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An equality condition for the norm of an elementary operator

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Abstract. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space H. The elementary operator $R_{\mathbf{A},\mathbf{B}}$ induced by two n-tuples $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ of elements of $\mathcal{B}(\mathcal{H})$ is defined by

$$R_{\mathbf{A},\mathbf{B}}(X) = \sum_{i=1}^{n} A_i X B_i \, (X \in \mathcal{B}(\mathcal{H})).$$

The aim of this paper is to give necessary and sufficient conditions under which the supremum $\sup \left\{ \|R_{\mathbf{A},\mathbf{B}}(X)\| : X \in \mathcal{B}(H), \|X\| = 1, \operatorname{rank} X = 1 \right\}$ attains its optimal value $\frac{1}{2} \sum_{i=1}^{n} (\|A_i\|^2 + \|B_i\|^2)$.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded linear operators from H into H. For two n-tuples $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ of elements of $\mathcal{B}(\mathcal{H})$, the elementary operator $R_{\mathbf{A},\mathbf{B}}$ induced by $\mathbf{A}$ and $\mathbf{B}$ is the linear operator defined by

$$R_{\mathbf{A},\mathbf{B}}(X) = \sum_{i=1}^{n} A_i X B_i \ \ (X \in \mathcal{B}(\mathcal{H})).$$

Well-know examples of elementary operators are the two sided multiplication $M_{S,T}$ and the generalized derivation $\delta_{S,T}$ generated by the operators $S, T \in \mathcal{B}(\mathcal{H})$ and which are defined by $M_{S,T}(X) = SXT$ and $\delta_{S,T}(X) = SX - TX$ $(X \in \mathcal{B}(\mathcal{H}))$, respectively.

Due to their numerous applications in different branches of mathematics, elementary operators have attracted the attention of many researchers. One of the famous problems relating to elementary operators is the norm problem, which consists of finding a formula that describes the norm of an elementary operator in terms of its coefficients, see [2, 3] for an overview. The first elegant formula for the norm of a derivation was

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established by Stampfli [6]. Later on, Timoney, in [8], obtained a first formula for the norm of an arbitrary elementary operator on a Hilbert space. Moreover, in [7], he used a generalization of a theorem of Stampfli to give an upper estimate of the norm of an elementary operator on a Hilbert space. In this line, we give another upper estimate of the norm of an elementary operator in the Hilbert space setting.

Before stating our main result, we recall some definitions and preliminaries.

Let $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ be n-tuples of operators in $\mathcal{B}(\mathcal{H})$. Define $d(R_{\mathbf{A},\mathbf{B}})$ by

$$d(R_{\mathbf{A},\mathbf{B}}) = \sup\Big\{\|R_{\mathbf{A},\mathbf{B}}(X)\| : X \in \mathcal{B}(H),\ \|X\| = 1,\ \text{rank}X = 1\Big\}.$$

For a tuple $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ of operators $A_i \in \mathcal{B}(\mathcal{H})$, the matrix numerical range of $\mathbf{A}$ is the subset of $n \times n$ matrices given by

$$W_m(\mathbf{A}) = \Big\{\big(\langle A_j^* A_i x, x\rangle\big)_{i,j=1}^n : x \in H,\ \|x\| = 1\Big\}.$$

The extremal matrix numerical range of $\mathbf{A}$ is the subset of the closure $\overline{W_m(\mathbf{A})}$ of $W_m(\mathbf{A})$ which is defined by

$$W_{m,e}(\mathbf{A}) = \Big\{\alpha \in \overline{W_m(\mathbf{A})} : \text{trace}(\alpha) = \|\sum_{i=1}^n A_i^* A_i\|\Big\}.$$

For $A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathcal{H})$, $W_m(\mathbf{A})$ is a nonempty subset consisting of $n \times n$ positive matrices. Moreover, $W_{m,e}(\mathbf{A})$ is nonempty and consists of those elements of $\overline{W_m(\mathbf{A})}$ of maximal trace, see [7].

Let $T \in \mathcal{B}(\mathcal{H})$. Recall that the maximal numerical range of T, denoted by $W_0(T)$, is defined by

$$W_0(T) = \Big\{\lambda \in \mathbb{C} : \text{ there exists a unit sequence } \{x_n\} \subseteq H \text{ such that } \lambda = \lim_n \langle Tx_n, x_n\rangle \text{ and } \lim_n \|Tx_n\| = \|T\|\Big\}.$$

The normalized maximal numerical range of T, denoted by $W_N(T)$, is given by

$$W_N(T) = \begin{cases} W_0\left(\frac{T}{\|T\|}\right) & \text{if } T \neq 0, \\ \{0\} & \text{if } T = 0. \end{cases}$$

The sets $W_0(T)$ and $W_N(T)$ are non-empty, closed, convex and contained in the closure of the numerical range of T, see [6].

By T^* we denote the adjoint of an operator $T \in \mathcal{B}(\mathcal{H})$. For $\lambda \in \mathbb{C}$, let $\overline{\lambda}$ denote its complex conjugate. If $\Omega \subseteq \mathbb{C}$, then we use the symbol $\overline{\Omega}$ to denote the subset of complex conjugates of elements of Ω. The identity operator on H is denoted by I.

2. Main results

Our main result is the following.

Theorem 2.1. *For two n-tuples $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ of nonzero elements in $\mathcal{B}(\mathcal{H})$, the following assertions are equivalent:*

1. $d(R_{\mathbf{A},\mathbf{B}}) = \frac{1}{2}\sum_{i=1}^n \left(\|A_i\|^2 + \|B_i\|^2\right)$;
2. $W_{m,e}(A_1^*, A_2^*, \cdots, A_n^*) \cap W_{m,e}(B_1, B_2, \cdots, B_n) \neq \emptyset$, $\|\sum_{i=1}^n \lambda_i A_i\| = \sum_{i=1}^n \|A_i\|$ *and* $\|\sum_{i=1}^n \overline{\lambda_i} B_i\| = \sum_{i=1}^n \|B_i\|$ *for some unit scalars* $\lambda_1, \lambda_2, \cdots, \lambda_n$.

For the proof of the above theorem, we need the following lemmas.

Lemma 2.2. *Let $A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathcal{H})$. If $\left\|R_{\mathbf{A},\mathbf{B}}\right\| = \frac{1}{2}\sum_{i=1}^n(\| A_i \|^2 + \| B_i \|^2)$, then*

$$\|R_{\mathbf{A},\mathbf{B}}\| = \frac{1}{2}\left(\left\|\sum_{i=1}^n A_i A_i^*\right\| + \left\|\sum_{i=1}^n B_i^* B_i\right\|\right). \tag{1}$$

Proof. Since $\left\|\sum_{i=1}^{n} T_iT_i^*\right\| \leq \sum_{i=1}^{n} \| T_i \|^2$ for any given operators $T_i \in \mathcal{B}(\mathcal{H})$, then the proof easily follows from the following well-known inequality due to Haagerup (see [8]):

$$\|R_{\mathbf{A},\mathbf{B}}\| \leq \sqrt{\left\|\sum_{i=1}^{n} A_iA_i^*\right\| \left\|\sum_{i=1}^{n} B_i^*B_i\right\|}\cdot$$

□

Lemma 2.3. *Let $A_1, A_2, \cdots, A_n \in \mathcal{B}(\mathcal{H})$. If $\| \sum_{i=1}^{n} \lambda_i A_i\| = \sum_{i=1}^{n} \|A_i\|$ for some unit scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, then*

$$\left\|\sum_{i=1}^{n} A_i^*A_i\right\| = \sum_{i=1}^{n} \| A_i \|^2 \cdot \tag{2}$$

Proof. Assume that $\| \sum_{i=1}^{n} \lambda_i A_i\| = \sum_{i=1}^{n} \|A_i\|$. Then, we have

$$\left\|\sum_{i=1}^{n} \lambda_i A_i\right\|^2 = \sum_{i=1}^{n} \|A_i\|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^{n} \|A_i\|\|A_j\|\cdot \tag{3}$$

On the other hand, we have

$$\left\|\sum_{i=1}^{n} \lambda_i A_i\right\|^2 = \left\|\Big(\sum_{i=1}^{n} \lambda_i A_i\Big)^*\Big(\sum_{i=1}^{n} \lambda_i A_i\Big)\right\| = \left\|\sum_{i=1}^{n} A_i^*A_i + \sum_{\substack{i\neq j\\ i,j=1}}^{n} \overline{\lambda_i}\lambda_j A_i^*A_j\right\|\cdot \tag{4}$$

Combining (3) and (4), we derive that

$$\sum_{i=1}^{n} \|A_i\|^2 + \sum_{\substack{i\neq j\\ i,j=1}}^{n} \|A_i\|\|A_j\| \leq \left\|\sum_{i=1}^{n} A_i^*A_i\right\| + \left\|\sum_{\substack{i\neq j\\ i,j=1}}^{n} \overline{\lambda_i}\lambda_j A_i^*A_j\right\|\cdot$$

This implies that

$$\left\|\sum_{i=1}^{n} A_i^*A_i\right\| = \sum_{i=1}^{n} \|A_i\|^2 \text{ and } \left\|\sum_{\substack{i\neq j\\ i,j=1}}^{n} \overline{\lambda_i}\lambda_j A_i^*A_j\right\| = \sum_{\substack{i\neq j\\ i,j=1}}^{n} \|A_i\|\|A_j\|\cdot$$

□

We also need the following results established by Timoney and Seddik, respectively.

Proposition 2.4. *[7, Proposition 3.1] If $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ are n-tuples of operators in $\mathcal{B}(\mathcal{H})$, then equality in (1) holds if and only if $W_{m,e}(A_1^*, A_2^*, \cdots, A_n^*) \cap W_{m,e}(B_1, B_2, \cdots, B_n) \neq \emptyset\cdot$*

Proposition 2.5. *[5, Theorem 2.2] If $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ are n-uples of nonzero operators in $\mathcal{B}(\mathcal{H})$, then the following assertions are equivalent:*

1. $d(R_{\mathbf{A},\mathbf{B}}) = \sum_{i=1}^{n} \|A_i\|\|B_i\|$,
2. $\| \sum_{i=1}^{n} \lambda_i A_i\| = \sum_{i=1}^{n} \|A_i\|$ *and* $\| \sum_{i=1}^{n} \overline{\lambda_i} B_i\| = \sum_{i=1}^{n} \|B_i\|$ *for some unit scalars* $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Proof of Theorem 2.1.

(1) $\Rightarrow$ (2): Suppose that $d(R_{\mathbf{A},\mathbf{B}}) = \frac{1}{2}\sum_{i=1}^{n}(\| A_i \|^2 + \| B_i \|^2)$. Since

$$d(R_{\mathbf{A},\mathbf{B}}) \leq \|R_{\mathbf{A},\mathbf{B}}\| \leq \frac{1}{2}\sum_{i=1}^{n}(\| A_i \|^2 + \| B_i \|^2),$$

it follows that

$$\|R_{\mathbf{A},\mathbf{B}}\| = \frac{1}{2}\sum_{i=1}^{n}(\| A_i \|^2 + \| B_i \|^2)\cdot$$

Applying Lemma 2.2, we get

$$\left\|R_{\mathbf{A},\mathbf{B}}\right\| = \frac{1}{2}\left(\left\|\sum_{i=1}^{n} A_iA_i^*\right\| + \left\|\sum_{i=1}^{n} B_i^*B_i\right\|\right)\cdot$$

So, by virtue of Proposition 2.4, we have

$$W_{m,e}(A_1^*, A_2^*, \cdots, A_n^*) \cap W_{m,e}(B_1, B_2, \cdots, B_n) \neq \emptyset\cdot$$

On the other hand, we have $\|A_i\|\|B_i\| \leq \frac{1}{2}(\| A_i \|^2 + \| B_i \|^2)$ for every i. Hence

$$\left\|R_{\mathbf{A},\mathbf{B}}\right\| \leq \sum_{i=1}^{n} \|A_i\|\|B_i\| \leq d(\mathbf{R}_{\mathbf{A},\mathbf{B}})\cdot$$

This implies that

$$d(\mathbf{R}_{\mathbf{A},\mathbf{B}}) = \left\|R_{\mathbf{A},\mathbf{B}}\right\| = \sum_{i=1}^{n} \|A_i\|\|B_i\|\cdot$$

Now, by using Proposition 2.5, we infer that there exist unit scalars $\lambda_1, \lambda_2, \cdot, \lambda_n$ such that $\| \sum_{i=1}^{n} \lambda_i A_i\| = \sum_{i=1}^{n} \|A_i\|$ and $\| \sum_{i=1}^{n} \overline{\lambda_i} B_i\| = \sum_{i=1}^{n} \|B_i\|\cdot$

(1) $\Leftarrow$ (2): Since $W_{m,e}(A_1^*, A_2^*, \cdots, A_n^*) \cap W_{m,e}(B_1, B_2, \cdots, B_n) \neq \emptyset$, then it follows from Proposition 2.4 that

$$\left\|R_{\mathbf{A},\mathbf{B}}\right\| = \frac{1}{2}\left(\left\|\sum_{j=1}^{n} A_jA_j^*\right\| + \left\|\sum_{j=1}^{n} B_j^*B_j\right\|\right)\cdot$$

On the other hand, since $\| \sum_{i=1}^{n} \lambda_i A_i\| = \sum_{i=1}^{n} \|A_i\|$ and $\| \sum_{i=1}^{n} \overline{\lambda_i} B_i\| = \sum_{i=1}^{n} \|B_i\|$ for some unit scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, then by Proposition 2.5, we have

$$d(R_{\mathbf{A},\mathbf{B}}) = \|R_{\mathbf{A},\mathbf{B}}\| = \sum_{i=1}^{n} \|A_i\|\|B_i\|\cdot$$

Thus, we get

$$d(R_{\mathbf{A},\mathbf{B}}) = \frac{1}{2}\left(\left\|\sum_{i=1}^{n} A_iA_i^*\right\| + \left\|\sum_{i=1}^{n} B_i^*B_i\right\|\right)\cdot$$

According to Lemma 2.3, we have the following equalities $\| \sum_{i=1}^{n} A_iA_i^*\| = \sum_{i=1}^{n} \| A_i \|^2$ and $\| \sum_{i=1}^{n} B_i^*B_i\| = \sum_{i=1}^{n} \| B_i \|^2$. Hence, we obtain

$$d(R_{\mathbf{A},\mathbf{B}}) = \frac{1}{2}\left(\sum_{i=1}^{n} \|A_i\|^2 + \sum_{i=1}^{n} \|B_i\|^2\right)$$

which completes the proof. □

As a consequence of Theorem 2.1, we have the following result.

Corollary 2.6. *If $d(R_{\mathbf{A},\mathbf{B}}) = \frac{1}{2}\sum_{i=1}^{n}\left(\|A_i\|^2 + \|B_i\|^2\right)$, then $\|A_i\| = \|B_i\|$ for all i.*

Proof. Suppose that $d(R_{\mathbf{A},\mathbf{B}}) = \frac{1}{2}\sum_{i=1}^{n}\left(\|A_i\|^2 + \|B_i\|^2\right)$. Since $\|R_{\mathbf{A},\mathbf{B}}\| \leq \sum_{i=1}^{n}\|A_i\|\|B_i\|$, then

$$\frac{1}{2}\sum_{i=1}^{n}(\| A_i \|^2 + \| B_i \|^2) \leq \sum_{i=1}^{n}\|A_i\|\|B_i\|\cdot$$

This implies that

$$\sum_{i=1}^{n}(\| A_i \| - \| B_i \|)^2 = 0.$$

Therefore, $\|A_i\| = \|B_i\|$ for all i. □

For a generalized derivation, we have the following characterization.

Proposition 2.7. *If $A, B \in \mathcal{B}(\mathcal{H})$, then the following assertions are equivalent:*

1. $d(\delta_{A,B}) = 1 + \frac{1}{2}(\|A\|^2 + \|B\|^2)$,
2. $\overline{W_N(A^*)} \cap W_N(-B) \neq \emptyset$, $\|A\| = \|B\| = 1$ *and there exists* $\lambda \in \mathbf{C}$ *with* $|\lambda| = 1$ *such that* $\|A + \lambda I\| = \|A\| + 1$ *and* $\|I - \overline{\lambda}B\| = 1 + \|B\|$.

The proof of this proposition hinges on the following lemma.

Lemma 2.8. *If $A, B \in \mathcal{B}(\mathcal{H})$, then the following assertions are equivalent:*

1. $W_{m,e}(A, I) \cap W_{m,e}(I, -B) \neq \emptyset$,
2. $\overline{W_N(A)} \cap W_N(-B) \neq \emptyset$ *and* $\|A\| = \|B\| = 1$.

Proof. To prove the implication (1) $\Longrightarrow$ (2), let $\alpha \in W_{m,e}(A, I) \cap W_{m,e}(I, -B)$. There exist sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in H such that

$$\begin{aligned} \alpha &= \lim_{n\to+\infty}\begin{pmatrix}\langle A^*Ax_n, x_n\rangle & \langle Ax_n, x_n\rangle \\ \langle A^*x_n, x_n\rangle & \langle x_n, x_n\rangle\end{pmatrix} \\ &= \lim_{n\to+\infty}\begin{pmatrix}\langle y_n, y_n\rangle & -\langle B^*y_n, y_n\rangle \\ -\langle By_n, y_n\rangle & \langle B^*By_n, y_n\rangle\end{pmatrix} \end{aligned} \tag{5}$$

with

$$\text{trace}(\alpha) = \|A^*A + I\| = \|I + B^*B\|\cdot$$

We claim that $\|A\| = \|B\| = 1$. Indeed, we deduce from (5) that

$$\lim_{n\to+\infty}\langle A^*Ax_n, x_n\rangle = \lim_{n\to+\infty}\|y_n\|^2 = 1 \text{ and } \lim_{n\to+\infty}\langle B^*By_n, y_n\rangle = \|x_n\|^2 = 1.$$

Thus

$$\|A^*A + I\| = \|I + B^*B\| = 2.$$

Since $A^*A + I$ and $I + B^*B$ are both positive operators, it follows that

$$\|A\| = \|B\| = 1.$$

Now, if we set $\lambda = \lim_{n\to+\infty}\langle A^*x_n, x_n\rangle$, then it follows from (5) that

$$\lambda = \lim_{n\to+\infty} -\langle By_n, y_n\rangle \in \overline{W_N(A)} \cap W_N(-B).$$

Conversely, let $\mu \in \overline{W_N(A)} \cap W_N(-B)$. There exist sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in H such that

$$\mu = \lim_{n\to+\infty} \overline{\langle Ax_n, x_n\rangle} = \lim_{n\to+\infty} \langle -By_n, y_n\rangle \text{ and } \lim_{n\to+\infty} \|Ax_n\| = \lim_{n\to+\infty} \|By_n\| = 1.$$

Consider the matrix $X=\begin{pmatrix} 1 & \overline{\mu} \\ \mu & 1 \end{pmatrix}$. Clearly, we have

$$\begin{aligned} X &= \lim_{n\to+\infty} \begin{pmatrix} \|Ax_n\|^2 & \langle Ax_n, x_n\rangle \\ \langle A^*x_n, x_n\rangle & \langle x_n, x_n\rangle \end{pmatrix} \\ &= \lim_{n\to+\infty} \begin{pmatrix} \langle y_n, y_n\rangle & -\langle B^*y_n, y_n\rangle \\ -\langle By_n, y_n\rangle & \langle B^*By_n, y_n\rangle \end{pmatrix}. \end{aligned}$$

Moreover, we can easily check that

$$\begin{aligned} \operatorname{trace}(X) &= \|A^*A + I\| \\ &= \|B^*B + I\| \\ &= 2. \end{aligned}$$

Thus $X \in W_{m,e}(A, I) \cap W_{m,e}(I, -B)$, which shows that

$$W_{m,e}(A, I) \cap W_{m,e}(I, -B) \neq \emptyset.$$

□

Now we are in a position to give the proof of Proposition 2.7.

Proof of Proposition 2.7.

The implication (2) ⇒ (1) follows easily from Lemma 2.8 and Theorem 2.1.

(1) ⇒ (2): Assume that $d(\delta_{A,B}) = 1+\frac{1}{2}(\|A\|^2+\|B\|^2)$. From Theorem 2.1, we have $W_{m,e}(A^*, I)\cap W_{m,e}(I, -B) \neq \emptyset$, and $\|\lambda_1 A + \lambda_2 I\| = \|A\| + 1$ and $\|\overline{\lambda_1} I - \overline{\lambda_2} B\| = 1 + \|B\|$ for some complex numbers λ_1, λ_2 with $|\lambda_1| = |\lambda_2| = 1$. Thus, we can write $\|A + \lambda I\| = \|A\| + 1$ and $\|I - \overline{\lambda} B\| = 1 + \|B\|$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Now, according to Lemma 2.8, it follows that $\overline{W_N(A^*)} \cap W_N(-B) \neq \emptyset$ and $\|A\| = \|B\| = 1$ which completes the proof.

□

Let $(J, \|\cdot\|_J)$ be a (symmetric) norm ideal in $\mathcal{B}(\mathcal{H})$ in the sense of R. Schatten [4, pp. 54-55]. If $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ and $\mathbf{B} = (B_1, B_2, \cdots, B_n)$ are n-tuples of operators in $\mathcal{B}(\mathcal{H})$, then $\|R_{\mathbf{A},\mathbf{B}}(X)\|_J \leq (\sum_{i=1}^{n} \|A_i\|\|B_i\|) \|X\|_J$ for every $X \in J$. Thus, the restriction of $R_{\mathbf{A},\mathbf{B}}$ to J, which we denote by $R_{J,\mathbf{A},\mathbf{B}}$, is a bounded linear operator on J and $\|R_{J,\mathbf{A},\mathbf{B}}\| \leq \sum_{i=1}^{n} \|A_i\|\|B_i\|$. Moreover, from [1, Remark 2.9] we have

$$d(R_{\mathbf{A},\mathbf{B}}) \leq \|R_{J,\mathbf{A},\mathbf{B}}\| \text{ for every } J. \tag{6}$$

In the special case where $J = C_{\in}(\mathcal{H})$, the Hilbert-Schmidt ideal of compact operators on H, we simply denote the operator $R_{J,\mathbf{A},\mathbf{B}}$ by $R_{2,\mathbf{A},\mathbf{B}}$. Here we recall that $C_{\in}(\mathcal{H})$ is a Hilbert space with the inner product defined by

$$\langle X, Y\rangle = \operatorname{tr}(XY^*),$$

where tr denotes the usual functional trace.

Theorem 2.9. *If $A, B, C, D \in \mathcal{B}(\mathcal{H})$ are nonzero, then the following assertions are equivalent:*

1. $\|M_{J,A,B} + M_{J,C,D}\| = \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right)$ *for every J;*
2. $\|M_{2,A,B} + M_{2,C,D}\| = \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right)$;
3. $d(M_{A,B} + M_{C,D}) = \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right)$.

Proof. The implications (3) ⇒ (1) ⇒ (2) are obvious as the inequality in (6) holds for every norm ideal J.

(2) ⇒ (3): Since

$$\begin{aligned}\|M_{2,A,B} + M_{2,C,D}\| &\leq \|A\|\|B\| + \|C\|\|D\| \\ &\leq \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right),\end{aligned}$$

then

$$\begin{aligned}\|M_{2,A,B} + M_{2,C,D}\| &= \|A\|\|B\| + \|C\|\|D\| \\ &= \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right)\cdot \end{aligned} \tag{7}$$

This implies that $\|A\| = \|B\|$ and $\|C\| = \|D\|$.
Now, from [1, Theorem 3.2], there exist a unit scalar $\lambda \in \mathbb{C}$, and sequences $\{x_n\}_n$ and $\{y_n\}_n$ of unit vectors in H such that

$$\lim_n \langle A^*Cx_n, x_n\rangle = \lambda\|A\|\|C\| \text{ and } \lim_n \langle DB^*y_n, y_n\rangle = \overline{\lambda}\|B\|\|D\|\cdot$$

Thus, we easily derive that

$$\lim_n \|Cx_n\| = \|C\| \text{ and } \lim_n \|D^*y_n\| = \|D\|\cdot$$

Set $X_n = x_n \otimes y_n$. Then rank $X_n = \|X_n\| = 1$ for every n. Thus

$$\begin{aligned}d(M_{A,B} + M_{C,D}) &\geq \lim_n \|(M_{A,B} + M_{C,D})(X_n)\| \\ &\geq \frac{1}{\|C\|\|D\|} \lim_n \left|\langle (M_{A,B} + M_{C,D})(X_n)D^*y_n, Cx_n\rangle\right| \\ &= \frac{1}{\|C\|\|D\|} \lim_n \left|\langle A^*Cx_n, x_n\rangle\langle DB^*y_n, y_n\rangle + \|Cx_n\|^2\|D^*y_n\|^2\right| \\ &= \|A\|\|B\| + \|C\|\|D\| \\ &= \frac{1}{2}\left(\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2\right),\end{aligned}$$

where the last equality follows from (7). □

References

[1] M. Boumazgour, *On the S-universal elementary operators*, Linear Algebra Appl., **507** (2016), 274-287.
[2] L. Fialkow, *Structural properties of elementary operators*, in: M. Mathieu (Ed.), Elementary Operators and Applications, Proc. Int. Workshop, Blaubern, 1991, World Scientific, Singapore, 1992, pp. 55-113.
[3] L. Fialkow, *Open problems*, in: Elementary Operators and Their Applications, in: Oper. Theory Adv. Appl., vol. 212, Birkhäuser/Springer, Basel AG, Basel, 2011, pp. 151-153.
[4] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer-Verlag, Berlin, 1960.
[5] A. Seddik, *Rank one operators and norm of elementary operators*, Linear Algebra Appl., **424**, (2007), 177-183.
[6] J. Stampfli, *The norm of a derivation*, Pacific J. Math., **33**, (1970), 737-747.
[7] R.M. Timoney, *Computing the norms of elementary operators*, Illinois J. Math. **47** (2003), 1207-1226.
[8] R.M. Timoney, *Some formulae for norms of elementary operators*, J. Operator Theory, **57**, (2007), 121-145.