ introduced in Proposition 3.1. Note that, if $\varphi(t) \leq t$, then $M_1 \geq 0$.

Proof. We have $A = A_1 + A_2$ and $\phi = \phi_1 + \phi_2$, where $A_1, \phi_1 \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ and $A_2, \phi_2 \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X})$. According to the theorem 2.27, we have $f(t, \phi(\varphi(t))) = g(t, \phi_1(\varphi(t)) + g(t, \phi(\varphi(t)) - g(t, \phi_1(\varphi(t)) + h(t, \phi(\varphi(t))$ where

$$g(t, \phi_1(\varphi(t)) \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$$

and

$$g(t, \phi(\varphi(t)) - g(t, \phi_1(\varphi(t)) + h(t, \phi(\varphi_r(t)) \in \mathbb{S}C_0(\mathbb{R}, \mathbb{X}).$$

We put

$$F(t) = \phi(\varphi(t)) + \int_{\varphi(t)}^{t} A(s)\phi(\varphi(s))ds + \int_{\varphi(t)}^{t} f(s, \phi(\varphi(s)))ds.$$

Therefore we can write $F(t) = G(t) + H(t)$ with

$$G(t) = \phi_1(\varphi(t)) + \int_{\varphi(t)}^{t} A_1(s)\phi_1(\varphi(s))ds + \int_{\varphi(t)}^{t} g(s, \phi_1(\varphi(s))ds$$

and

$$\begin{aligned} H(t) = \phi_2(\varphi(t)) &+ \int_{\varphi(t)}^{t} A_1(s)\phi_2(\varphi(s))ds + \int_{\varphi(t)}^{t} A_2(s)\phi_1(\varphi(s))ds \\ &+ \int_{\varphi(t)}^{t} A_2(s)\phi_2(\varphi(s))ds + \int_{\varphi(t)}^{t} g(s, \phi(\varphi(s)) - g(s, \phi_1(\varphi(s))ds \\ &+ \int_{\varphi(t)}^{t} h(s, \phi(\varphi(s))ds. \end{aligned}$$

According to the lemma 3.3 of [13], we have that $G \in \mathbb{S}AA(\mathbb{R}, \mathbb{X})$ because $\phi_1 \in SAA(\mathbb{R}, \mathbb{X})$. Let $\epsilon > 0$. There exist $T > 0$ such that if $t > T$ then

$$||\phi_2(\varphi(t))|| \leq \frac{\epsilon}{6},\ ||\phi_2(\varphi(t))|| \leq \frac{\epsilon}{6M_1||A_1||_\infty},\ ||\phi_2(\varphi(t))|| \leq \frac{\epsilon}{6M_1K_g}$$

$$||A_2(t)|| \leq \frac{\epsilon}{6M_1||\phi_1||_\infty},\ ||A_2(t)|| \leq \frac{\epsilon}{6M_1||\phi_2||_\infty}\ \textit{and}\ ||h(t, \phi(\varphi(t)))|| \leq \frac{\epsilon}{6}.$$

According to **(P2)**, there exist T_γ such that if $t > T_\gamma$, then $\varphi(t) > T$. If $t > T_\gamma$, we observe that

$$\begin{aligned} ||H(t)|| \leq ||\phi_2(\varphi(t))|| &+ \int_{\varphi(t)}^{t} ||A_1(s)\phi_2(\varphi(s))||ds + \int_{\varphi(t)}^{t} ||A_2(s)\phi_1(\varphi(s))||ds \\ &+ \int_{\varphi(t)}^{t} ||A_2(s)\phi_2(\varphi(s))||ds + \int_{\varphi(t)}^{t} ||g(s, \phi(\varphi(s)) - g(s, \phi_1(\varphi(s))||ds \\ &+ \int_{\varphi(t)}^{t} ||h(s, \phi(\varphi(s))||ds. \\ \leq ||\phi_2(\varphi(t))|| &+ \int_{\varphi(t)}^{t} ||A_1||_\infty ||\phi_2(\varphi(s))||ds + \int_{\varphi(t)}^{t} ||A_2(s)||\, ||\phi_1||_\infty ds \\ + \int_{\varphi(t)}^{t} ||A_2(s)||\, ||\phi_2||_\infty ds &+ \int_{\varphi(t)}^{t} K_g ||\phi_2(\varphi(s))||ds + \int_{\varphi(t)}^{t} ||h(s, \phi(\varphi(s))||ds \\ \leq \frac{\epsilon}{6} &+ \int_{\varphi(t)}^{t} ||A_1||_\infty \frac{\epsilon}{6M_1||A_1||_\infty} ds + \int_{\varphi(t)}^{t} \frac{\epsilon}{6M_1||\phi_1||_\infty} ||\phi_1||_\infty ds \end{aligned}$$

$$+\int_{\varphi(t)}^{t}\frac{\epsilon}{6M_1||\phi_2||_\infty}||\phi_2||_\infty ds+\int_{\varphi(t)}^{t}K_g\frac{\epsilon}{6M_1K_g}ds+\int_{\varphi(t)}^{t}\frac{\epsilon}{6M_1}ds$$
$$\leq\frac{\epsilon}{6}+\frac{\epsilon}{6M_1}M_1+\frac{\epsilon}{6M_1}M_1+\frac{\epsilon}{6M_1}M_1+\frac{\epsilon}{6M_1}M_1+\frac{\epsilon}{6M_1}M_1$$
$$\leq\epsilon.$$

Therefore $H\in\mathbb{S}C_0(\mathbb{R},\mathbb{X})$. □

Theorem 3.5. *Assume that* **(P1)**, **(P2)**, **(P3)** *and* **(P4)** *are satisfied and that $y\to\varphi(y)$ is constant on the interval $[\varphi(t),t]$. If*

$$\|I+\int_{\varphi(t)}^{t}A(s)ds\|+M_1L<1,$$

then (1) *has a unique $\mathbb{S}$-Asymptotically Almost Automorphic solution which is also the unique Asymptotically Almost Automorphic solution of* (1).

Proof. First Step

We define the nonlinear operator Γ by the expression

$$(\Gamma\phi)(t)=\phi(\varphi(t))+\int_{\varphi(t)}^{t}A(s)\phi(\varphi(s))ds+\int_{\varphi(t)}^{t}f(s,\phi(\varphi(s)))ds.$$

According to Theorem 2.27, the function $t\mapsto f(t,\phi(\varphi(t)))$ belongs to $\mathbb{S}AAA(\mathbb{R},\mathbb{X})$. According to Lemma 3.4 the operator Γ maps $\mathbb{S}AAA(\mathbb{R},\mathbb{X})$ into itself. Since $t-\varphi(t)\leq M_1$ for all $t\in\mathbb{R}$, we have:

$$\|(\Gamma\phi)(t)-(\Gamma\psi)(t)\|=\|\Big(I+\textstyle\int_{\varphi(t)}^{t}A(s)ds\Big)\Big(\phi(\varphi(t))-\psi(\varphi(t))\Big)$$
$$+\textstyle\int_{\varphi(t)}^{t}f(s,\phi(\varphi(s)))-f(s,\psi(\varphi(s)))ds\|$$
$$\leq\|I+\textstyle\int_{\varphi(t)}^{t}A(s)ds\|\,\|\phi(\varphi(t))-\psi(\varphi(t))\|$$
$$+\|\textstyle\int_{\varphi(t)}^{t}f(s,\phi(\varphi(s)))-f(s,\psi(\varphi(s)))ds\|$$
$$\leq\|I+\textstyle\int_{\varphi(t)}^{t}A(s)ds\|\,\|\phi-\psi\|_\infty$$
$$+\textstyle\int_{\varphi(t)}^{t}L\|\phi(\varphi(s)))-\psi(\varphi(s))\|ds$$
$$\|(\Gamma\phi)(t)-(\Gamma\psi)(t)\|\leq\Big(\|I+\textstyle\int_{\varphi(t)}^{t}A(s)ds\|+M_1L\Big)\|\phi-\psi\|_\infty.$$

This proves that Γ is a contraction. We conclude that Γ has a unique fixed point in $\mathbb{S}AAA(\mathbb{R},\mathbb{X})$. We denote by z the unique $\mathbb{S}$-asymptotically almost automorphic solution of (1).

Second Step

We show that z is an asymptotically almost automorphic solution of (1). Since z is $\mathbb{S}$-asymptotically almost automorphic, it suffices to prove that z is uniformly continuous. According to the second step of the proof oh the theorem of [13], z is uniformly continuous. Therefore z is an asymptotically almost automorphic solution of (1).

The function z is necessarily the unique asymptotically almost automorphic solution of (1). In fact, an asymptotically almost automorphic function is also $\mathbb{S}$-asymptotically almost automorphic and (1) has a unique such solution. The theorem is thus proved. □

Corollary 3.6. *Let $A(t)$ be a $\mathbb{Z}$-asymptotically almost automorphic operator and assume that* **(P4)** *is satisfied. If*

$$\|I + \int_{[t]}^{t} A(s)ds\| + L < 1,$$

then the following equation

$$x'(t) = A(t)x([t]) + f(t, x([t]))dt, \ t \in \mathbb{R}$$

has a unique $\mathbb{Z}$-asymptotically almost automorphic solution which is also his unique asymptotically almost automorphic solution.

Proof. We have $\varphi(t) = [t] \le t$, $K_0 = K_1 = [0, 1[$ and $M_1 = 1$. let $T > 0$. We put $T_\gamma = [T] + 1$. If $t > T_\gamma$ then $[t] > T$. □

Corollary 3.7. *Suppose that $A(t)$ is a $\alpha h\mathbb{Z}$-asymptotically almost automorphic operator and that* **(P4)** *is satisfied. If*

$$\|I + \int_{[\frac{t}{\alpha h}]\alpha h}^{t} A(s)ds\| + \alpha h L < 1,$$

then the following equation

$$x'(t) = A(t)x(\left[\frac{t}{\alpha h}\right]\alpha h) + f(t, x(\left[\frac{t}{\alpha h}\right]\alpha h))dt, \ t \in \mathbb{R},$$

has a unique $\alpha h\mathbb{Z}$-asymptotically almost automorphic solution which is also its unique asymptotically almost automorphic solution.

Proof. We have that $\varphi(t) = [\frac{t}{\alpha h}]\alpha h$. Then φ is constant on each interval $[n\alpha h, (n+1)\alpha h[$ where $n \in \mathbb{Z}$. We observe also that

$$\begin{aligned} \varphi(t + \alpha h n) &= \Big[\frac{t + \alpha h n}{\alpha h}\Big]\alpha h = \Big[\frac{t}{\alpha h} + n\Big]\alpha h \\ &= \Big[\frac{t}{\alpha h}\Big]\alpha h + \alpha h n = \varphi(t) + \alpha h n. \end{aligned}$$

If $t \in [n\alpha h, (n+1)\alpha h[$ where $n \in \mathbb{Z}$, then $\varphi(t) = \alpha h n$, $\varphi(t) \le t$ $t - \varphi(t) \in [0, \alpha h]$ and $M_0 = \alpha h$. All real t can be written as $t = \beta + \zeta$ where $\beta \in [0, \alpha h]$ and $\zeta \in \alpha h\mathbb{Z}$. Let $T > 0$. We put $T_\gamma = \big([\frac{T}{\alpha h}] + 1\big)\alpha h$. If $t > T_\gamma$ then $[\frac{t}{\alpha h}]\alpha h > T$. □

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