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Limit theorems for lacunary weighted Cesàro methods

Elif N. Yldrm[a], Fatih Nuray[b]

[a] *Istanbul Commerce University*
[b] *Afyon Kocatepe University*

Abstract. In this paper, we study the asymptotic behavior of lacunary weighted Cesàro means. Building upon classical results in Cesàro summability, we extend the theory to the lacunary setting and establish sufficient conditions for the convergence of such means. The analysis includes both one-dimensional and multi-dimensional sequences, with applications to ergodic-type limits and central limit behavior. Several counterexamples are presented to illustrate the necessity of the assumptions and to highlight differences between classical and lacunary Cesàro methods.

1. Introduction

In [15], the limits of weighted Cesàro means were studied in the classical setting, focusing on their convergence properties and their connections to ergodic theorems and central limit theorems. The concept of *p-mean convergence* was analyzed in detail, and necessary and sufficient conditions for convergence were established. Additionally, multi-dimensional extensions and counterexamples were provided to illustrate the limits of the approach.

In our study, we extend these results to the *lacunary Cesàro means* framework. We adapt all fundamental results from [15] to the lacunary setting, including p-mean convergence, integral transformations, and multi-dimensional extensions. Furthermore, we provide counterexamples that demonstrate the limitations of lacunary Cesàro means, offering a novel perspective on the convergence behavior of such averages.

This work explores the asymptotic behavior of lacunary weighted Cesàro means, motivated by their applications in diverse areas of mathematics. Specifically, we investigate their convergence properties under various conditions, aiming to generalize classical Cesàro means to a lacunary framework.

A sequence $\theta = \{k_r\}$ of positive integers is called a *lacunary sequence* if it satisfies the following conditions [11]:

- $k_0 = 0$ and $k_{r-1} < k_r$ for all $r \geq 1$.
- The sequence of gaps $h_r = k_r - k_{r-1}$ satisfies $h_r \to \infty$ as $r \to \infty$.

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Email addresses: enuray@ticaret.edu.tr (Elif N. Yldrm), fnuray@aku.edu.tr (Fatih Nuray)

The intervals induced by the lacunary sequence are defined as $I_r := (k_{r-1}, k_r]$ [11].

In [11], the concepts of lacunary Cesàro convergence and strong lacunary Cesàro convergence were introduced. Additionally, the inclusion relationships between strong lacunary Cesàro convergence and strong Cesàro convergence were analyzed, providing insights into their structural connections.

We define the lacunary Cesàro mean as follows:

$$\mathcal{A}_{b,\Phi;p,\theta}(r) = \frac{1}{h_r^p} \sum_{k\in I_r} b_k \Phi(k/k_r), \tag{1}$$

where $p > 0$, the function $\Phi : (0,1] \to \mathbb{R}$ governs the transformation, and $(b_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{C}$ exhibiting *p-mean lacunary convergence*

$$\lim_{r\to\infty} \frac{1}{h_r^p} \sum_{k\in I_r} b_k = b \in \mathbb{C}. \tag{2}$$

The nature of the convergence in (1) is contingent upon the characteristics of the function Φ. Our primary results in Section 2 establish sufficient conditions ensuring the limit's existence. In particular, explicit convergence is demonstrated for specific instances, such as:

$$\Phi(\xi) = \xi^q, \quad \Phi(\xi) = (1-\xi)^q, \quad q > 0. \tag{3}$$

In Section 3, we extend the analysis to multi-indexed sequences in the lacunary setting, examining their role in quantum probability, particularly in the study of central limit theorems. The results in Propositions 3.1 and 3.2 offer insight into mixed moments and their structural properties, particularly under monotonic and anti-monotonic conditions (see, e.g., [13]). Readers interested in quantum probability and its interactions with functional analysis may refer to [1, 2].

The final section presents counterexamples that clarify the necessity of imposed assumptions. Additionally, we discuss connections between the classical Cesàro means and their lacunary counterparts, situating our results within the broader literature (see, e.g., [4]).

2. Limits of Lacunary Weighted Cesàro Means

In this study, we consider the set of natural numbers excluding zero

$$\mathbb{N} := \{1, 2, \ldots, n, \ldots\}.$$

To establish a clear framework, we introduce fundamental notation. Let $\Phi : (0,1] \to \mathbb{R}$ be a function, $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{C}$, and $p \in (0, +\infty)$ a given parameter. We define the lacunary weighted Cesàro mean as

$$\mathcal{A}_{\mathbf{b},\Phi;p,\theta}(r) = \frac{1}{h_r^p} \sum_{k\in I_r} b_k \Phi(k/k_r).$$

For any sequence $\mathbf{b}$, we define its modulus sequence as $|\mathbf{b}| := (|b_n|)_{n\in\mathbb{N}}$. A sequence $\mathbf{b}$ is termed *p-mean lacunary convergent* if the sequence $(\mathcal{A}_{\mathbf{b},1;p,\theta}(r))_{r\in\mathbb{N}}$ of its lacunary Cesàro p-means converges, where the function Φ is taken as identically 1.

When $p = 1$, this reduces to the conventional lacunary arithmetic means. Moreover, a straightforward argument shows that if $\mathbf{b}$ exhibits p-mean lacunary convergence, then it satisfies

$$b_n = o(h_r^p) \quad \text{as } r \to +\infty.$$

Lemma 2.1. *Let* $\mathbf{a} = (a_n)$ *and* $\mathbf{b} = (b_n)$ *be sequences such that* (a_n) *is convergent and* (b_n) *is lacunary p-mean convergent with respect to* $\theta = (k_r)$*, i.e.*

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{r\to\infty} \mathcal{A}_{\mathbf{b},1;p,\theta}(r) = b.$$

Assume further that

$$\mathcal{A}_{|\mathbf{b}|,1;p,\theta}(r) = \frac{1}{h_r^p} \sum_{k\in I_r} |b_k| \le B, \quad r \in \mathbb{N}. \tag{4}$$

Then the sequence $(a_n b_n)$ *is also lacunary p-mean convergent, and*

$$\lim_{r\to\infty} \mathcal{A}_{\mathbf{ab},1;p,\theta}(r) = ab.$$

Proof. We have

$$\mathcal{A}_{\mathbf{ab},1;p,\theta}(r) = \frac{1}{h_r^p} \sum_{k\in I_r} a_k b_k.$$

Write $a_k = a + (a_k - a)$. Then

$$\mathcal{A}_{\mathbf{ab},1;p,\theta}(r) - ab = \frac{1}{h_r^p} \sum_{k\in I_r} (a_k - a) b_k + a\Big(\mathcal{A}_{\mathbf{b},1;p,\theta}(r) - b\Big).$$

Hence

$$\Big|\mathcal{A}_{\mathbf{ab},1;p,\theta}(r) - ab\Big| \le \frac{1}{h_r^p} \sum_{k\in I_r} |a_k - a|\,|b_k| + |a|\,\Big|\mathcal{A}_{\mathbf{b},1;p,\theta}(r) - b\Big|.$$

Fix $\varepsilon > 0$. Since $a_n \to a$, there exists N_ε such that $|a_k - a| \le \varepsilon$ for all $k \ge N_\varepsilon$. Because $k_{r-1} \to \infty$, there exists r_ε such that $k_{r-1} \ge N_\varepsilon$ for all $r \ge r_\varepsilon$. Thus, for $r \ge r_\varepsilon$ and all $k \in I_r$ we have $k \ge k_{r-1} \ge N_\varepsilon$, hence $|a_k - a| \le \varepsilon$. It follows that

$$\frac{1}{h_r^p} \sum_{k\in I_r} |a_k - a|\,|b_k| \le \varepsilon\, \frac{1}{h_r^p} \sum_{k\in I_r} |b_k| \le \varepsilon B,$$

where we used (4) in the last step.

Therefore, for $r \ge r_\varepsilon$,

$$\Big|\mathcal{A}_{\mathbf{ab},1;p,\theta}(r) - ab\Big| \le \varepsilon B + |a|\,\Big|\mathcal{A}_{\mathbf{b},1;p,\theta}(r) - b\Big|.$$

Taking lim sup as $r \to \infty$ and using the lacunary p-mean convergence of (b_n), we obtain

$$\limsup_{r\to\infty} \Big|\mathcal{A}_{\mathbf{ab},1;p,\theta}(r) - ab\Big| \le \varepsilon B.$$

Since $\varepsilon > 0$ is arbitrary, the limit on the left-hand side must be zero, which proves that

$$\lim_{r\to\infty} \mathcal{A}_{\mathbf{ab},1;p,\theta}(r) = ab.$$

□

Theorem 2.2. *Let* $\mathbf{b}$ *be a lacunary p-mean convergent sequence satisfying*

$$\lim_{r\to\infty} \mathcal{A}_{\mathbf{b},1;p,\theta}(r) = b,$$

and let $\Phi : (0,1] \to \mathbb{R}$ *be a monotone function such that*

$$\Phi \in L^1((0,1], \xi^{p-1}d\xi) \quad \text{and} \quad \xi^p \in L^1((0,1], |d\Phi|).$$

Then, the following holds:

$$\lim_{r\to\infty} \mathcal{A}_{\mathbf{b},\Phi;p,\theta}(r) = bp\int_0^1 \xi^{p-1}\Phi(\xi)d\xi.$$

Proof. Without any loss of generality, we may assume that Φ is a decreasing function, as one can always consider its negation if needed. Additionally, we can ensure that Φ remains positive by shifting it with a suitable constant. Given these assumptions, for every $\epsilon > 0$, there exists an index r_0 such that for all $r > r_0$

$$0 \le \sum_{k\in I_r} \Phi\left(\frac{k}{k_r}\right)\left[\left(\frac{k}{k_r}\right)^p - \left(\frac{k-1}{k_r}\right)^p\right] \le p\int_0^{1/r_0} \xi^{p-1}\Phi(\xi)d\xi \le \epsilon.$$

Since I_r is a lacunary interval, summation over k provides a Riemann-Stieltjes type sum approximation

$$0 \le p\int_0^1 \xi^{p-1}\Phi(\xi)d\xi - \sum_{k\in I_r} \Phi\left(\frac{k}{k_r}\right)\left[\left(\frac{k}{k_r}\right)^p - \left(\frac{k-1}{k_r}\right)^p\right] \le 2\epsilon.$$

Taking the limit as $r \to \infty$, we obtain:

$$\lim_{r\to\infty} \sum_{k\in I_r} \Phi\left(\frac{k}{k_r}\right)\left[\left(\frac{k}{k_r}\right)^p - \left(\frac{k-1}{k_r}\right)^p\right] = p\int_0^1 \xi^{p-1}\Phi(\xi)d\xi. \tag{5}$$

Defining

$$c_r := \mathcal{A}_{\mathbf{b},1;p,\theta}(r) - b,$$

we obtain the decomposition

$$\begin{aligned}\mathcal{A}_{\mathbf{b},\Phi;p,\theta}(r) &= c_r\Phi(1) + b\sum_{k\in I_r} \Phi\left(\frac{k}{k_r}\right)\left[\left(\frac{k}{k_r}\right)^p - \left(\frac{k-1}{k_r}\right)^p\right] \\ &\quad + \sum_{k\in I_r} c_{k-1}\left(\frac{k-1}{k_r}\right)^p\left[\Phi\left(\frac{k-1}{k_r}\right) - \Phi\left(\frac{k}{k_r}\right)\right].\end{aligned}$$

For any $\epsilon > 0$, there exists r_0 such that $|c_r| < \epsilon$ for all $r > r_0$. Then, for sufficiently large r

$$\left|\sum_{k\in I_r} c_{k-1}\left(\frac{k-1}{k_r}\right)^p\left[\Phi\left(\frac{k-1}{k_r}\right) - \Phi\left(\frac{k}{k_r}\right)\right]\right|$$

$$\le \sup_r |c_r|\int_0^1 \xi^p|d\Phi(\xi)| + \epsilon\int_0^1 \xi^p|d\Phi(\xi)|.$$

Since $\epsilon > 0$ is arbitrary, the term vanishes as $r \to \infty$. Collecting this result with (5), we conclude:

$$\lim_{r\to\infty} \mathcal{A}_{\mathbf{b},\Phi;p,\theta}(r) = bp \int_0^1 \xi^{p-1}\Phi(\xi)d\xi.$$

□

3. Multi-Dimensional Extensions

In this section, we analyze the asymptotic behavior of multi-dimensional lacunary Cesàro means, particularly in the context of ergodic limits. Our goal is to extend the classical framework of Cesàro means to higher-dimensional sequences, exploring their structure and potential convergence properties.

Auxiliary notation

Let $p > 0$, $q > 0$, $\theta = (k_r)$ be a lacunary sequence with $h_r = k_r - k_{r-1}$ and $I_r = (k_{r-1}, k_r]$. For a complex sequence $\mathbf{b} = (b_k)$ we set

$$L_{b,\theta}(p,q) := \lim_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k\, k^q,$$

whenever this limit exists and is finite. Similarly, we write

$$L_{|b|,\theta}(p,q) := \limsup_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} |b_k|\, k^q,$$

and analogously

$$M_{b,\theta}(p,q) := \lim_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k\,(h_r - k)^q,$$

$$M_{|b|,\theta}(p,q) := \limsup_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} |b_k|\,(h_r - k)^q.$$

Theorem 3.1. *Let $\mathbf{b} = (b_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers that is* lacunary p-mean convergent *with respect to a lacunary sequence $\theta = (k_r)$, i.e.*

$$\lim_{r\to\infty} \mathcal{A}_{\mathbf{b},1;p,\theta}(r) = \lim_{r\to\infty} \frac{1}{h_r^p} \sum_{k\in I_r} b_k = b,$$

where $p > 0$, $h_r = k_r - k_{r-1}$ and $I_r = (k_{r-1}, k_r]$. Assume moreover that the weighted lacunary sums

$$\frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k\, k^q$$

converge for some $q > 0$, and that the corresponding absolute-value averages are uniformly bounded, i.e.

$$\sup_{r\in\mathbb{N}} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} |b_k|\, k^q < +\infty. \tag{4'}$$

Denote

$$L_{b,\theta}(p,q) = \lim_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k\, k^q.$$

Let $(a_{i_1,\dots,i_m})_{i_1,\dots,i_m\in\mathbb{N}}$ be a multi-indexed sequence of complex numbers for which there exists $q > 0$ and $a \in \mathbb{C}$ such that

$$\lim_{n\to\infty} \frac{1}{n^q} \sum_{1\le i_1,\dots,i_m\le n} a_{i_1,\dots,i_m} = a.$$

Define

$$S(n) := \sum_{1\le i_1,\dots,i_m\le n} a_{i_1,\dots,i_m} = a\, n^q + R(n), \qquad \frac{R(n)}{n^q} \to 0 \quad \text{as } n\to\infty.$$

Then the following limit exists and satisfies

$$\lim_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k \sum_{1\le i_1,\dots,i_m\le k} a_{i_1,\dots,i_m} = a\, L_{b,\theta}(p,q).$$

Proof. We write

$$L_r := \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k S(k) = \frac{a}{h_r^{p+q}} \sum_{k\in I_r} b_k k^q + \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k R(k) =: M_r + E_r.$$

By the very definition of $L_{b,\theta}(p,q)$ we have

$$M_r = a\, \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k k^q, \qquad \lim_{r\to\infty} M_r = a\, L_{b,\theta}(p,q).$$

It remains to prove that the error term E_r tends to 0. Let $\varepsilon > 0$. Since $R(n) = o(n^q)$, there exists $K_\varepsilon \in \mathbb{N}$ such that

$$|R(k)| \le \varepsilon\, k^q \quad \text{for all } k \ge K_\varepsilon.$$

We split the sum over $k \in I_r$ into the part with $k \ge K_\varepsilon$ and the (at most K_ε) remaining terms:

$$|E_r| \le \frac{1}{h_r^{p+q}} \sum_{\substack{k\in I_r\\ k\ge K_\varepsilon}} |b_k|\, |R(k)| + \frac{1}{h_r^{p+q}} \sum_{\substack{k\in I_r\\ k< K_\varepsilon}} |b_k|\, |R(k)| =: E_r^{(1)} + E_r^{(2)}.$$

For $E_r^{(1)}$ we use $|R(k)| \le \varepsilon k^q$ and the boundedness assumption (4′):

$$E_r^{(1)} \le \frac{\varepsilon}{h_r^{p+q}} \sum_{k\in I_r} |b_k|\, k^q \le \varepsilon \sup_{s\in\mathbb{N}} \frac{1}{h_s^{p+q}} \sum_{k\in I_s} |b_k|\, k^q =: \varepsilon C,$$

where $C < \infty$ is independent of r.

For $E_r^{(2)}$, note that the number of indices with $k < K_\varepsilon$ is bounded by K_ε, and $R(k)$ is bounded on the finite set $\{1, \dots, K_\varepsilon\}$; say $|R(k)| \le M_\varepsilon$ there. Hence

$$E_r^{(2)} \le \frac{M_\varepsilon}{h_r^{p+q}} \sum_{\substack{k\in I_r\\ k< K_\varepsilon}} |b_k| \le \frac{M'_\varepsilon}{h_r^{p+q}},$$

for some constant M'_ε independent of r. Since $h_r \to \infty$, we obtain $E_r^{(2)} \to 0$ as $r \to \infty$.

Consequently,

$$\limsup_{r\to\infty} |E_r| \le \varepsilon C.$$

Letting $\varepsilon \to 0$ shows that $\lim_{r\to\infty} E_r = 0$. Combining this with the limit of M_r gives

$$\lim_{r\to\infty} L_r = a\, L_{b,\theta}(p,q),$$

which completes the proof. □

Theorem 3.2. *Assume all hypotheses of Theorem 3.1. Suppose in addition that the multi-indexed sequence* $(a_{i_1,\dots,i_m})$ *is shift-invariant in the sense that*

$$a_{i_1-h,\dots,i_m-h} = a_{i_1,\dots,i_m} \qquad \text{for all } i_1,\dots,i_m \in \mathbb{N} \text{ and every } h < \min\{i_1,\dots,i_m\}. \tag{6}$$

Assume also that, for the same $p, q > 0$, *the limit*

$$M_{b,\theta}(p,q) = \lim_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k\,(h_r-k)^q$$

exists and that the corresponding absolute-value averages are uniformly bounded:

$$\sup_{r\in\mathbb{N}} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} |b_k|\,(h_r-k)^q < +\infty. \tag{4''}$$

Then

$$\lim_{r\to\infty} \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k \sum_{k+1\le i_1,\dots,i_m\le h_r} a_{i_1,\dots,i_m} = a\, M_{b,\theta}(p,q).$$

Proof. By (6),

$$\sum_{k+1\le i_1,\dots,i_m\le h_r} a_{i_1,\dots,i_m} = \sum_{1\le i_1,\dots,i_m\le h_r-k} a_{i_1,\dots,i_m} = S(h_r-k),$$

where $S(n) = an^q + R(n)$ with $R(n) = o(n^q)$.

Define

$$T_r = \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k S(h_r-k) = \frac{a}{h_r^{p+q}} \sum_{k\in I_r} b_k (h_r-k)^q + \frac{1}{h_r^{p+q}} \sum_{k\in I_r} b_k R(h_r-k) =: U_r + F_r.$$

By definition,

$$U_r \to a\, M_{b,\theta}(p,q).$$

To show $F_r \to 0$, fix $\varepsilon > 0$. Choose K_ε so that $|R(n)| \le \varepsilon n^q$ for $n \ge K_\varepsilon$. Split

$$F_r = F_r^{(1)} + F_r^{(2)},$$

where

$$F_r^{(1)} = \frac{1}{h_r^{p+q}} \sum_{\substack{k\in I_r\\ h_r-k\ge K_\varepsilon}} b_k R(h_r-k), \qquad F_r^{(2)} = \frac{1}{h_r^{p+q}} \sum_{\substack{k\in I_r\\ h_r-k< K_\varepsilon}} b_k R(h_r-k).$$

Then

$$|F_r^{(1)}| \le \varepsilon \sup_{s\in\mathbb{N}} \frac{1}{h_s^{p+q}} \sum_{k\in I_s} |b_k|\,(h_s-k)^q = \varepsilon C' \quad (C' < \infty).$$

Since $h_r - k < K_\varepsilon$ occurs for finitely many k,

$$|F_r^{(2)}| \le \frac{M_\varepsilon}{h_r^{p+q}} \sum_{\substack{k\in I_r \\ h_r-k<K_\varepsilon}} |b_k| \to 0.$$

Thus,

$$\limsup_{r\to\infty} |F_r| \le \varepsilon C' \Rightarrow F_r \to 0,$$

and the result follows. □

4. Some Counterexamples

We conclude this note by presenting some counterexamples concerning the lacunary Cesàro average-convergence of sequences.

4.1. Failure of Lemma 2.1 Without Condition (4)

Consider the sequences (b_k) and (a_k) defined by

$$b_{2k-1} = -\sqrt{2k}, \quad b_{2k} = 1 + \sqrt{2k}, \qquad k \in \mathbb{N},$$

$$a_{2k-1} = -\frac{1}{\sqrt{2k}}, \quad a_{2k} = \frac{1}{\sqrt{2k}}, \qquad k \in \mathbb{N}.$$

Then $\lim_{n\to\infty} a_n = 0$, while the lacunary mean-limit of (b_k) exists:

$$\lim_{r\to\infty} \mathcal{A}_{b,1;1,\theta}(r) = \frac{1}{2}.$$

However, the lacunary Cesro averages of the absolute values diverge:

$$\frac{1}{h_r}\sum_{k\in I_r} |b_k| = \frac{1}{h_r}\sum_{k\in I_r}\left(\sqrt{2\left\lceil\frac{k}{2}\right\rceil}\right) \longrightarrow +\infty \qquad (r\to\infty).$$

Finally, the lacunary Cesro means of the product fail to converge to the product of limits:

$$\lim_{r\to\infty} \mathcal{A}_{ab,1;1,\theta}(r) = \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r} b_k a_k = 1 \ne 0 = \left(\lim_{r\to\infty}\mathcal{A}_{b,1;1,\theta}(r)\right)\cdot\left(\lim_{n\to\infty} a_n\right).$$

Thus, Lemma 2.1 does not hold without the boundedness condition (4).

4.2. Failure of Theorem 3.2 Without Condition (7)

In conclusion, we examine whether Equation (8) holds under the assumptions of Proposition 3.1, with the exception of condition (7). The outcome is negative, as evidenced by the following counterexample in the particular case where $p = 1$, $m = 2$, and $q = m$.

Consider

$$b_k = 1, \qquad a_{k_1,k_2} = (\sqrt{k_1} - \sqrt{k_1 - 1})\sqrt{k_2}, \qquad k, k_1, k_2 \in \mathbb{N}.$$

Then

$$b = 1, \qquad a = \frac{2}{3},$$

since

$$\frac{1}{h_r^2}\sum_{1\leq k_1,k_2\leq h_r} a_{k_1,k_2} = \frac{1}{h_r^{3/2}}\sum_{k_2=1}^{h_r}\sqrt{k_2} = \frac{1}{h_r}\sum_{k_2=1}^{h_r}\sqrt{\frac{k_2}{h_r}} \longrightarrow \int_0^1 \xi^{1/2}\,d\xi = \frac{2}{3}.$$

However, computing the left-hand side of (8), we obtain

$$\lim_{r\to\infty}\frac{1}{h_r^3}\sum_{k\in I_r} b_k \sum_{k+1\leq k_1,k_2\leq h_r} a_{k_1,k_2} = \frac{4}{15}ab \neq aM_{b,\theta}(p,q).$$

Thus, (8) does not hold without condition (7).

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