Functional Analysis, Approximation and Computation
17:2 (2025), 1–7

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Convergence theorems for strongly pseudocontractive operators

Priya Raphael[a,]

[a]*Department of Mathematics, Christ college, Pune, India*

Abstract. In this paper, the convergence of Ishikawa type iteration process to the unique fixed point of a strongly pseudocontractive operator in real Banach space is established. The convergence of a Mann type iteration scheme to a fixed point of a continuous strongly pseudocontractive operator with pseudocontractive parameter $k \in (0, \frac{1}{8})$ in Hilbert space is also proved.

1. Introduction

Let X be a real Banach space and X^* be the dual space of X. Let J denote the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^2 = \left\|f\right\|^2 \right\}, \quad \text{for all} \quad x \in X,$$

where $\langle ., . \rangle$ denotes the generalized duality pairing between X and X^*. In the sequel, we shall denote the domain and the range of $T : X \to X$ by $D(T)$ and $R(T)$ respectively. It is well known that

$$\langle x, j(y) \rangle \leq \|x\| \left\|y\right\|, \quad \text{for all} \quad x, y \in X \quad \text{and each} \quad j(y) \in J(y.) \tag{1}$$

An operator $T : D(T) \subset X \to X$ is called a *strongly pseudocontractive* operator if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq k \left\|x - y\right\|^2 .$$

For k=1, such operators are called *pseudocontractive*.

An equivalent definition of strongly pseudocontractive mapping in Hilbert space [1] is as follows:
Let H be a Hilbert space. A mapping $T : H \to H$ is said to be *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + k \left\|(I - T)x - (I - T)y\right\|^2 , \qquad \text{for all} \qquad x, y \in H.$$

2020 *Mathematics Subject Classification*. Primary 47H10, 47J26.
Keywords. Strongly pseudocontractive operator; Mann type iteration; Ishikawa type iteration.
Received: 2 May 2025; Accepted: 12 October 2025
Communicated by Dragan S. Djordjević
* Corresponding author
Email address: priyaraphael77@gmail.com (Priya Raphael)

This alternative definition of strongly pseudocontractive operator is used in Theorem 2.2 of this paper.

An operator $T : X \to X$ is called *strongly accretive* if there exists a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq k \left\|x - y\right\|^2 ,$$

holds for all $x, y \in X$ and some $j(x - y) \in J(x - y)$.

T is called *accretive* if $k = 0$ in the above inequality.

T is strongly pseudocontractive if and only if (I-T) is strongly accretive, where I denotes the identity mapping. This gives that if T is strongly pseudocontractive, then for every $x, y \in K$, there exists $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq k \left\|x - y\right\|^2 . \tag{2}$$

One of the effective methods for approximating fixed points of an operator $T : X \to X$ is the Ishikawa iteration process [7] which is defined as follows:
For any initial point $x_0 \in X$, let $(x_n)_{n=0}^{\infty} := (x_0, x_1, x_2, ...)$: be the sequence given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n = 0, 1, 2, ...,$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ satisfy certain appropriate conditions. If $\beta_n = 0$, then Ishikawa iteration process reduces to Mann iteration [8].

In the past few decades, several authors like Chang et al. [2], Chidume [3], Chidume and Osilike [4], Ciric [5] , Deng [6], Morales [9], Osilike [10], Soltuz [12], Xu [13], Zhou and Jia [14] and Zhou [15] have studied the convergence of Ishikawa iteration process associated with strongly accretive or strongly pseudocontractive operators under different conditions. Zhou [15] considered Ishikawa iteration process with parameters $0 < \alpha < \alpha_n$ and proved the stability of the iteration procedure for strongly pseudocontractive operators in real uniformly smooth Banach spaces. In 2002, Soltuz [12] presented a correction for the main result of Zhou [15] for srongly pseudocontractive operators with bounded range and pseudocontractive parameter $k \in (0, \frac{1}{2})$. This result was extended by Ciric [5] to all strongly pseudocontractive operators with $0 < k < 1$ satisfying the restriction that the sequences $\{Tx_n\}, \{Ty_n\}$ being bounded and α_n satisfying the condition $0 < a \leq \alpha_n \leq b < 2(1 - k)$, where constants $a, b \in (0, 1]$.

In this paper, we consider the Ishikawa type iteration process defined as

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_n T y_n, \tag{3}$$

$$y_n = (1 - \beta_n)x_{n-1} + \beta_n T x_n, \qquad n \geq 1,$$

where $\alpha_n, \beta_n \in (0, 1)$. We prove the strong convergence of this iteration process to the unique fixed point of the strongly pseudocontractive operator T with pseudocontractive parameter $k \in (0, 1)$ in real Banach spaces.

In 2007, Rafiq [11] introduced the following Mann type implicit iteration process:
Let K be a closed, convex subset of a real normed space and $T : K \to K$ be a mapping. For a sequence $\{v_n\}$ in K, define the sequence $\{x_n\}$ as

$$x_0 \in K, \tag{4}$$

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T v_n, \qquad n \geq 1,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying some appropriate conditions. Using the above iteration he proved the following theorem:

Theorem 1.1. [11, *Theorem 3*] *Let K be a compact convex subset of a real Hilbert space H and $T : K \to K$ a continuous hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in* $[0, 1]$ *satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by (4) satisfying $\sum_{n\geq 1} \|v_n - x_n\| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Inspired and motivated by this, the convergence of the iteration process given by (4) for a continuous, strongly pseudocontractive operator with pseudocontractive parameter $k \in (0, \frac{1}{8})$ in Hilbert space is established.

The following lemmas are required to prove the results:

Lemma 1.2. *Let X be a real Banach space and let $J : X \to 2^{X^*}$ be a normalized duality mapping. Then*

$$\left\|x + y\right\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle,$$

for all $x, y \in K$ and each $j(x + y) \in J(x + y)$.

Lemma 1.3. *Let $\{\rho_n\}$ be a sequence of non-negative real numbers which satisfy*

$$\rho_{n+1} \leq (1 - \omega)\rho_n + \sigma_n$$

where $\omega \in (0, 1)$ is a fixed number and $\sigma_n \geq 0$ is such that $\sigma_n \to 0$ as $n \to \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

Lemma 1.4. *Let H be a Hilbert space. Then*

$$\left\|x + y\right\|^2 + \left\|x - y\right\|^2 = 2\|x\|^2 + 2\left\|y\right\|^2, \qquad \text{for all} \quad x, y \in H.$$

2. Main Results

Theorem 2.1. *Let X be a real Banach space, K a nonempty, closed, convex subset of X and $T : K \to K$ be a strongly pseudocontractive operator with pseudocontractive parameter $k \in (0, 1)$ and $F(T) \neq \phi$ where $F(T)$ is the set of fixed points of T. For an initial point $x_0 \in K$, let the Ishikawa type iteration be defined by (3) where $\alpha_n, \beta_n \in (0, 1)$ and constants $a, b \in (0, 1-k)$ are such that*

$$0 < a \leq \alpha_n < 1 - k - b, \qquad n \geq 1. \tag{5}$$

If

$$\left\|Tx_n - Ty_n\right\| \to 0, \quad \text{as} \quad n \to \infty, \tag{6}$$

then the sequence $\{x_n\}$ converges strongly to the unique fixed point of T in K.

Proof. From the assumption $F(T) \neq \phi$, it follows that T has a fixed point in K, say x^*.
Using (3) and applying Lemma 1.2, we get that

$$\begin{aligned}
\|x_n - x^*\|^2 &= \left\|(1-\alpha_n)(x_{n-1} - x^*) + \alpha_n(Ty_n - x^*)\right\|^2 \\
&\leq (1-\alpha_n)^2\|x_{n-1} - x^*\|^2 + 2\alpha_n\langle Ty_n - x^*, j(x_n - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_{n-1} - x^*\|^2 + 2\alpha_n\langle Ty_n - Tx_n, j(x_n - x^*)\rangle \\
&\quad + 2\alpha_n\langle Tx_n - x^*, j(x_n - x^*)\rangle.
\end{aligned} \tag{7}$$

Using (1) and the inequality $ab \leq \frac{a^2 + b^2}{2}$, we have

$$\begin{aligned}
2\alpha_n\langle Ty_n - Tx_n, j(x_n - x^*)\rangle &\leq 2\alpha_n\left\|Ty_n - Tx_n\right\|\|x_n - x^*\| \\
&\leq \left\|Ty_n - Tx_n\right\|^2 + \alpha_n{}^2\|x_n - x^*\|^2.
\end{aligned} \tag{8}$$

By strong pseudocontractivity of T, we obtain

$$2\alpha_n \langle Tx_n - x^*, j(x_n - x^*)\rangle \le 2\alpha_n k \|x_n - x^*\|^2 . \tag{9}$$

Using (8) and (9) in (7), we have

$$\|x_n - x^*\|^2 \le (1-\alpha_n)^2 \|x_{n-1} - x^*\|^2 + \left\|Ty_n - Tx_n\right\|^2 + {\alpha_n}^2 \|x_n - x^*\|^2 + 2\alpha_n k \|x_n - x^*\|^2 ,$$

which gives

$$\|x_n - x^*\|^2 \le \frac{(1-\alpha_n)^2}{1-{\alpha_n}^2 - 2\alpha_n k} \|x_{n-1} - x^*\|^2 + \frac{\left\|Ty_n - Tx_n\right\|^2}{1-{\alpha_n}^2 - 2\alpha_n k}. \tag{10}$$

From (5), it follows that

$$\begin{aligned} 1-{\alpha_n}^2 - 2\alpha_n k &> 1-(1-k-b)^2 - 2k(1-k-b) \\ &= k^2 + b(2-b) > k^2 > 0. \end{aligned} \tag{11}$$

Since $\alpha_n \in (0,1)$ and $k \in (0,1)$ we get

$$1-{\alpha_n}^2 - 2\alpha_n k < 1$$

and combining with (11), this inequality gives

$$0 < 1-{\alpha_n}^2 - 2\alpha_n k < 1. \tag{12}$$

Now, by applying (5) and (11), (10) becomes

$$\begin{aligned} \|x_n - x^*\|^2 &\le \left\{1 - \frac{2\alpha_n(1-k-\alpha_n)}{1-{\alpha_n}^2 - 2\alpha_n k}\right\} \|x_{n-1} - x^*\|^2 + \frac{\left\|Ty_n - Tx_n\right\|^2}{1-{\alpha_n}^2 - 2\alpha_n k} \\ &\le \left\{1 - \frac{2\alpha_n(1-k-(1-k-b))}{1-{\alpha_n}^2 - 2\alpha_n k}\right\} \|x_{n-1} - x^*\|^2 + \frac{1}{k^2}\left\|Ty_n - Tx_n\right\|^2 \\ &= \left\{1 - \frac{2\alpha_n b}{1-{\alpha_n}^2 - 2\alpha_n k}\right\} \|x_{n-1} - x^*\|^2 + \frac{1}{k^2}\left\|Ty_n - Tx_n\right\|^2 \\ &\le (1-2\alpha_n b) \|x_{n-1} - x^*\|^2 + \frac{1}{k^2}\left\|Ty_n - Tx_n\right\|^2 \\ &\le (1-ab) \|x_{n-1} - x^*\|^2 + \frac{1}{k^2}\left\|Ty_n - Tx_n\right\|^2 . \end{aligned} \tag{13}$$

Set $\omega = ab$, $\rho_n = \|x_{n-1} - x^*\|^2$ and $\sigma_n = \frac{1}{k^2}\left\|Ty_n - Tx_n\right\|^2$.
From (6), we have

$$\lim_{n\to\infty} \sigma_n = 0.$$

Hence, by applying Lemma 1.3, we conclude that sequence $\{x_n\}$ converges strongly to a fixed point of T in K.
Next, we prove the uniqueness of the fixed point of T. Suppose that x^{**} is another fixed point of T. Since $T : K \to K$ is strongly pseudocontractive, $I - T$ is strongly accretive and hence by (2), we have

$$0 = \langle (x^* - Tx^*) - (x^{**} - Tx^{**}), j(x^* - x^{**})\rangle \ge k \|x^* - x^{**}\|^2 ,$$

which implies that $x^* = x^{**}$.

Theorem 2.2. *Let K be a nonempty, closed, convex subset of a real Hilbert space. Let $T : K \to K$ be a continuous, strongly pseudocontractive operator with pseudocontractive parameter $k \in (0, \frac{1}{8})$. Then for an arbitrary $x_0 \in K$ and $\{v_n\}$ in K, the sequence defined as in (4) satisfying the condition $\sum_{n\geq 1} \|v_n - x_n\| < \infty$ where $\{\alpha_n\}$ is a real sequence in $(0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$, if convergent, converges to a fixed point of T.*

Proof. Suppose that $\{x_n\}$ converges to x^*. Since K is closed, $x^* \in K$. Now, we prove that x^* is a fixed point of T. Assume that x^* is not a fixed point of T. We have,

$$\|x^* - Tx^*\|^2 = \|(x^* - x_n) + (x_n - Tx^*\|^2 .$$

By applying Lemma (1.4) and using (4), we get

$$\begin{aligned} \|x^* - Tx^*\|^2 &\leq 2\|x^* - x_n\|^2 + 2\|x_n - Tx^*\|^2 \\ &= 2\|x^* - x_n\|^2 + 2\|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - Tx^*\|^2 \\ &= 2\|x^* - x_n\|^2 + 2\|\alpha_n(x_{n-1} - Tx^*) + (1-\alpha_n)(Tv_n - Tx^*)\|^2 \\ &\leq 2\|x^* - x_n\|^2 + 2\left\{2\alpha_n^2\|x_{n-1} - Tx^*\|^2 + 2(1-\alpha_n)^2\|Tv_n - Tx^*\|^2\right\} \\ &= 2\|x^* - x_n\|^2 + 4\alpha_n^2\|x_{n-1} - Tx^*\|^2 + 4(1-\alpha_n)^2\|Tv_n - Tx^*\|^2 . \end{aligned}$$

Since T is strongly pseudocontractive and again by applying Lemma (1.4), the above inequality becomes

$$\begin{aligned} \|x^* - Tx^*\|^2 &\leq 2\|x^* - x_n\|^2 + 4\alpha_n^2\|x_{n-1} - Tx^*\|^2 \\ &\quad +4(1-\alpha_n)^2\left\{\|v_n - x^*\|^2 + k\|(v_n - Tv_n) - (x^* - Tx^*)\|^2\right\} \\ &\leq 2\|x^* - x_n\|^2 + 4\alpha_n^2\|x_{n-1} - Tx^*\|^2 \\ &\quad +4(1-\alpha_n)^2\left\{\|v_n - x^*\|^2 + 2k\|v_n - Tv_n\|^2 + 2k\|x^* - Tx^*\|^2\right\} \\ &= 2\|x^* - x_n\|^2 + 4\alpha_n^2\|x_{n-1} - Tx^*\|^2 + 4(1-\alpha_n)^2\|v_n - x^*\|^2 \\ &\quad +4(1-\alpha_n)^2\left\{2k\|v_n - Tv_n\|^2 + 2k\|x^* - Tx^*\|^2\right\}. \end{aligned} \tag{14}$$

We have

$$\begin{aligned} \|v_n - x^*\|^2 &= \|(v_n - x_n) + (x_n - x^*\|^2 \\ &\leq 2\|v_n - x_n\|^2 + 2\|x_n - x^*\|^2 . \end{aligned}$$

Using the above inequality in (13), we obtain

$$\begin{aligned} \|x^* - Tx^*\|^2 &\leq 2\|x^* - x_n\|^2 + 4\alpha_n{}^2\|x_{n-1} - Tx^*\|^2 \\ &\quad +4(1-\alpha_n)^2\left\{2\|v_n - x_n\|^2 + 2\|x_n - x^*\|^2\right\} \\ &\quad +4(1-\alpha_n)^2\left\{2k\|v_n - Tv_n\|^2 + 2k\|x^* - Tx^*\|^2\right\}. \end{aligned} \tag{15}$$

Here we observe that $\{x_n\}$ is bounded. Continuity of T implies that $\{Tx_n\}$ is bounded which in turn implies that $\{\|x_{n-1} - Tx_n\|\}$ is a bounded sequence.
Now, we prove that

$$\lim_{n\to\infty} \|x_{n-1} - Tx^*\| < \infty.$$

Considering the expression

$$\|x_{n-1} - Tx^*\| \leq \|x_{n-1} - Tx_n\| + \|Tx_n - Tx^*\| ,$$

and applying limit on both sides, we have

$$\lim_{n\to\infty} \|x_{n-1} - Tx^*\| \leq \lim_{n\to\infty} \|x_{n-1} - Tx_n\| + \lim_{n\to\infty} \|Tx_n - Tx^*\| .$$

By the assumption $\{x_n\}$ converges to x^* we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and continuity of T gives that $\lim_{n\to\infty} \|Tx_n - Tx^*\| = 0$. Now using the boundedness of the sequence $\{\|x_{n-1} - Tx_n\|\}$, we obtain that

$$\lim_{n\to\infty} \|x_{n-1} - Tx_n\| < \infty,$$

which further implies that

$$\lim_{n\to\infty} \|x_{n-1} - Tx^*\| < \infty.$$

By applying Lemma (1.4), we get

$$\begin{aligned} \|v_n - Tv_n\|^2 &= \|(v_n - x_n) + (x_n - Tv_n)\|^2 \\ &\leq 2\|v_n - x_n\|^2 + 2\|x_n - Tv_n\|^2, \end{aligned}$$

and

$$\begin{aligned} \|x_n - Tv_n\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - Tv_n\|^2 \\ &= {\alpha_n}^2 \|x_{n-1} - Tv_n\|^2. \end{aligned} \tag{16}$$

Next, we prove that

$$\lim_{n\to\infty} \|x_{n-1} - Tv_n\| < \infty.$$

Considering the expression

$$\|x_{n-1} - Tv_n\| \leq \|x_{n-1} - Tx_n\| + \|Tx_n - Tv_n\|,$$

and taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|x_{n-1} - Tv_n\| \leq \lim_{n\to\infty} \|x_{n-1} - Tx_n\| + \lim_{n\to\infty} \|Tx_n - Tv_n\|.$$

The condition $\sum_{n\geq 1} \|v_n - x_n\| < \infty$ implies that $\lim_{n\to\infty} \|v_n - x_n\| = 0$ and continuity of T gives that $\lim_{n\to\infty} \|Tv_n - Tx_n\| = 0$ and hence we conclude that

$$\lim_{n\to\infty} \|x_{n-1} - Tv_n\| < \infty.$$

Now by using the above inequality and by applying the condition $\lim_{n\to\infty} \alpha_n = 0$, on (15) we get

$$\lim_{n\to\infty} \|x_n - Tv_n\|^2 = 0,$$

which implies that

$$\lim_{n\to\infty} \|v_n - Tv_n\|^2 = 0.$$

By letting $n \to \infty$ and applying the condition $\lim_{n\to\infty} \alpha_n = 0$, (14) reduces to

$$\|x^* - Tx^*\|^2 \leq 4.2k \|x^* - Tx^*\|^2$$

which is a contradiction since $k \in (0, \frac{1}{8})$ which in turn implies that $x^* = Tx^*$. Thus, we conclude that $\{x_n\}$, if convergent, converges to a fixed point of T.

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Springer, New York, 2009.

[2] S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung and S. M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudocontractive mappings in Banach spaces*, J. Math. Anal. Appl. **224** (1998), 149-165.

[3] C. E. Chidume, *Approximation of fixed points of strongly pseudocontractive mappings*, Proc. Amer. Math. Soc. **120** (1994), 545-551.

[4] C. E. Chidume and M. O. Osilike, *Ishikawa iteration process for nonlinear Lipschitz strongly accretive mappings*, J. Math. Anal. Appl. **192** (1995), 727-741.
[5] L. J. Ciric and J. S. Ume, *Ishikawa iterative process for strongly pseudocontractive operators in arbitrary Banach spaces*, Math. Commun. **8** (2003), 43-48.
[6] L. Deng, *An iterative process for nonlinear Lipschitzian and strongly accretive mappings in uniformly convex and uniformly smooth Banach spaces*, Acta Appl. Math. **32** (1993), 183-196.
[7] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.
[8] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.
[9] C. H. Morales, J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc. **120 (11)** (2000), 3411-3419.
[10] M. O. Osilke, *A note on the stability of iteration procedures for strongly pseudocontractions and strongly accretive type equations*, J. Math. Anal. Appl. **250** (2000), 726-730.
[11] A. Rafiq, *Implicit fixed point iterations*, Rostock. Math. Kolloq. **62** (2007), 21-39.
[12] S. M. Soltuz, *A correction for a result on convergence of Ishikawa iteration for strongly pseudocontractive maps*, Math. Commun. **7** (2002), 61-64.
[13] Y. G. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 91-101.
[14] H. Y. Zhou and Y. Jia, *Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption*, Proc. Amer. Math. Soc. **125** (1997), 1705-1709.
[15] H. Y. Zhou, *Stable iteration procedures for strong pseudocontractions and nonlinear equations involving accretive operators without Lipschitz assumption*, J. Math. Anal. Appl. **230** (1999), 1-10.