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FG-coupled fixed point theorems for contractive type mappings in partial cone metric spaces

Gurucharan Singh Saluja[a]

[a]*H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India*

Abstract. The concept of FG-coupled fixed point is a generalization of coupled fixed point introduced by *Guo and Lakshmikantham* [7] in 1987. In this paper, we define FG-coupled fixed point in partial cone metric spaces for the mappings $F\colon X \times Y \to X$ and $G\colon Y \times X \to Y$ and prove some FG-coupled fixed point theorems for contractive type mappings in the setting of complete partial cone metric spaces. Furthermore, we provide some consequences of the established results. An illustrative example and an application to the system of nonlinear integral equation are given. Our results extend and generalize several results in the existing literature, mainly our results extend and generalize the results of *Prajisha and Shaini* [18] and *Sabetghadam et al.* [25].

1. Introduction

Banach contraction principle (in short BCP) is one of the earlier and main results in fixed point theory. Banach contraction principle [3] was proved in complete metric spaces. Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, the concept of partial metric space was introduced by *Matthews* [14, 15]. In such space, the distance of a point in the self may not be zero. A motivation behind introducing the concept of partial metric space was to give a modified version of a Banach contraction mapping principle [3], more appropriate to solve certain problems arising in computer science [15].

As an extension of fixed point, a new concept called coupled fixed point is introduced by *Guo and Lakshmikantham* [7]. They investigated some coupled fixed point theorems of mixed monotone operators, and they applied their results to solve initial value problem of ordinary differential equations with discontinuous right hand sides. Later, *Bhaskar and Lakshmikantham* [4] established existence and uniqueness theorems of coupled fixed point for mixed monotone mappings defined on partially ordered complete metric spaces satisfying contractive type condition and applied their result to solve periodic boundary value problems. After this work of [4], *Ćirić and Lakshmikantham* [5] in 2009, introduced a new mapping called mixed g-monotone mapping. Using this idea, they established coupled coincidence and common

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Communicated by Dragan S. Djordjević
Email address: saluja1963@gmail.com (Gurucharan Singh Saluja)
ORCID iD: https://orcid.org/ 0009-0006-1595-8133 (Gurucharan Singh Saluja)

coupled fixed point theorems which generalized the results of [4]. In [21], Radenovic proved coupled fixed point theorems for monotone mappings in partially ordered metric spaces.

In 2007, *Huang and Zhang* [8] introduced cone metric spaces as generalization of metric spaces and they gave a generalized fixed point theorem for contractive type mappings on a cone metric space provided that the given cone is normal. In 2008, *Rezapour and Hamlbarani* [24] improved the result of [8] by omitting the normality of the given cone induced the partial order relation. Following them several authors have proved various fixed point theorems in cone metric spaces (see, e.g., [1, 2, 9, 12, 22, 23]). Later in 2009, *Sabetghadam et al.* [25] introduced the notion of coupled fixed point in cone metric spaces and established several coupled fixed point theorems for different contractive type mappings. In [16], Olatinwo proved coupled fixed point theorems by considering two different cone metrics on the same ambient space. Partial cone metric space and fixed point theorem have investigated by *Mahlotra et al.* [13] (see, also [6], [26]).

Recently, *Prajisha and Shaini* [19] introduced the notion of *FG*-coupled fixed point (which is a generalization of coupled fixed point) and they proved some *FG*-coupled fixed point theorems for various types of contractive mappings (see, also [10, 11]). Later on, many researchers have established *FG*-coupled fixed point results in generalized metric spaces (see, e.g. [17], [20]).

Very recently, *Prajisha and Shaini* [18] (Carpathian Math. Publ., 2017, 9(2), 163-170) proved some coupled fixed point theorems for various contractive type mappings in the framework of cone metric spaces which generalize several results in the literature. They also provided an example in support of the main result.

In this paper, we establish some *FG*-coupled fixed point theorems for contractive type mappings in the setting of complete partial cone metric spaces. Our results extend and generalize several results in the existing literature (see, e.g., [18], [25] and many others).

2. Preliminaries

In this section, the following definitions and results will be needed in the sequel.

Definition 2.1. *([15]) (Partial Metric Space) A partial metric on a non-void set X is a function $p\colon X\times X \to [0,+\infty)$ such that for all $x, y, z \in X$ the followings hold:*

(**P1**) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(**P2**) $p(x,x) \leq p(x,y)$,

(**P3**) $p(x,y) = p(y,x)$,

(**P4**) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then the pair (X, p) is called a partial metric space (in short PMS).

It is clear that if $p(x, y) = 0$, then (**P1**) and (**P2**) imply that $x = y$. But if $x = y$, then $p(x, y)$ may not be 0 (see [15]). A basic example of a partial metric space is the $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$ and $\mathbb{R}^+ = [0,+\infty)$.

Definition 2.2. *(see [8]) (Cone) Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if the following conditions hold:*

(**C1**) *P is closed, nonempty and $P \neq \{0\}$;*

(**C2**) *$a\,x + b\,y \in P$ for all $x, y \in P$ where a, b are nonnegative real numbers;*

(**C3**) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ in E with respect to P by $x \leq y \Leftrightarrow y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int\,P$, in which $int\,P$ denotes the interior of P. The cone P is called normal if there is a constant number $M > 0$ such that for all $x, y \in P$, $0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$. The least positive number M satisfying the inequality $\|x\| \leq M\|y\|$ is called the normal constant of the cone P.

Remark 2.3. *(see [27]) If $int\,(P) \neq \emptyset$, then P is called a solid cone.*

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with *int* $P \neq \emptyset$ and $\leq$ is partial ordering in E with respect to P.

Example 2.4. *(see [12]) Let $M > 1$ be given. Consider the real vector space*

$$E = \Big\{a\,x + b : a, b \in \mathbb{R};\, x \in \Big[1 - \frac{1}{M}, 1\Big]\Big\}$$

with supremum norm and the cone

$$P = \Big\{a\,x + b \in E : a \geq 0, b \geq 0\Big\}$$

in E. The cone P is regular and so normal.

Definition 2.5. *(see [8, 28]) (Cone Metric Space) Let $X \neq \emptyset$ be a set. The mapping $d\colon X \times X \to E$ is said to be a cone metric on X if for all $x, y, z \in X$ the followings hold:*

(**CM1**) $0 \leq d(x, y)$ *and* $d(x, y) = 0 \Leftrightarrow x = y$;

(**CM2**) $d(x, y) = d(y, x)$;

(**CM3**) $d(x, y) \leq d(x, z) + d(z, y)$.

Then d is called a cone metric [8] on X and the pair (X, d) is called a cone metric space [8] or simply CMS.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Mahlotra et al. [13] and *Sönmez* [26] introduced the concept of partial cone metric space and its topological properties. Now, we give the definition of partial cone metric space as follows.

Definition 2.6. *(see [13]) (Partial Cone Metric Space) A partial cone metric on a non-void set X is a function $p\colon X \times X \to E$ such that for all $x, y, z \in X$ satisfying the followings:*

(**PCM1**) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(**PCM2**) $0 \leq p(x, x) \leq p(x, y)$;

(**PCM3**) $p(x, y) = p(y, x)$;

(**PCM4**) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial cone metric space (in short PCMS) is a pair (X, p) such that X is a non-void set and p is a partial cone metric on X.

It is clear that, if $p(x, y) = 0$, then (**PCM1**) and (**PCM2**) imply that $x = y$. But the converse is not true in general. A cone metric space is a partial cone metric space, but there exists partial cone metric spaces which are not cone metric space, we give the following example from [26].

Example 2.7. *(see [26]) Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}^+$ and $p\colon X \times X \to E$ defined by $p(x, y) = \big(\max\{x, y\}, \beta \max\{x, y\}\big)$, where $\beta \geq 0$ is a constant. Then (X, p) is a partial cone metric space which is not a cone metric space.*

Remark 2.8. *Suppose (X, p) is a partial cone metric space, then*

$$d(x, y) = 2p(x, y) - p(x, x) - p(y, y) \text{ for all } x, y \in X,$$

defines a cone metric on X.

Example 2.9. *(see [26]) Let $E = \ell_1$, $P = \{\{x_n\} \in \ell_1 : x_n \geq 0\}$. Also, let $X = \{\{x_n\} : x_n \in [0, \frac{\pi}{4}], \sum_n x_n < \infty\}$ and $p\colon X \times X \to E$ defined by $p(x, y) = \big(x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n, \ldots\big)$, where the symbol $\vee$ denotes the maximum, i.e. $x \vee y = \max\{x, y\}$. Then (X, p) is a partial cone metric space which is not a cone metric space.*

Example 2.10. *(see [2]) Let $\mathbb{R}$ be the real Banach space of real numbers with the usual absolute value metric, $P = \mathbb{R}^+$ be the cone of non-negative real numbers, $X = \{[a,b] : a,b \in \mathbb{R}, a \le b\}$, and define $p\colon X \times X \to \mathbb{R}^+$ by*

$$p(x,y) = p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}, \ \forall\, x = [a,b], \ y = [c,d] \in X.$$

Then (X,p) is a partial cone metric space.

Following, we give some properties of partial cone metric spaces (for more details see [26]).

Theorem 2.11. *([26])) Every partial cone metric space (X,p) is a topological space.*

Theorem 2.12. *([26])) Let (X,p) be a partial cone metric space and P be a normal cone with normal constant M, then (X,p) is T_0.*

Theorem 2.13. *([26])) Let (X,p) be a partial cone metric space. If $\{x_n\}$ is a Cauchy sequence in (X,p), then it is a Cauchy sequence in the cone metric space (X,d).*

Definition 2.14. *([26]) Let (X,p) be a partial cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.*

(**1**) *$\{x_n\}$ is said to be convergent to x and x is called a limit of $\{x_n\}$ if*

$$\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = p(x,x).$$

(**2**) *$\{x_n\}$ is called a Cauchy sequence if there is $x \in P$ such that for every $\varepsilon > 0$ there exists a natural number N such that for all $n,m > N$, $\|p(x_n,x_m) - x\| < \varepsilon$.*

(**3**) *(X,p) is said to be complete if every Cauchy sequence in (X,p) is convergent in (X,p).*

Definition 2.15. *([4, 5]) An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping $F\colon X\times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.*

Definition 2.16. *([19]) Let $F\colon X \times Y \to X$ and $G\colon Y \times X \to Y$ be two mappings, then for $n \ge 1$, $F^n(x,y) = F(F^{n-1}(x,y), G^{n-1}(y,x))$ and $G^n(y,x) = G(G^{n-1}(y,x), F^{n-1}(x,y))$ where $F^0(x,y) = x$ and $G^0(y,x) = y$ for all $x \in X$ and $y \in Y$.*

In the next section, we prove existence and uniqueness theorems of FG-coupled fixed point for different contractive type mappings in the framework of partial cone metric spaces. We consider $p_X\colon X\times X \to E$ and $p_Y\colon Y \times Y \to E$ where E is a real Banach space equipped with the partial ordering $\le$ with respect to the cone $P \subseteq E$.

3. Main Results

Here, we shall investigate FG-coupled fixed point theorems for contractive type mappings in the setting of partial cone metric spaces. We define FG-coupled fixed point in partial cone metric space as follows.

Definition 3.1. *Let (X,p_X) and (Y,p_Y) be two partial cone metric spaces and $F\colon X\times Y \to X$ and $G\colon Y \times X \to Y$ are two mappings. An element $(x,y) \in X \times Y$ is said to be an FG-coupled fixed point if $F(x,y) = x$ and $G(y,x) = y$.*

Remark 3.2. *([19])*

If $(x,y) \in X \times Y$ is an FG-coupled fixed point then $(y,x) \in Y \times X$ is a GF-coupled fixed point.

Theorem 3.3. *Let (X,p_X) and (Y,p_Y) be two complete partial cone metric spaces, P be a normal cone with constant M. Suppose that the mappings $F\colon X \times Y \to X$ and $G\colon Y\times X \to Y$ satisfy the following conditions for all $x,u \in X$, $y,v \in Y$:*

$$p_X(F(x,y),F(u,v)) \le k\,p_X(x,u) + l\,p_Y(y,v), \tag{1}$$

$$p_Y(G(y,x),G(v,u)) \le k\,p_Y(y,v) + l\,p_X(x,u), \tag{2}$$

where k,l are nonnegative constants such that $k+l < 1$. Then there exists a unique FG-coupled fixed point.

Proof. Let $(x_0, y_0) \in X \times Y$ be arbitrary point. Construct sequences $\{x_n\}$ and $\{y_n\}$ by defining $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$ and $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \geq 0$. Now, from equations (1) and (2), we have

$$\begin{aligned} p_X(x_{n+1}, x_n) &= p_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq k\, p_X(x_n, x_{n-1}) + l\, p_Y(y_n, y_{n-1}), \end{aligned} \tag{3}$$

and likewise we obtain

$$\begin{aligned} p_Y(y_{n+1}, y_n) &= p_Y(G(y_n, x_n), G(y_{n-1}, x_{n-1})) \\ &\leq k\, p_Y(y_n, y_{n-1}) + l\, p_X(x_n, x_{n-1}). \end{aligned} \tag{4}$$

Adding equations (3) and (4), we obtain

$$p_X(x_{n+1}, x_n) + p_Y(y_{n+1}, y_n) \leq (k + l)\,[p_X(x_n, x_{n-1}) + p_Y(y_n, y_{n-1})],$$

or,

$$A_n \leq (k + l)\, A_{n-1}, \tag{5}$$

where $A_n = p_X(x_{n+1}, x_n) + p_Y(y_{n+1}, y_n)$.

Continuing the same process as above, we obtain

$$A_n \leq \delta\, A_{n-1} \leq \delta^2\, A_{n-2} \leq \cdots \leq \delta^n\, A_0, \tag{6}$$

where $\delta = k + l < 1$. If $A_0 = 0$ then (x_0, y_0) is a FG-coupled fixed point. So, we assume that $A_0 > 0$. For $m > n$, we have

$$\begin{aligned} p_X(x_m, x_n) &\leq p_X(x_m, x_{m-1}) + p_X(x_{m-1}, x_{m-2}) + \cdots + p_X(x_{n+1}, x_n) \\ &\quad - \sum_{i=1}^{m-n-1} p_X(x_{n-i}, x_{n-i}) \\ &\leq p_X(x_m, x_{m-1}) + p_X(x_{m-1}, x_{m-2}) + \cdots + p_X(x_{n+1}, x_n), \end{aligned}$$

and likewise

$$p_Y(y_m, y_n) \leq p_Y(y_m, y_{m-1}) + p_Y(y_{m-1}, y_{m-2}) + \cdots + p_Y(y_{n+1}, y_n),$$

that is,

$$\begin{aligned} p_X(x_m, x_n) + p_Y(y_m, y_n) &\leq A_{m-1} + A_{m-2} + \cdots + A_n \\ &\leq \delta^{m-1}\, A_0 + \delta^{m-2}\, A_0 + \cdots + \delta^n\, A_0 \\ &\leq \delta^n \Big(\frac{1 - \delta^{m-n}}{1 - \delta}\Big) A_0 \\ &\leq \Big(\frac{\delta^n}{1 - \delta}\Big) A_0. \end{aligned}$$

Hence

$$\begin{aligned} \|p_X(x_m, x_n) + p_Y(y_m, y_n)\| &\leq M\Big(\frac{\delta^n}{1 - \delta}\Big)\|A_0\| \\ &\to 0 \text{ as } m, n \to \infty, \text{ since } \delta < 1. \end{aligned}$$

This implies that $\lim_{m,n\to\infty} p_X(x_m, x_n) + p_Y(y_m, y_n) = 0$ and hence

$$\lim_{m,n\to\infty} p_X(x_m, x_n) = 0 = \lim_{m,n\to\infty} p_Y(y_m, y_n).$$

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences such that $\lim_{m,n\to\infty} p_X(x_m, x_n) = 0$ and $\lim_{m,n\to\infty} p_Y(y_m, y_n) = 0$. Since (X, p_X) and (Y, p_Y) are complete partial cone metric spaces, so there exist $x \in X$ and $y \in Y$ such that

$$p_X(x, x) = \lim_{n\to\infty} p_X(x_n, x) = \lim_{n\to\infty} p_X(x_n, x_n) = 0,$$

and

$$p_Y(y, y) = \lim_{n\to\infty} p_Y(y_n, y) = \lim_{n\to\infty} p_Y(y_n, y_n) = 0.$$

From equation (1), we have

$$\begin{aligned} p_X(F(x, y), x) &\leq p_X(F(x, y), x_{n+1}) + p_X(x_{n+1}, x) - p_X(x_{n+1}, x_{n+1}) \\ &\leq p_X(F(x, y), x_{n+1}) + p_X(x_{n+1}, x) \\ &= p_X(F(x, y), F(x_n, y_n) + p_X(x_{n+1}, x) \\ &\leq k\, p_X(x, x_n) + l\, p_Y(y, y_n) + p_X(x_{n+1}, x). \end{aligned}$$

Hence

$$\begin{aligned} \|p_X(F(x, y), x)\| &\leq M\Big[k\,\|p_X(x, x_n)\| + l\,\|p_Y(y, y_n)\| + \|p_X(x_{n+1}, x)\|\Big] \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that $p_X(F(x, y), x) = 0$ and hence $F(x, y) = x$. Similarly, we can show that $G(y, x) = y$.

Now we prove the uniqueness of FG-coupled fixed point. Suppose that $(x', y') \in X \times Y$ is another FG-coupled fixed point such that $F(x', y') = x'$ and $G(y', x') = y'$ with $(x, y) \neq (x', y')$. Then, we have

$$p_X(x, x') = p_X(F(x, y), F(x', y')) \leq k\, p_X(x, x') + l\, p_Y(y, y'),$$

and

$$p_Y(y, y') = p_Y(G(y, x), G(y', x')) \leq k\, p_Y(y, y') + l\, p_X(x, x'),$$

that is,

$$\begin{aligned} p_X(x, x') + p_Y(y, y') &\leq (k + l)\,[p_X(x, x') + p_Y(y, y')], \\ &< p_X(x, x') + p_Y(y, y'), \text{ since } k + l < 1, \end{aligned}$$

which is a contradiction. Hence $p_X(x, x') + p_Y(y, y') = 0$ and so $x = x'$ and $y = y'$. Thus the FG-coupled fixed point is unique. This completes the proof. □

Example 3.4. *Let $X = [0, \infty)$ and $Y = (-\infty, 0]$. $E = C^1_{\mathbb{R}}$ and $P = \{\varphi \in E : \varphi \geq 0\}$. Define partial cone metric $p\colon X \times X \to E$ by $p(x, y) = \big(|x - y|, \beta|x - y|\big)e^t$, where $\beta \geq 0$ is a constant and $e^t \in E$ on $P = \{\varphi \in E : \varphi \geq 0\}$. Consider the mappings $F\colon X \times Y \to X$ and $G\colon Y \times X \to Y$ defined as $F(x, y) = \frac{x-4y}{6}$ and $G(y, x) = \frac{y-4x}{6}$. Clearly F and G satisfy all the conditions given in Theorem 3.3, and it is easy to see that $(0, 0)$ is a unique FG-coupled fixed point.*

Remark 3.5. *Theorem 3.3 extends Theorem 1 of [18] from complete cone metric space to complete partial cone metric space.*

Remark 3.6. *Theorem 3.3 also extends and generalizes Theorem 2.2 of [25] from complete cone metric space to complete partial cone metric space.*

Corollary 3.7. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$p(F(x, y), F(u, v)) \leq k\, p(x, u) + l\, p(y, v),$$

where k, l are nonnegative constants such that $k + l < 1$. Then F has a unique coupled fixed point.

Remark 3.8. *Corollary 3.7 extends Theorem 2.2 of [25] from complete cone metric space to complete partial cone metric space.*

Corollary 3.9. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$p(F(x, y), F(u, v)) \le \frac{k}{2}[p(x, u) + p(y, v)],$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Remark 3.10. *Corollary 3.9 extends Corollary 2.3 of [25] from complete cone metric space to complete partial cone metric space.*

If we define as $Tx = F(x, x)$ and $k + l = h$ where $h \in [0, 1)$ in Corollary 3.7, then we have the following result.

Corollary 3.11. *[26] (Banach's fixed point theorem) Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $T\colon X \to X$ satisfies the following condition for all $x, u \in X$:*

$$p(Tx, Tu) \le h\, p(x, u),$$

where $h \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Example 3.12. *Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \ge 0\}$, $X = [0, \frac{\pi}{4}]$ and $p\colon X \times X \to E$ defined by $p(x, y) = \big(\max\{x, y\}, k \max\{x, y\}\big)$ where $k > 0$ is a constant. Define $T\colon X \to X$ as $T(x) = \frac{x}{2}$ for all $x \in X$. Let $x \le y$. Then*

$$p(x, y) = (y, ky),\ p(Tx, Ty) = p\Big(\frac{x}{2}, \frac{y}{2}\Big) = \Big(\frac{y}{2}, k\frac{y}{2}\Big).$$

Now, we have

$$p(Tx, Ty) \ = \ \Big(\frac{y}{2}, k\frac{y}{2}\Big) = \frac{1}{2}(y, ky) \le h\,(y, ky),$$

or $h \ge \frac{1}{2}$. If we take $0 \le h < 1$, then all the conditions of Corollary 3.11 are satisfied. Hence by application of Corollary 3.11, T has a unique fixed point. Here 0 is the unique fixed point of T.

Theorem 3.13. *Let (X, p_X) and (Y, p_Y) be two complete partial cone metric spaces, P be a normal cone with constant M. Suppose that the mappings $F\colon X \times Y \to X$ and $G\colon Y \times X \to Y$ satisfy the following conditions for all $x, u \in X$, $y, v \in Y$:*

$$p_X(F(x, y), F(u, v)) \le k\, p_X(x, F(x, y)) + l\, p_X(u, F(u, v)), \tag{7}$$

$$p_Y(G(y, x), G(v, u)) \le k\, p_Y(y, G(y, x)) + l\, p_Y(v, G(v, u)), \tag{8}$$

where k, l are nonnegative constants such that $k + l < 1$. Then there exists a unique FG-coupled fixed point.

Proof. As in the proof of Theorem 3.3 construct sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$, $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \ge 0$. Then, from equations (7) and (8), we have

$$\begin{aligned} p_X(x_{n+1}, x_n) &= p_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\le k\, p_X(x_n, F(x_n, y_n)) + l\, p_X(x_{n-1}, F(x_{n-1}, y_{n-1})) \\ &= k\, p_X(x_n, x_{n+1}) + l\, p_X(x_{n-1}, x_n). \end{aligned}$$

Therefore,

$$p_X(x_{n+1}, x_n) \le \Big(\frac{l}{1-k}\Big) p_X(x_{n-1}, x_n). \tag{9}$$

Similarly, we obtain

$$p_Y(y_{n+1}, y_n) \le \Big(\frac{l}{1-k}\Big) p_Y(y_{n-1}, y_n). \tag{10}$$

Repeating the above process, we get

$$p_X(x_{n+1}, x_n) \le \theta^n\, p_X(x_0, x_1) \text{ and } p_Y(y_{n+1}, y_n) \le \theta^n\, p_Y(y_0, y_1), \tag{11}$$

where $\theta = \Big(\frac{l}{1-k}\Big) < 1$, since $k + l < 1$.

If $x_1 = x_0$ and $y_1 = y_0$, then the result follows. Otherwise for $m > n$ consider,

$$\begin{aligned} p_X(x_m, x_n) &\le p_X(x_m, x_{m-1}) + p_X(x_{m-1}, x_{m-2}) + \cdots + p_X(x_{n+1}, x_n) \\ &\quad - \sum_{i=1}^{m-n-1} p_X(x_{n-i}, x_{n-i}) \\ &\le p_X(x_m, x_{m-1}) + p_X(x_{m-1}, x_{m-2}) + \cdots + p_X(x_{n+1}, x_n) \\ &\le \theta^{m-1} p_X(x_1, x_0) + \theta^{m-2} p_X(x_1, x_0) + \cdots + \theta^n p_X(x_1, x_0) \\ &\le \theta^n \Big(\frac{1-\theta^{m-n}}{1-\theta}\Big) p_X(x_1, x_0) \\ &\le \Big(\frac{\theta^n}{1-\theta}\Big) p_X(x_1, x_0). \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Similarly, one can prove that $\{y_n\}$ is a Cauchy sequence in Y.

Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences such that $\lim_{m,n\to\infty} p_X(x_m, x_n) = 0$ and $\lim_{m,n\to\infty} p_Y(y_m, y_n) = 0$. Now, by the completeness of the spaces X and Y, there exist $(x, y) \in X \times Y$ such that

$$p_X(x, x) = \lim_{n\to\infty} p_X(x_n, x) = \lim_{n\to\infty} p_X(x_n, x_n) = 0,$$

and

$$p_Y(y, y) = \lim_{n\to\infty} p_Y(y_n, y) = \lim_{n\to\infty} p_Y(y_n, y_n) = 0.$$

Therefore, from equation (7), we have

$$\begin{aligned} p_X(F(x, y), x) &\le p_X(F(x, y), x_{n+1}) + p_X(x_{n+1}, x) - p_X(x_{n+1}, x_{n+1}) \\ &\le p_X(F(x, y), x_{n+1}) + p_X(x_{n+1}, x) \\ &= p_X(F(x, y), F(x_n, y_n)) + p_X(x_{n+1}, x) \\ &\le k\, p_X(x, F(x, y)) + l\, p_X(x_n, F(x_n, y_n)) + p_X(x_{n+1}, x) \\ &= k\, p_X(x, F(x, y)) + l\, p_X(x_n, x_{n+1}) + p_X(x_{n+1}, x), \end{aligned}$$

that is,

$$p_X(F(x, y), x) \le \Big(\frac{l}{1-k}\Big) p_X(x_n, x_{n+1}) + \Big(\frac{1}{1-k}\Big) p_X(x_{n+1}, x).$$

Hence

$$\begin{aligned} \|p_X(F(x, y), x)\| &\le M\Big[\Big(\frac{l}{1-k}\Big)\|p_X(x_n, x_{n+1})\| + \Big(\frac{1}{1-k}\Big)\|p_X(x_{n+1}, x)\|\Big] \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that $p_X(F(x, y), x) = 0$ and hence $F(x, y) = x$. Similarly, we can show that $G(y, x) = y$. If (x', y') is another FG-coupled fixed point such that $(x, y) \ne (x', y')$, then we have

$$\begin{aligned} p_X(x, x') &= p_X(F(x, y), F(x', y')) \le k\, p_X(x, F(x, y)) + l\, p_X(x', F(x', y')) \\ &= k\, p_X(x, x) + l\, p_X(x', x') = 0. \end{aligned}$$

Thus, $x = x'$. Similarly, we show that $y = y'$. This shows the uniqueness of FG-coupled fixed point. The proof is completed. □

Remark 3.14. *Theorem 3.13 extends Theorem 2 of [18] from complete cone metric space to complete partial cone metric space.*

Remark 3.15. *Theorem 3.13 also extends and generalizes Theorem 2.5 of [25] from complete cone metric space to complete partial cone metric space.*

Corollary 3.16. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$p(F(x,y), F(u,v)) \le k\, p(x, F(x,y)) + l\, p(u, F(u,v)),$$

where k, l are nonnegative constants such that $k + l < 1$. Then F has a unique coupled fixed point.

Remark 3.17. *Corollary 3.16 extends Theorem 2.5 of [25] from complete cone metric space to complete partial cone metric space.*

Corollary 3.18. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$p(F(x,y), F(u,v)) \le \frac{k}{2}[p(x, F(x,y)) + p(u, F(u,v))],$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Remark 3.19. *Corollary 3.18 extends Corollary 2.7 of [25] from complete cone metric space to complete partial cone metric space.*

Theorem 3.20. *Let (X, p_X) and (Y, p_Y) be two complete partial cone metric spaces, P be a normal cone with constant M. Suppose that the mappings $F\colon X \times Y \to X$ and $G\colon Y \times X \to Y$ satisfy the following conditions for all $x, u \in X$, $y, v \in Y$:*

$$p_X(F(x,y), F(u,v)) \le k\, p_X(x, F(u,v)) + l\, p_X(u, F(x,y)), \tag{12}$$

$$p_Y(G(y,x), G(v,u)) \le k\, p_Y(y, G(v,u)) + l\, p_Y(v, G(y,x)), \tag{13}$$

where k, l are nonnegative constants such that $k + 2l < 1$. Then there exists a unique FG-coupled fixed point.

Proof. As in the proof of above theorems construct sequences $\{x_n\}$ and $\{y_n\}$ by defining $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$ and $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \ge 0$. Now using equations (12), (13) and conditions (**PCM2**)-(**PCM4**), we have

$$\begin{aligned} p_X(x_{n+1}, x_n) &= p_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\le k\, p_X(x_n, F(x_{n-1}, y_{n-1})) + l\, p_X(x_{n-1}, F(x_n, y_n)) \\ &= k\, p_X(x_n, x_n) + l\, p_X(x_{n-1}, x_{n+1}) \\ &\le k\, p_X(x_n, x_{n+1}) + l\, [p_X(x_{n-1}, x_n) + p_X(x_n, x_{n+1}) \\ &\quad - p_X(x_n, x_n)] \\ &\le (k + l)\, p_X(x_n, x_{n+1}) + l\, p_X(x_n, x_{n-1}). \end{aligned}$$

Therefore,

$$p_X(x_{n+1}, x_n) \le \Big(\frac{l}{1 - k - l}\Big) p_X(x_n, x_{n-1}). \tag{14}$$

Similarly, we obtain

$$p_Y(y_{n+1}, y_n) \le \Big(\frac{l}{1-k-l}\Big) p_Y(y_n, y_{n-1}). \tag{15}$$

Repeating the above process, we get

$$p_X(x_{n+1}, x_n) \le \sigma^n\, p_X(x_1, x_0) \text{ and } p_Y(y_{n+1}, y_n) \le \sigma^n\, p_Y(y_1, y_0), \tag{16}$$

where $\sigma = \Big(\frac{l}{1-k-1}\Big) < 1$, since $k + 2l < 1$.

Along the same lines of Theorem 3.13 see that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Since $\lim_{m,n\to\infty} p_X(x_m, x_n) = 0$, $\lim_{m,n\to\infty} p_Y(y_m, y_n) = 0$ and (X, p_X), (Y, p_Y) are complete, there exist $(x, y) \in X \times Y$ such that

$$p_X(x, x) = \lim_{n\to\infty} p_X(x_n, x) = \lim_{n\to\infty} p_X(x_n, x_n) = 0,$$

and

$$p_Y(y, y) = \lim_{n\to\infty} p_Y(y_n, y) = \lim_{n\to\infty} p_Y(y_n, y_n) = 0.$$

Now, from equation (13), we have

$$\begin{aligned} p_X(F(x,y), x) &\le p_X(F(x,y), x_{n+1}) + p_X(x_{n+1}, x) - p_X(x_{n+1}, x_{n+1}) \\ &\le p_X(F(x,y), x_{n+1}) + p_X(x_{n+1}, x) \\ &= p_X(F(x,y), F(x_n, y_n)) + p_X(x_{n+1}, x) \\ &\le k\, p_X(x, F(x_n, y_n)) + l\, p_X(x_n, F(x,y)) + p_X(x_{n+1}, x) \\ &= k\, p_X(x, x_{n+1}) + l\, p_X(x_n, F(x,y)) + p_X(x_{n+1}, x). \end{aligned}$$

Hence

$$\begin{aligned} \|p_X(F(x,y), x)\| &\le M\Big[k\, \|p_X(x, x_{n+1})\| + l\, \|p_X(x_n, F(x,y))\| + \|p_X(x_{n+1}, x)\|\Big] \\ &\le M\, l\, \|p_X(F(x,y), x)\| \text{ as } n \to \infty, \end{aligned}$$

which is a contradiction since $M > 0$ and $l < 1$. Hence $p_X(F(x, y), x) = 0$ and so $F(x, y) = x$. Similarly, we can show that $G(y, x) = y$. Now, we have to show that uniqueness of FG-coupled fixed point. Assume that $(x, y) \ne (x', y')$ is another FG-coupled fixed point, then we have

$$\begin{aligned} p_X(x, x') &= p_X(F(x,y), F(x',y')) \le k\, p_X(x, F(x',y')) + l\, p_X(x', F(x,y)) \\ &= k\, p_X(x, x') + l\, p_X(x', x) = (k + l) p_X(x, x') \\ &< p_X(x, x'), \end{aligned}$$

which is a contradiction, since $k + l < 1$. Thus, $x = x'$. Similarly, we show that $y = y'$. This proves the uniqueness of FG-coupled fixed point. The proof of Theorem 3.20 is completed. □

Remark 3.21. *Theorem 3.20 extends Theorem 3 of [18] from complete cone metric space to complete partial cone metric space.*

Remark 3.22. *Theorem 3.20 also extends and generalizes Theorem 2.6 of [25] from complete cone metric space to complete partial cone metric space.*

Corollary 3.23. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$p(F(x,y), F(u,v)) \le k\, p(x, F(u,v)) + l\, p(u, F(x,y)),$$

where k, l are nonnegative constants such that $k + 2l < 1$. Then F has a unique coupled fixed point.

Remark 3.24. *Corollary 3.23 extends Theorem 2.6 of [25] from complete cone metric space to complete partial cone metric space.*

Corollary 3.25. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$p(F(x,y), F(u,v)) \leq \frac{k}{2}[p(x, F(u,v)) + p(u, F(x,y))],$$

where $k \in [0,1)$ is a constant. Then F has a unique coupled fixed point.

Remark 3.26. *Corollary 3.25 extends Corollary 2.8 of [25] from complete cone metric space to complete partial cone metric space.*

Remark 3.27. *If $X = Y$ and $F = G$ in Theorems 3.3, 3.13 and 3.20, then we get Corollaries 3.7, 3.16 and 3.23. In addition to this, if $k = l$ in Theorems 3.3, 3.13 and 3.20, then we get Corollaries 3.9, 3.18 and 3.25.*

Theorem 3.28. *Let (X, p_X) and (Y, p_Y) be two complete partial cone metric spaces, P be a normal cone with constant M. Suppose that the mappings $F\colon X \times Y \to X$ and $G\colon Y \times X \to Y$ satisfy the following conditions for all $x, u \in X$, $y, v \in Y$:*

$$\begin{aligned} p_X(F(x,y), F(u,v)) \leq\ & k_1\, p_X(x,u) + k_2\, p_X(x, F(x,y)) + k_3\, p_X(u, F(u,v)) \\ & + k_4\, p_X(x, F(u,v)) + k_5\, p_X(u, F(x,y)) \end{aligned} \tag{17}$$

$$\begin{aligned} p_Y(G(y,x), G(v,u)) \leq\ & k_1\, p_Y(y,v) + k_2\, p_Y(y, G(y,x)) + k_3\, p_Y(v, G(v,u)) \\ & + k_4\, p_Y(y, G(v,u)) + k_5\, p_Y(v, G(y,x)), \end{aligned} \tag{18}$$

where k_1, k_2, k_3, k_4, k_5 are nonnegative constants such that $k_1 + k_2 + k_3 + k_4 + 2k_5 < 1$. Then there exists a unique FG-coupled fixed point.

Proof. As in the proof of above theorems construct sequences $\{x_n\}$ and $\{y_n\}$ by defining $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$ and $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \geq 0$. Now using equations (17), (18) and condition (**PCM2**)-(**PCM4**), we have

$$\begin{aligned} p_X(x_{n+1}, x_n) &= p_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq k_1\, p_X(x_n, x_{n-1}) + k_2\, p_X(x_n, F(x_n, y_n)) + k_3\, p_X(x_{n-1}, F(x_{n-1}, y_{n-1})) \\ &\quad + k_4\, p_X(x_n, F(x_{n-1}, y_{n-1})) + k_5\, p_X(x_{n-1}, F(x_n, y_n)) \\ &\leq k_1\, p_X(x_n, x_{n-1}) + k_2\, p_X(x_n, x_{n+1}) + k_3\, p_X(x_{n-1}, x_n) \\ &\quad + k_4\, p_X(x_n, x_n) + k_5\, p_X(x_{n-1}, x_{n+1}) \\ &\leq k_1\, p_X(x_n, x_{n-1}) + k_2\, p_X(x_{n+1}, x_n) + k_3\, p_X(x_{n-1}, x_n) \\ &\quad + k_4\, p_X(x_{n+1}, x_n) + k_5\, [p_X(x_{n-1}, x_n) + p_X(x_n, x_{n+1}) - p_X(x_n, x_n)] \\ &\leq k_1\, p_X(x_n, x_{n-1}) + k_2\, p_X(x_{n+1}, x_n) + k_3\, p_X(x_{n-1}, x_n) \\ &\quad + k_4\, p_X(x_{n+1}, x_n) + k_5\, [p_X(x_{n-1}, x_n) + p_X(x_n, x_{n+1})] \\ &= (k_1 + k_3 + k_5) p_X(x_n, x_{n-1}) + (k_2 + k_4 + k_5) p_X(x_{n+1}, x_n). \end{aligned}$$

Therefore,

$$\begin{aligned} p_X(x_{n+1}, x_n) &\leq \Big(\frac{k_1 + k_3 + k_5}{1 - k_2 - k_4 - k_5}\Big) p_X(x_n, x_{n-1}) \\ &= \zeta\, p_X(x_n, x_{n-1}) \end{aligned} \tag{19}$$

Similarly, we obtain

$$p_Y(y_{n+1}, y_n) \leq \zeta\, p_Y(y_n, y_{n-1}), \tag{20}$$

where $\zeta = \left(\frac{k_1+k_3+k_5}{1-k_2-k_4-k_5}\right) < 1$, since by assumption $k_1 + k_2 + k_3 + k_4 + 2k_5 < 1$. Continuing the same process as above, we obtain

$$p_X(x_{n+1}, x_n) \leq \zeta^n \, p_X(x_1, x_0) \text{ and } p_Y(y_{n+1}, y_n) \leq \zeta^n \, p_Y(y_1, y_0). \tag{21}$$

In the same lines of Theorem 3.13 see that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Since $\lim_{m,n\to\infty} p_X(x_m, x_n) = 0$, $\lim_{m,n\to\infty} p_Y(y_m, y_n) = 0$ and (X, p_X), (Y, p_Y) are complete PCM spaces, there exist $(x, y) \in X \times Y$ such that

$$p_X(x, x) = \lim_{n\to\infty} p_X(x_n, x) = \lim_{n\to\infty} p_X(x_n, x_n) = 0,$$

and

$$p_Y(y, y) = \lim_{n\to\infty} p_Y(y_n, y) = \lim_{n\to\infty} p_Y(y_n, y_n) = 0.$$

Now, from equation (17), we have

$$\begin{aligned}
p_X(F(x, y), x) &\leq p_X(F(x, y), x_{n+1}) + p_X(x_{n+1}, x) - p_X(x_{n+1}, x_{n+1}) \\
&\leq p_X(F(x, y), x_{n+1}) + p_X(x_{n+1}, x) \\
&= p_X(F(x, y), F(x_n, y_n)) + p_X(x_{n+1}, x) \\
&\leq k_1 \, p_X(x, x_n) + k_2 \, p_X(x, F(x, y)) + k_3 \, p_X(x_n, F(x_n, y_n)) \\
&\quad + k_4 \, p_X(x, F(x_n, y_n)) + k_5 \, p_X(x_n, F(x, y)) + p_X(x_{n+1}, x) \\
&= k_1 \, p_X(x, x_n) + k_2 \, p_X(x, F(x, y)) + k_3 \, p_X(x_n, x_{n+1}) \\
&\quad + k_4 \, p_X(x, x_{n+1}) + k_5 \, p_X(x_n, F(x, y)) + p_X(x_{n+1}, x).
\end{aligned}$$

Hence

$$\begin{aligned}
\|p_X(F(x, y), x)\| &\leq M\Big[k_1 \|p_X(x, x_n)\| + k_2 \|p_X(x, F(x, y))\| + k_3 \|p_X(x_n, x_{n+1})\| \\
&\quad + k_4 \|p_X(x, x_{n+1})\| + k_5 \|p_X(x_n, F(x, y))\| + \|p_X(x_{n+1}, x)\|\Big] \\
&\leq M(k_2 + k_5) \|p_X(F(x, y), x)\| \text{ as } n \to \infty,
\end{aligned}$$

which is a contradiction since $M > 0$ and $k_2 + k_5 < 1$. Hence $p_X(F(x, y), x) = 0$ and so $F(x, y) = x$. Similarly, we can show that $G(y, x) = y$. Now, we prove the uniqueness of FG-coupled fixed point. Suppose that $(x, y) \neq (x', y')$ is another FG-coupled fixed point, then we have

$$\begin{aligned}
p_X(x, x') &= p_X(F(x, y), F(x', y')) \\
&\leq k_1 \, p_X(x, x')) + k_2 \, p_X(x, F(x, y)) + k_3 \, p_X(x', F(x', y')) \\
&\quad + k_4 \, p_X(x, F(x', y')) + k_5 \, p_X(x', F(x, y)) \\
&= k_1 \, p_X(x, x')) + k_2 \, p_X(x, x) + k_3 \, p_X(x', x') \\
&\quad + k_4 \, p_X(x, x') + k_5 \, p_X(x', x) \\
&= (k_1 + k_4 + k_5) \, p_X(x, x') \\
&< p_X(x, x'),
\end{aligned}$$

which is a contradiction, since $k_1 + k_4 + k_5 < 1$. Thus, $x = x'$. Similarly, we show that $y = y'$. This proves the uniqueness of FG-coupled fixed point. Hence the proof. □

If we take $X = Y$ and $F = G$ in Theorem 3.28, then we have the following result.

Corollary 3.29. *Let (X, p) be a complete partial cone metric space, P be a normal cone with constant M. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following condition for all $x, y, u, v \in X$:*

$$\begin{aligned}
p(F(x, y), F(u, v)) &\leq k_1 \, p(x, u) + k_2 \, p(x, F(x, y)) + k_3 \, p(u, F(u, v)) \\
&\quad + k_4 \, p(x, F(u, v)) + k_5 \, p(u, F(x, y))
\end{aligned}$$

where k_1, k_2, k_3, k_4, k_5 are nonnegative constants such that $k_1 + k_2 + k_3 + k_4 + 2k_5 < 1$. Then F has a unique coupled fixed point.

Remark 3.30. *Theorem 3.28 and Corollary 3.29 extend and generalize the corresponding results of [18, 25] from complete cone metric space to complete partial cone metric space.*

4. Application to integral equations

Here, in this section, as an application of Corollary 3.9, we find an existence and uniqueness result for a type of the system of nonlinear integral equations.

Consider the following system of integral equations, for $t \in [0, T]$, $T > 0$,

$$\begin{aligned} \mu(t) &= \gamma(t) + \int_0^T K(t,s)[f(s,\mu(s)) + \phi(s,\nu(s))]ds, \\ \nu(t) &= \gamma(t) + \int_0^T K(t,s)[f(s,\nu(s)) + \phi(s,\mu(s))]ds, \end{aligned} \tag{22}$$

where the space $X = C([0, T], \mathbb{R})$ of continuous functions defined in $[0, T]$.

Define $p\colon X \times X \to [0, +\infty)$ by

$$p(\mu, \nu) = \sup_{t\in[0,T]} |\mu(t) - \nu(t)|, \tag{23}$$

for all $\mu, \nu \in X$.

Theorem 4.1. *Consider the Corollary 3.9 and assume that the following conditions hold:*

(O_1) *the mappings $f, \phi\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous;*

(O_2) $K\colon [0, T] \times \mathbb{R} \to [0, +\infty)$ *is continuous;*

(O_3) *the mapping $\gamma\colon [0, T] \to \mathbb{R}$ is continuous;*

(O_4) *there exists $0 \le k < 1$ such that for all $\mu, \nu \in X$:*

$$\begin{aligned} |f(s,\mu) - f(s,\nu)| &\le \frac{k}{2}|\mu - \nu| \\ |\phi(s,\mu) - \phi(s,\nu)| &\le \frac{k}{2}|\mu - \nu|; \end{aligned} \tag{24}$$

(O_5)

$$\sup_{t\in[0,T]} \int_0^T \Big|K(t,s)\Big| ds \le 1.$$

Then the system of nonlinear integral equations (22) has a unique solution in $X = C([0, T], \mathbb{R})$.

Proof. Define the mapping $F\colon X \times X \to X$ by

$$F(\mu,\nu)(t) = \gamma(t) + \int_0^T K(t,s)[f(s,\mu(s)) + \phi(s,\nu(s))]ds, \tag{25}$$

for all $\mu, \nu \in X$ and $t \in [0, T]$.

Obviously, $(\mu(t), \nu(t))$ is a solution of system of nonlinear integral equations (22) if and only if $(\mu(t), \nu(t))$ is a coupled fixed point of F. Existence of such a point follows from Corollary 3.9.

Consider

$$
\begin{aligned}
p(F(\mu,\nu),F(\rho,\sigma)) &= \sup_{t\in[0,T]}\Big|F(\mu,\nu)(t)-F(\rho,\sigma)(t)\Big| \\
&= \sup_{t\in[0,T]}\Big|\gamma(t)+\int_0^T K(t,s)[f(s,\mu(s))+\phi(s,\nu(s))]ds \\
&\qquad -\Big(\gamma(t)+\int_0^T K(t,s)[f(s,\rho(s))+\phi(s,\sigma(s))]ds\Big)\Big| \\
&= \sup_{t\in[0,T]}\Big|\int_0^T K(t,s)\Big\{[f(s,\mu(s))+\phi(s,\nu(s))] \\
&\qquad -[f(s,\rho(s))+\phi(s,\sigma(s))]\Big\}ds\Big| \\
&= \sup_{t\in[0,T]}\Big|\int_0^T K(t,s)\Big\{[f(s,\mu(s))-f(s,\rho(s))] \\
&\qquad +[\phi(s,\nu(s))-\phi(s,\sigma(s))]\Big\}ds\Big| \\
&\le \sup_{t\in[0,T]}\int_0^T |K(t,s)|\Big\{|f(s,\mu(s))-f(s,\rho(s))| \\
&\qquad +|\phi(s,\nu(s))-\phi(s,\sigma(s))|\Big\}ds \\
&\le \frac{k}{2}\sup_{t\in[0,T]}\int_0^T |K(t,s)|[|\mu(s)-\rho(s)|+|\nu(s)-\sigma(s)|]ds \\
&\le \frac{k}{2}\Big(\sup_{t\in[0,T]}\int_0^T |K(t,s)|ds\Big)\Big(\sup_{t\in[0,T]}|\mu(t)-\rho(t)| \\
&\qquad +\sup_{t\in[0,T]}|\nu(t)-\sigma(t)|\Big) \\
&\le \frac{k}{2}\Big(\sup_{t\in[0,T]}|\mu(t)-\rho(t)|+\sup_{t\in[0,T]}|\nu(t)-\sigma(t)|\Big) \\
&= \frac{k}{2}\Big[p(\mu,\rho)+p(\nu,\sigma)\Big].
\end{aligned}
$$

Therefore

$$
p(F(\mu,\nu),F(\rho,\sigma)) \le \frac{k}{2}\Big[p(\mu,\rho)+p(\nu,\sigma)\Big],
$$

where $0 \le k < 1$. This shows that the contractive condition of Corollary 3.9 holds. Hence F has a unique coupled fixed point in X, where $X = C([0,T],\mathbb{R})$. That is, the system of nonlinear integral equations (22) has a unique solution.

□

Example 4.2. *Consider $X = C([0,T],\mathbb{R})$. Now, consider the integral equation in X as*

$$
\begin{aligned}
\mu(t) &= \frac{6t^2}{5}+\int_0^1 \frac{ts}{23(t+5)}\Big[\frac{1}{1+\mu(s)}+\frac{1}{2+\nu(s)}\Big]ds, \\
\nu(t) &= \frac{6t^2}{5}+\int_0^1 \frac{ts}{23(t+5)}\Big[\frac{1}{1+\nu(s)}+\frac{1}{2+\mu(s)}\Big]ds.
\end{aligned} \tag{26}
$$

Then, clearly the above equation in the form of the following equation:

$$\mu(t) = \gamma(t) + \int_0^T K(t,s)[f(s,\mu(s)) + \phi(s,\nu(s)]ds,$$

$$\nu(t) = \gamma(t) + \int_0^T K(t,s)[f(s,\nu(s)) + \phi(s,\mu(s))]ds, \ t \in [0,T], \tag{27}$$

where

$$\gamma(t) = \frac{6t^2}{5}, \ K(t,s) = \frac{ts}{23(t+5)}, \ f(s,t) = \frac{1}{1+t}, \ \phi(s,t) = \frac{1}{2+t}, \ T = 1.$$

Thus, equation (26) is a special case of equation (22) in Theorem 4.1. Hence it is easy to verify that the functions $\gamma(t)$*,* $K(t,s)$*,* $f(s,t)$ *and* $\phi(s,t)$ *are continuous. Again, there exists* $0 \leq k < 1$ *such that*

$$|f(s,\mu) - f(s,\nu)| \leq \frac{k}{2}|\mu - \nu|,$$
$$|\phi(s,\mu) - \phi(s,\nu)| \leq \frac{k}{2}|\mu - \nu|,$$

and

$$\int_0^T |K(t,s)|ds = \int_0^1 \frac{ts}{23(t+5)}ds = \frac{t}{46(t+5)} < 1.$$

Therefore, all the conditions of Theorem 4.1 are satisfied. Hence, the system of nonlinear integral equations (22) has a unique solution $(\mu^*, \nu^*) \in X \times X$*, where* $X = C([0,T], \mathbb{R})$*.*

5. Conclusion

In this paper, we prove some *FG*-coupled fixed point theorems for contractive type conditions in the framework of complete partial cone metric spaces using normal cone with constant M. Moreover, we provide some consequences of the established results. An illustrative example and an application to the system of nonlinear integral equation are given. Our results extend, generalize and enrich several results from the existing literature (see, [18], [25] and many others).

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