Functional Analysis,
Approximation and
Computation
17:2 (2025), 43–61

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

q-A-numerical range of an operator

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Abstract. The aim of this paper is to introduce and investigate the concept of the q-A-numerical range of an operator on a Hilbert space. Basic properties of this set are studied, and several estimates of the related q-A-numerical radius are established. As applications, many q-A-numerical radius inequalities for some operator matrices on a Hilbert space are given. In addition, some results about the center of the q-A-numerical range of an operator are established.

1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. By $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators from H into H. For $T \in \mathcal{B}(H)$, we denote by $\mathcal{N}(T)$, $\mathcal{R}(T)$ and T^* the kernel, the range and the adjoint of T, respectively. If F is a linear subspace of H, then $\overline{F}$ stands for its closure in the topology induced by the norm $\| \cdot \|$.

Recall that an operator $A \in \mathcal{B}(H)$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for every $x \in H$. In this case we let $A^{\frac{1}{2}}$ denote the square root of A.

If $A \in \mathcal{B}(H)$ is positive, then it induces a semi-inner product on H defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$ for every $x, y \in H$. Let $\| \cdot \|_A$ be the semi-norm on H induced by $\langle \cdot , \cdot \rangle_A$. Obviously, we have $\|x\|_A = \|A^{\frac{1}{2}}x\|$ for all $x \in H$. Moreover, we can easily check that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. This shows $\| \cdot \|_A$ is a norm if and only if A is an injective operator. Also, we can easily prove that the space $(H, \| \cdot \|_A)$ is complete if and only if the range $\mathcal{R}(A)$ is closed in H.

In all what follows, we shall suppose that A is a positive operator. We say that an operator $T \in \mathcal{B}(H)$ is A-bounded if there exists $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in H$. By $\mathcal{B}_A(H)$ we denote the set of all operators $T \in \mathcal{B}(H)$ which are A-bounded. Namely,

$$\mathcal{B}_A(H) = \Big\{T \in \mathcal{B}(H) : \exists c > 0 \text{ such that } \|Tx\|_A \leq c\|x\|_A \text{ for all } x \in H\Big\}.$$

2020 *Mathematics Subject Classification*. 47A12, 47A30, 47A63.
Keywords. Numerical range; Numerical radius; Positive operator; Inequality; Operator matrix.
Received: 30 August 2025; Accepted: 12 October 2025
Communicated by Dragan S. Djordjević
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We equip the subalgebra $\mathcal{B}_A(H)$ with the semi-norm $\|\cdot\|_A$ given by

$$\begin{aligned}\|T\|_A &= \sup\left\{\frac{\|Tx\|_A}{\|x\|_A} : x \in \overline{\mathcal{R}(A)}, x \neq 0\right\}\\ &= \sup\left\{\|Tx\|_A : x \in \overline{\mathcal{R}(A)}, \|x\|_A = 1\right\}.\end{aligned}$$

It can be easily seen that $\|T\|_A < \infty$ for all operators $T \in \mathcal{B}_A(H)$. Moreover, from [11], there is equality

$$\|T\|_A = \sup\left\{|\langle Tx, y\rangle_A| : x, y \in H, \|x\|_A = \|y\|_A = 1\right\}.$$

Notice that if $T \notin \mathcal{B}_A(H)$, then the supremum of $\left\{\frac{\|Tx\|_A}{\|x\|_A} : x \in \overline{\mathcal{R}(A)}, x \neq 0\right\}$ may be infinite (see [12]). However, if the operator A is supposed to be injective with closed range, then $\mathcal{B}_A(H) = \mathcal{B}(H)$. We refer to [6, 11, 12, 21, 23] for more details about A-bounded operators.

Let $T \in \mathcal{B}_A(H)$. An operator $S \in \mathcal{B}_A(H)$ is called A-adjoint of T if for every $x, y \in H$, there is equality

$$\langle Tx, y\rangle_A = \langle x, Sy\rangle_A.$$

Notice that, in general, an operator $T \in \mathcal{B}_A(H)$ may admit none, only one or many A-adjoints. In fact, by the Douglas Range Inclusion Theorem [9], such T admits an A-adjoint if and only if $T^*\mathcal{R}(T) \subseteq \mathcal{R}(T)$, or equivalently S is solution of the operator equation $AX = T^*A$. Moreover, it is not difficult to see that $T \in \mathcal{B}_A(H)$ admits an A-adjoint if and only if $T \in \mathcal{B}_{A^2}(H)$. In this case there exists a distinguished A-adjoint of T which we shall denote by T_A (see [21]).

Let P denote the orthogonal projection onto the space $\overline{\mathcal{R}(A^{\frac{1}{2}})}$. Equip the space $\mathcal{R}(A^{\frac{1}{2}})$ with the inner product defined by

$$(A^{\frac{1}{2}}x, A^{\frac{1}{2}}y) := \langle Px, Py\rangle \text{ for all } x, y \in H.$$

Throughout the remainder of this paper, we shall use the symbols $\mathbf{R}(A^{\frac{1}{2}})$ and $\|\cdot\|_{\mathbf{R}(A^{\frac{1}{2}})}$ to denote the Hilbert space $(\mathcal{R}(A^{\frac{1}{2}}), (\cdot,\cdot))$ and the norm induced by the inner product $(\cdot,\cdot)$, respectively. Notice that a simple calculation shows that $\|Ax\|_{\mathbf{R}(A^{\frac{1}{2}})} = \|x\|_A$ for every $x \in H$.

Let $T \in \mathcal{B}(H)$. The A-numerical range and the A-numerical radius of T are defined respectively by

$$W_A(T) = \left\{\langle Tx, x\rangle_A : x \in H, \|x\|_A = 1\right\}$$

and

$$w_A(T) = \sup\left\{|\lambda| : \lambda \in W_A(T)\right\}.$$

It is worthwhile to note that $W_A(T)$ is a nonempty convex subset of $\mathbb{C}$ which may not be closed (see [5]). Moreover, the A-numerical radius $w_A(\cdot)$ is a semi-norm which is equivalent to $\|\cdot\|_A$; more precisely, $\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A$ for all operators $T \in \mathcal{B}_A(H)$. For more details about the A-numerical range and A-numerical radius, we refer to [1, 5, 7, 26] and references therein. The A-Crawford number of T is given by

$$c_A(T) = \inf\left\{|\lambda| : \lambda \in W_A(T)\right\}.$$

For $T \in \mathcal{B}(H)$ and $|q| \leq 1$, the q-numerical range of T is defined by

$$W_q(T) = \left\{\langle Tx, y\rangle : (x, y) \in C_q\right\},$$

where

$$C_q = \left\{(x, y) \in H \times H : \|x\| = \|y\| = 1;\ \langle x, y\rangle = q\right\}.$$

The q-numerical radius of T is defined by

$$w_q(T) = \sup\left\{|\lambda| : \lambda \in W_q(T)\right\}.$$

For $q = 1$, $W_q(T)$ and $w_q(T)$ reduce to the classical numerical range $W(T)$ and classical numerical radius $w(T)$ of T, respectively. For more results about the q-numerical range, the reader is referred to [18, 20] and references therein.

The numerical range and its variants are useful tools in operator theory, matrix analysis and numerical analysis, see for example [16, 25] and references therein. The A-numerical range and the q-numerical range of an operator have been studied by many authors, see [1, 5, 7, 8, 10, 11, 15, 18–20, 26] and references therein for an overview about the subject. In this paper, we introduce the concept of q-A-numerical range of an operator and study its basic properties. The remainder of the paper is organized as follows. In the beginning of Section 2, we introduce the q-A-numerical range of an operator $T \in \mathcal{B}(H)$ and investigate its basic properties. Next, we give several estimates of the q-A-numerical radius of T. Our q-A-numerical radius inequalities are natural generalizations of some existing A-numerical radius and q-numerical radius inequalities, see [10, 15]. Section 3 is devoted to estimates of q-A-numerical radius for some $n \times n$ operator matrices regarded as operators on the direct sum $\oplus_{i=1}^{n} H$. The obtained results generalize many inequalities established in [15]. In Section 4, we study the center of the q-A-numerical range of an operator and give some related results.

For a scalar $z \in \mathbb{C}$, let $\overline{z}$ and $\Re(z)$ denote the complex conjugate and the real part of z, respectively. If $K \subseteq \mathbb{C}$, we set $K^* = \{\overline{\lambda} : \lambda \in K\}$.

2. q-A-numerical range

In this section, we define the q-A-numerical range of an operator $T \in \mathcal{B}(H)$ and describe its basic properties, extending known results on the q-numerical range.

Definition 2.1. Let $T \in \mathcal{B}(H)$ and $|q| \leq 1$. We define the q-A-numerical range of T by

$$W_{q,A}(T) = \left\{\langle Tx, y\rangle_A : x, y \in H, \|x\|_A = \|y\|_A = 1, \langle x, y\rangle_A = q\right\}.$$

The q-A-numerical radius of T is given by

$$w_{q,A}(T) = \sup\left\{|\lambda| : \lambda \in W_{q,A}(T)\right\}.$$

When $A = I$ (I: the identity operator), $W_{q,A}(T)$ reduces to the q-numerical range $W_q(T)$ of T.

We begin with the following theorem.

Theorem 2.2. *If $T \in \mathcal{B}(H)$ and $|q| \leq 1$, then the following assertions hold.*

1. *If $\dim H = 1$, then $W_{q,A}(T)$ is nonempty if and only if $|q| = 1$, in which case if $A = [a]$ and $T = [t]$, then $W_{q,A}(T) = \{atq\}$,*
2. *$W_{q,A}(\alpha I + \lambda T) = \alpha q + \lambda W_{q,A}(T)$ for all $\alpha, \lambda \in \mathbb{C}$,*
3. *$W_{\lambda q,A}(T) = \lambda W_{q,A}(T)$ for all $|\lambda| = 1$,*
4. *If $U \in \mathcal{B}(H)$ is unitary and $AU = UA$, then $W_{q,A}(U^*TU) = W_{q,A}(T)$,*
5. *If $AT = TA$, then $(W_{q,A}(T))^* = W_{\overline{q},A}(T^*)$,*
6. *If $AT = TA$, then $W_{q,A}(T) \subseteq W_q(T)$.*

Proof. The assertions (1), (2) and (3) are easy to prove, so their proofs will be omitted.

To prove (4), consider $\lambda \in W_{q,A}(U^*TU)$. There exist two vectors $x, y \in H$ such that $\|x\|_A = \|y\|_A = 1$, $\langle x, y\rangle_A = q$ and $\lambda = \langle U^*TUx, y\rangle_A$. Since $AU = UA$, then $AU^* = U^*A$, and so

$$\lambda = \langle AU^*TUx, y\rangle = \langle ATUx, Uy\rangle =< TUx, Uy >_A .$$

On the other hand, since U is unitary, then

$$\|Ux\|_A^2 = \langle AUx, Ux\rangle = \langle UAx, Ux\rangle = \|x\|_A^2 = 1.$$

By using a similar argument, it follows that $\|Uy\|_A = 1$ and $\langle Ux, Uy\rangle_A = \langle x, y\rangle_A = q$. Thus $\lambda \in W_{q,A}(T)$. This shows that

$$W_{q,A}(U^*TU) \subseteq W_{q,A}(T).$$

Since $W_{q,A}(T) = W_{q,A}(U^*(UTU^*)U)$, we derive the equality

$$W_{q,A}(U^*TU) = W_{q,A}(T).$$

To prove (5), let $\lambda = \langle Tx, y\rangle_A \in W_{q,A}(T)$, where $x, y \in H$ with $\|x\|_A = \|y\|_A = 1$ and $\langle x, y\rangle_A = q$. Since $AT = TA$ and $\langle y, x\rangle_A = \langle y, Ax\rangle = \overline{q}$, then $\overline{\lambda} = \langle T^*y, x\rangle_A \in W_{\overline{q},A}(T^*)$. This implies that

$$(W_{q,A}(T))^* \subseteq W_{\overline{q},A}(T^*).$$

The reverse inclusion obviously follows from the first inclusion since $T^{**} = T$ and $\overline{\overline{q}} = q$. This completes the proof of (5).

To prove (6), suppose that $AT = TA$. Then $A^{\frac{1}{2}}T = TA^{\frac{1}{2}}$. Now, if $x, y \in H$ are given such that $\|x\|_A = \|y\|_A = 1$ and $\langle x, y\rangle_A = q$, then $\|A^{\frac{1}{2}}x\| = \|A^{\frac{1}{2}}y\| = 1$ and $\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\rangle = \langle Ax, y\rangle = q$. Thus

$$\langle Tx, y\rangle_A = \langle TA^{\frac{1}{2}}x, A^{\frac{1}{2}}y\rangle \in W_q(T).$$

From this we derive that

$$W_{q,A}(T) \subseteq W_q(T).$$

□

For an arbitrary operator $T \in \mathcal{B}(H)$, the set $W_{q,A}(T)$ may not be bounded. To see this, consider the following example.

Example 2.3. *Let A and T be the 2×2 matrices defined by $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively. Then*

$$W_{1,A}(T) = \left\{ y\bar{a} : x, a, y \in \mathbb{C}, |x|^2 = |a|^2 = 1, x\bar{a} = 1 \right\}.$$

Thus

$$W_{1,A}(T) = \mathbb{C}.$$

In particular, $w_{1,A}(T) = +\infty$.
Since $w_q(T) \leq \|T\| = 1$, then we deduce that the inequality $w_{q,A}(T) \leq w_q(T)$ does not hold in general. Moreover, this example shows that the commutativity condition in Part (6) *of the above theorem cannot be dropped.*

Remark 2.4. *The above example also shows that $W_{q,A}(T)$ may be empty. Indeed, consider the operators A and T defined in Example 2.3. If $q = \frac{1}{2}$, then a straightforward computation shows that $W_{q,A}(T) = \emptyset$.*

Proposition 2.5. *Suppose* $\dim H \geq 2$ *and let $T \in \mathcal{B}(H)$. If $|q| \leq 1$ and A is invertible, then $W_{q,A}(T)$ is nonempty.*

Proof. It is not difficult to show that there exist unit vectors $a, b \in H$ such that $\langle a, b\rangle = q$. Since A is invertible, then $A^{\frac{1}{2}}$ is also invertible. Thus, there exist $x, y \in H$ such that $a = A^{\frac{1}{2}}x$ and $b = A^{\frac{1}{2}}y$. Hence $\|x\|_A = \|y\|_A = 1$ and $\langle x, y\rangle_A = \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}y\rangle = \langle a, b\rangle = q$. Thus $\langle Tx, y\rangle_A \in W_{q,A}(T)$. This implies that $W_{q,A}(T)$ is nonempty, which ends the proof. □

Proposition 2.6. *Let $T \in \mathcal{B}(H)$ and $|q| \leq 1$. If A is invertible and $AT = TA$, then*

$$W_{q,A}(T) = W_q(T).$$

Proof. By Part (6) of Theorem 2.2, it suffices to show $W_q(T) \subseteq W_{q,A}(T)$. To do so let $\lambda \in W_q(T)$. Then there exists $(a,b) \in C_q$ such that $\lambda = \langle Ta, b\rangle$. Since A is invertible, there exist $x, y \in H$ such that $a = A^{\frac{1}{2}}x$ and $b = A^{\frac{1}{2}}y$. Thus

$$\lambda = \langle Ta, b\rangle = \langle TA^{\frac{1}{2}}x, A^{\frac{1}{2}}y\rangle = \langle A^{\frac{1}{2}}Tx, A^{\frac{1}{2}}y\rangle = \langle ATx, y\rangle = \langle Tx, y\rangle_A.$$

Also, we easily check that $\|x\|_A = \|y\|_A = 1$ and $\langle x, y\rangle_A = \langle a, b\rangle = q$. Hence

$$\lambda = \langle Tx, y\rangle_A \in W_{q,A}(T).$$

This completes the proof. □

Proposition 2.7. *Let $T \in \mathcal{B}_A(H)$ and $|q| \leq 1$. Then*

$$W_{q,A}(T_A) = (W_{\bar{q},A}(T))^*.$$

Proof. Let $\lambda \in W_{q,A}(T_A)$. We can write $\lambda = \langle T_A x, y\rangle_A$, where $x, y \in H$ satisfy $\|x\|_A = \|y\|_A = 1$ and $\langle x, y\rangle_A = q$. Hence

$$\lambda = \langle AT_A x, y\rangle = \overline{\langle Ty, x\rangle_A}.$$

Since $\langle y, x\rangle_A = \overline{q}$, then $\lambda \in (W_{\bar{q},A}(T))^*$. Therefore

$$W_{q,A}(T_A) \subseteq (W_{\bar{q},A}(T))^*.$$

Using a similar argument, we can also prove the reverse inclusion. Thus we get the desired equality. □

An immediate consequence of the above theorem is the following equality.

Corollary 2.8. *If $T \in \mathcal{B}_A(H)$ and $0 \leq q \leq 1$, then*

$$w_{q,A}(T_A) = w_{q,A}(T).$$

Proposition 2.9. *If $T \in \mathcal{B}_A(H)$, then*

$$w_{q,A}(T) \leq \|T\|_A.$$

Proof. Since $T \in \mathcal{B}_A(H)$, then

$$\|T\|_A = \sup\left\{|\langle Tx, y\rangle_A| : x, y \in H,\ \|x\|_A = \|y\|_A = 1\right\}.$$

Hence

$$w_{q,A}(T) \leq \|T\|_A.$$

□

The following theorem is a generalization of [12, Theorem 2]. The proof is similar to that given in [12], we give it here for the sake of completness.

Theorem 2.10. *Let $T \in \mathcal{B}(H)$ be given such that $T(\mathcal{N}(A)) \nsubseteq \mathcal{N}(A)$. Then*

$$w_{q,A}(T) = +\infty \text{ for every } |q| \leq 1.$$

Proof. With respect to the decomposition $H = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, we have

$$\begin{aligned} W_{q,A}(T) = &\Big\{\langle Tx_1, y_2\rangle_A + \langle Tx_2, y_2\rangle_A : x_1 \in \mathcal{N}(A), x_2, y_2 \in \overline{\mathcal{R}(A)}, \\ &\|x_2\|_A = \|y_2\|_A = 1, \langle x_2, y_2\rangle_A = q\Big\}. \end{aligned} \tag{2.1}$$

Since $T(\mathcal{N}(A)) \nsubseteq \mathcal{N}(A)$, then there exists $a \in \mathcal{N}(A)$ such that $ATa \neq 0$. This implies that $A^{\frac{1}{2}}ATa \neq 0$. Indeed, if $A^{\frac{1}{2}}ATa = 0$, then $A^2Ta = 0$. Hence $\|ATa\|^2 = \langle A^2Ta, Ta\rangle = 0$ and so $ATa = 0$ leading to a contradiction. Next, let $b = \frac{ATa}{\|ATa\|_A} \in \mathcal{R}(A)$. Clearly we have $\|b\|_A = 1$. Then it follows from (2.1) that

$$\begin{aligned} W_{q,A}(T) &\supseteq \left\{\lambda\langle Ta, b\rangle_A + \langle Tx_2, b\rangle_A : x_2 \in \overline{\mathcal{R}(A)}, \|x_2\|_A = 1, \langle x_2, b\rangle_A = q, \lambda \in \mathbb{C}\right\} \\ &\supseteq \left\{\lambda \frac{\|ATa\|^2}{\|ATa\|_A} + \langle Tx_2, b\rangle_A : x_2 \in \overline{\mathcal{R}(A)}, \|x_2\|_A = 1, \langle x_2, b\rangle_A = q, \lambda \in \mathbb{C}\right\} \\ &= \mathbb{C}. \end{aligned}$$

Therefore

$$w_{q,A}(T) = +\infty.$$

□

The following theorem shows that $W_{q,A}(T)$ is a convex subset of$\mathbb{C}$ for all $T \in \mathcal{B}(H)$ and $0 \leq q \leq 1$.

Theorem 2.11. *If $0 \leq q \leq 1$, then $W_{q,A}(T)$ is convex for all $T \in \mathcal{B}(H)$.*

Proof. Consider the following two cases:

1. A is injective : In this case $(H, \langle ., .\rangle_A)$ is a pre-Hilbert space. Thus the convexity of $W_{q,A}(T)$ follows from [24, Theorem 1].
2. A is not injective : Suppose first that $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. From (2.1), we have

$$\begin{aligned} W_{q,A}(T) &= \left\{\langle PTx_2, y_2\rangle_A : x_2, y_2 \in \overline{\mathcal{R}(A)}, \|x_2\|_A = \|y_2\|_A = 1, \langle x_2, y_2\rangle_A = q\right\} \\ &= W_{q,A_1}(T_1), \end{aligned}$$

   where P denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$, $A_1 = A \mid_{\overline{\mathcal{R}(A)}}$ and $T_1 = PT \mid_{\overline{\mathcal{R}(A)}}$. Since A_1 is an injective operator, we deduce from the first case above that $W_{q,A_1}(T_1)$ is convex. Hence $W_{q,A}(T)$ is convex.

   Next, if $T(\mathcal{N}(A)) \nsubseteq \mathcal{N}(A)$, then by Theorem 2.10 we have $W_{q,A}(T) = \mathbb{C}$ which is convex.

□

Define the operator $Z_A : H \to \mathbf{R}(A^{\frac{1}{2}})$ by $Z_A(x) = Ax$. If $T \in \mathcal{B}(\mathbf{R}(A^{\frac{1}{2}}))$, then by $\|T\|_{\mathbf{R}(A^{\frac{1}{2}})}$ we denote its norm as an operator acting on the Hilbert space $\mathbf{R}(A^{\frac{1}{2}})$.

To prove our upcoming result, we need the following proposition which holds by combining Proposition 3.6 and Proposition 3.9 in [2].

Proposition 2.12. *Let $T \in \mathcal{B}(H)$. Then $T \in \mathcal{B}_A(H)$ if and only if there exists a unique $\widetilde{T} \in \mathcal{B}(\mathbf{R}(A^{\frac{1}{2}}))$ such that $Z_AT = \widetilde{T}Z_A$. Moreover, $\|\widetilde{T}\|_{\mathbf{R}(A^{\frac{1}{2}})} = \|T\|_A$.*

Suppose that $T \in \mathcal{B}(H)$ is A-bounded. In all what follows we denote by $\widetilde{T} \in \mathcal{B}(\mathbf{R}(A^{\frac{1}{2}}))$ the unique operator induced by T in Proposition 2.12.

Theorem 2.13. *If $T \in \mathcal{B}_A(H)$ and $|q| \leq 1$, then*

$$W_{q,A}(T) \subseteq W_q(\widetilde{T}) \subseteq \overline{W_{q,A}(T)}.$$

In particular,

$$w_{q,A}(T) = w_q(\widetilde{T}).$$

Proof. Let $\lambda \in W_{q,A}(T)$. Then we can find $x, y \in H$ with $\|x\|_A = \|y\|_A = 1$ and $\langle x, y\rangle_A = q$ such that $\lambda = \langle Tx, y\rangle_A$. A straightforward calculation shows that $\|Ax\|_{\mathbf{R}(A^{\frac{1}{2}})} = \|Ay\|_{\mathbf{R}(A^{\frac{1}{2}})} = 1$ and $(Ax, Ay) = q$. Thus, by virtue of Proposition 2.12, we have

$$\begin{aligned}\lambda &= \langle A^{\frac{1}{2}}Tx, A^{\frac{1}{2}}y\rangle \\ &= (ATx, Ay) \\ &= (Z_A Tx, Ay) \\ &= (\widetilde{T}Z_A x, Ay) \\ &= (\widetilde{T}Ax, Ay) \in W_q(\widetilde{T}).\end{aligned}$$

This shows that

$$W_{q,A}(T) \subseteq W_q(\widetilde{T})\cdot$$

For the second inclusion, assume that $\lambda \in W_q(\widetilde{T})$. Then $\lambda = (\widetilde{T}A^{\frac{1}{2}}x, A^{\frac{1}{2}}y)$, where x and y are two vectors in H with $\|A^{\frac{1}{2}}x\|_{\mathbf{R}(A^{\frac{1}{2}})} = \|A^{\frac{1}{2}}y\|_{\mathbf{R}(A^{\frac{1}{2}})} = 1$ and $(A^{\frac{1}{2}}x, A^{\frac{1}{2}}y) = q$. Since, by [2], the subspace $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{\frac{1}{2}})$, there exist sequences $\{x_n\}_n$, $\{y_n\}_n \subseteq H$ such that $A^{\frac{1}{2}}x = \lim_n Ax_n$ and $A^{\frac{1}{2}}y = \lim_n Ay_n$. Hence

$$\lim_n \|x_n\|_A = \lim_n \|Ax_n\|_{\mathbf{R}(A^{\frac{1}{2}})} = \|A^{\frac{1}{2}}x\|_{\mathbf{R}(A^{\frac{1}{2}})} = 1.$$

Similarly, we have $\lim_n \|y_n\|_A = 1$. Furthermore,

$$\lim_n \langle x_n, y_n\rangle_A = \lim_n (Ax_n, Ay_n) = (A^{\frac{1}{2}}x, A^{\frac{1}{2}}y) = q.$$

For each n, set $x'_n = \frac{x_n}{\|x_n\|_A}$, $y'_n = \frac{y_n}{\|y_n\|_A}$ and $q_n = \langle x'_n, y'_n\rangle_A$. Then $\|x'_n\|_A = \|y'_n\|_A = 1$ and q_n converges to q. Now, by Proposition 2.12 we have

$$\lambda = \lim_n (\widetilde{T}Ax_n, Ay_n) = \lim_n (ATx_n, Ay_n) = \lim_n \langle Tx'_n, y'_n\rangle_A \in \overline{W_{q_n,A}(T)}.$$

If the operator A is injective, then $(H, \langle\cdot,\cdot\rangle_A)$ is a pre-Hilbert space. Hence we derive from [20, Theorem 2.9] that

$$\lambda \in \overline{W_{q,A}(T)}.$$

Next, if A is not injective, then $W_{q_n,A}(T) = W_{q_n,A_1}(T_1)$, where $A_1 = A \mid_{\overline{\mathcal{R}(A)}}$ and $T_1 = PT \mid_{\overline{\mathcal{R}(A)}}$. Since A_1 is injective, we deduce as in the above that $\lambda \in \overline{W_{q,A}(T)}$.

This completes the proof. □

As a consequence of Theorem 2.13, we obtain the following corollary.

Corollary 2.14. *If $T \in \mathcal{B}_A(H)$ and $|q| \le 1$, then*

$$\overline{W_q(\widetilde{T})} = \overline{W_{q,A}(T)}.$$

In particular,

$$w(\widetilde{T}) = w_A(T) \text{ and } c(\widetilde{T}) = c_A(T).$$

Theorem 2.15. *Let $T \in \mathcal{B}_A(H)$ be injective. If $|q| \le 1$, then*

$$\frac{q}{2}\left(\|T\|_A + \frac{c_A^2(T)}{\|T\|_A}\right) \le w_{q,A}(T) \le \|T\|_A.$$

If in addition, T is A-normaloid then

$$q\|T\|_A \le w_{q,A}(T).$$

Proof. By [10, Theorem 2.3], we have

$$\frac{q}{2}\left(\|\widetilde{T}\| + \frac{c^2(\widetilde{T})}{\|\widetilde{T}\|}\right) \leq w_q(\widetilde{T}) \leq \|\widetilde{T}\|$$

and if $\widetilde{T}$ is normaloid, then

$$q\|\widetilde{T}\| \leq w_q(\widetilde{T}).$$

Thus the proof follows by combining Proposition 2.12 and Theorem 2.13 and the fact that T is A-normaloid if and only if $\widetilde{T}$ is normaloid. □

In [5], it was shown that, for all $T \in \mathcal{B}_A(H)$,

$$\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A.$$

The following corollary gives a generalization of this inequality.

Corollary 2.16. *If $T \in \mathcal{B}_A(H)$ and $|q| \leq 1$, then*

$$\frac{q}{2}\|T\|_A \leq w_{q,A}(T) \leq \|T\|_A.$$

The next theorem provides an upper bound for the set $\left\{w_{q,A}(T) : q \in [0,1]\right\}$, where $T \in \mathcal{B}_A(H)$.

Theorem 2.17. *If $T \in \mathcal{B}_A(H)$, then*

$$\|T\|_A = \sup\left\{w_{q,A}(T) : q \in [0,1]\right\}.$$

Proof. By virtue of [10, Theorem 2.1], we derive that

$$\|\widetilde{T}\|_{\mathbf{R}(A^{\frac{1}{2}})} = \sup\left\{w_q(\widetilde{T}) : q \in [0,1]\right\}.$$

Thus the proof follows by combining Proposition 2.12 and Theorem 2.13. □

The next theorem provides some estimates for the q-A-numerical radius of an operator.

Theorem 2.18. *Let $T \in \mathcal{B}_A(H)$ and $q \in [0,1]$. The following assertions hold.*

1. *If $q \geq \frac{1}{\sqrt{2}}$, then*

$$w_{q,A}^2(T) \leq (2q^2-1)w_A^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|_A^2.$$

2. *If $q < \frac{1}{\sqrt{2}}$, then*

$$w_{q,A}^2(T) \leq (2q^2-1)c_A^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|_A^2.$$

Proof. 1. Assume that $q \geq \frac{1}{\sqrt{2}}$. By applying [10, Theorem 2.2] for the operator $\widetilde{T}$, we get

$$w_q^2(\widetilde{T}) \leq (2q^2-1)w^2(\widetilde{T}) + (1-q^2+q\sqrt{1-q^2})\|\widetilde{T}\|_{\mathbf{R}(A^{\frac{1}{2}})}^2. \tag{2.2}$$

Since by Proposition 2.12, $\|\widetilde{T}\|_{\mathbf{R}(A^{\frac{1}{2}})} = \|T\|_A$, and by Theorem 2.13, $w_{q,A}(T) = w_q(\widetilde{T})$, it follows from (2.2) that

$$w_{q,A}^2(T) \leq (2q^2-1)w_A^2(T) + (1-q^2+q\sqrt{1-q^2})\|T\|_A^2.$$

2. If $q < \frac{1}{\sqrt{2}}$, then by applying [10, Theorem 2.2] once again, we get

$$w_q^2(\widetilde{T}) \leq (2q^2 - 1)c^2(\widetilde{T}) + (1 - q^2 + q\sqrt{1 - q^2})\|\widetilde{T}\|^2_{\mathbf{R}(A^{\frac{1}{2}})}.$$

Therefore, by Proposition 2.12, Theorem 2.13 and Corollary 2.14, we deduce that

$$w_{q,A}^2(T) \leq (2q^2 - 1)c_A^2(T) + \left(1 - q^2 + q\sqrt{1 - q^2}\right)\|T\|_A^2$$

which completes the proof.
□

Let $T \in \mathcal{B}_A(H)$. Feki, in [12, Theorem 7], proved that

$$w_A(T) \leq \frac{1}{2}(\|T\|_A + \|T^2\|_A^{\frac{1}{2}}). \tag{2.3}$$

By using Theorem 2.18, Part (1) and the inequality in (2.3), we get the following upper bound of $w_{q,A}(T)$ for all $T \in \mathcal{B}_A(H)$.

Corollary 2.19. *If $T \in \mathcal{B}_A(H)$ and $\frac{1}{\sqrt{2}} \leq q \leq 1$, then*

$$w_{q,A}^2(T) \leq \Big(\frac{3}{4} - \frac{q^2}{2} + q\sqrt{1 - q^2}\,\Big)\|T\|_A^2 + \Big(\frac{q^2}{2} - \frac{1}{4}\Big)\Big(\|T^2\|_A + 2\|T\|_A\|T^2\|_A^{\frac{1}{2}}\Big).$$

If, in particular $AT^2 = 0$, then

$$w_{q,A}(T) \leq \sqrt{\frac{3}{4} - \frac{q^2}{2} + q\sqrt{1 - q^2}}\,\|T\|_A.$$

Combining the above theorem and [14, Theorem 2.2], we also have the following result.

Corollary 2.20. *If $T \in \mathcal{B}_A(H)$ and $\frac{1}{\sqrt{2}} \leq q \leq 1$, then*

$$w_{q,A}^2(T) \leq \Big(\frac{3}{4} - \frac{q^2}{2} + q\sqrt{1 - q^2}\,\Big)\|T\|_A^2 + \Big(\frac{q^2}{2} - \frac{1}{4}\Big)w_A(T^2).$$

Let $S, T \in \mathcal{B}(H)$ and $|q| \leq 1$. Obviously $w_{q,A}(T + S) \leq w_{q,A}(T) + w_{q,A}(S)$. So it seems interesting to find conditions under which there is equality $w_{q,A}(T + S) = w_{q,A}(T) + w_{q,A}(S)$. Such equality has been studied by Kaadoud and Moulaharabbi [18, Theorem 3.1] in the special case where A is the identity operator.

Theorem 2.21. *Let $S, T \in \mathcal{B}_A(H)$ and $|q| \leq 1$. The following assertions are equivalent:*

1. $w_{q,A}(T + S) = w_{q,A}(T) + w_{q,A}(S)$,
2. *There exit sequences $\{x_n\}_n$ and $\{y_n\}_n$ of vectors in H with $\|x_n\|_A = \|y_n\|_A = 1$ and $\langle x_n, y_n\rangle_A = q$ such that*

$$\lim_n \langle T_A y_n, x_n\rangle_A \langle S x_n, y_n\rangle_A = w_{q,A}(T)w_{q,A}(S).$$

Proof. Assume that $w_{q,A}(T + S) = w_{q,A}(T) + w_{q,A}(S)$. Then there exist sequences $\{x_n\}_n$ and $\{y_n\}_n$ of vectors in H with $\|x_n\|_A = \|y_n\|_A = 1$ and $\langle x_n, y_n\rangle_A = q$ such that

$$w_{q,A}(T) + w_{q,A}(S) = \lim_n |\langle (T + S)x_n, y_n\rangle_A|.$$

For every n, we have

$$\begin{aligned} |\langle (T+S)x_n, y_n\rangle_A|^2 &= |\langle Tx_n, y_n\rangle_A|^2 + |\langle Sx_n, y_n\rangle_A|^2 + 2\Re(\overline{\langle Tx_n, y_n\rangle_A}\langle Sx_n, y_n\rangle_A) \\ &= |\langle Tx_n, y_n\rangle_A|^2 + |\langle Sx_n, y_n\rangle_A|^2 + 2\Re(\langle T_A y_n, x_n\rangle_A\langle Sx_n, y_n\rangle_A) \\ &\leq |\langle Tx_n, y_n\rangle_A|^2 + |\langle Sx_n, y_n\rangle_A|^2 + 2|\langle T_A y_n, x_n\rangle_A||\langle Sx_n, y_n\rangle_A| \\ &= \Big(|\langle Tx_n, y_n\rangle_A| + |\langle Sx_n, y_n\rangle_A|\Big)^2 \\ &\leq \Big(w_{q,A}(T) + w_{q,A}(S)\Big)^2. \end{aligned}$$

Hence, we derive that

$$\lim_n \langle T_A y_n, x_n\rangle_A\langle Sx_n, y_n\rangle_A = w_{q,A}(T)w_{q,A}(S)\cdot$$

Conversely, suppose that (2) holds. Then

$$\lim_n \Re(\langle T_A y_n, x_n\rangle_A\langle Sx_n, y_n\rangle_A) = w_{q,A}(T)w_{q,A}(S)\cdot$$

Since $|\langle T_A y_n, x_n\rangle_A| \leq w_{q,A}(T)$ and $|\langle Sx_n, y_n\rangle_A| \leq w_{q,A}(S)$, it follows that

$$\lim_n |\langle T_A y_n, x_n\rangle_A| = w_{q,A}(T) \text{ and } \lim_n |\langle Sx_n, y_n\rangle_A| = w_{q,A}(S)\cdot$$

This implies that

$$\begin{aligned} w_{q,A}(T) + w_{q,A}(S) &= \lim_n |\langle (T+S)x_n, y_n\rangle_A| \\ &\leq w_{q,A}(T+S) \\ &\leq w_{q,A}(T) + w_{q,A}(S)\cdot \end{aligned}$$

Therefore (1) holds and the proof is complete. □

3. *q-A*-numerical range of an operator matrix

In this section, we investigate the q-A-numerical range of some operator matrices. Observe that since A is positive, then for every n, the $n \times n$ operator matrix

$$\mathbb{A}_n = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & 0 & \vdots \\ \vdots & \dots & \dots & 0 \\ 0 & \dots & 0 & A \end{pmatrix} \tag{3.4}$$

defined on the space $\oplus_{i=1}^n H$ is also positive. Furthermore, it is not difficult to show that if $T_{ij} \in \mathcal{B}_A(H)$, $i, j = 1, \cdots, n$, then the operator matrix $T = \Big(T_{ij}\Big)_{i,j} \in \mathcal{B}_{\mathbb{A}}(\oplus_{i=1}^n H)$.

In all what follows we shall denote $\mathbb{A}_n$ the positive operator defined by (3.4). For simplicity we omit n and simply write $\mathbb{A} = \mathbb{A}_n$.

In the sequel we will need the following lemma.

Lemma 3.1. *Let* $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$, *where* $T_{11}, T_{12}, T_{21}, T_{22} \in \mathcal{B}_A(H)$. *Then*

$$\widetilde{T} = \begin{pmatrix} \widetilde{T_{11}} & \widetilde{T_{12}} \\ \widetilde{T_{21}} & \widetilde{T_{22}} \end{pmatrix}\cdot$$

Proof. We have

$$\begin{aligned}\begin{pmatrix}\widetilde{T_{11}} & \widetilde{T_{12}}\\ \widetilde{T_{21}} & \widetilde{T_{22}}\end{pmatrix}\mathbb{A} &= \begin{pmatrix}\widetilde{T_{11}}A & \widetilde{T_{12}}A\\ \widetilde{T_{21}}A & \widetilde{T_{22}}A\end{pmatrix}\\ &= \begin{pmatrix}AT_{11} & AT_{12}\\ AT_{21} & AT_{22}\end{pmatrix}\\ &= \mathbb{A}T.\end{aligned}$$

Since, by [2, Proposition 3.6], $\widetilde{T}$ is the unique operator that satisfies the equation $X\mathbb{A} = \mathbb{A}T$, we deduce that

$$\widetilde{T} = \begin{pmatrix}\widetilde{T_{11}} & \widetilde{T_{12}}\\ \widetilde{T_{21}} & \widetilde{T_{22}}\end{pmatrix}.$$

□

To prove the upcoming theorem, we also need the following lemma.

Lemma 3.2. *[22, Theorem 3.3] Let H and K be Hilbert spaces. If $T = \begin{pmatrix}B & C\\ D & E\end{pmatrix} \in \mathcal{B}(H \oplus K)$ and $q \in [0, 1]$, then*

1. $w_q(T) \leq \max\{\|B\|, \|E\|\} + \left(1 - \frac{3q^2}{4} + q\sqrt{1-q^2}\right)^{\frac{1}{2}}\left(\|C\| + \|D\|\right).$
2. $$\begin{aligned}w_q(T) \leq& \sqrt{1-q^2}\left(\|B\|^2 + \|C\|^2 + \|D\|^2 + \|E\|^2\right)^{\frac{1}{2}}\\ &+ q\left(\max\{w(B), w(E)\} + \frac{\|C\| + \|D\|}{2}\right).\end{aligned}$$
3. $\max\left\{w_q(B), w_q(E), w_q\begin{pmatrix}0 & C\\ D & 0\end{pmatrix}\right\} \leq w_q(T).$

For the $q - A-$numerical radius, we have the following analogous inequalities.

Theorem 3.3. *Let $T = \begin{pmatrix}B & C\\ D & E\end{pmatrix} \in \mathcal{B}(H \oplus H)$. If $q \in [0, 1]$, then*

1. $w_{q,\mathbb{A}}(T) \leq \max\{\|B\|_A, \|E\|_A\} + \left(1 - \frac{3q^2}{4} + q\sqrt{1-q^2}\right)^{\frac{1}{2}}\left(\|C\|_A + \|D\|_A\right).$
2. $$\begin{aligned}w_{q,\mathbb{A}}(T) \leq& \sqrt{1-q^2}\left(\|B\|_A^2 + \|C\|_A^2 + \|D\|_A^2 + \|E\|_A^2\right)^{\frac{1}{2}}\\ &+ q\left(\max\{w_A(B), w_A(E)\} + \frac{\|C\|_A + \|D\|_A}{2}\right).\end{aligned}$$
3. $\max\left\{w_{q,\mathbb{A}}(B), w_{q,\mathbb{A}}(E), w_{q,\mathbb{A}}\begin{pmatrix}0 & C\\ D & 0\end{pmatrix}\right\} \leq w_{q,\mathbb{A}}(T).$

Proof. By Lemma 3.1, we know that

$$\widetilde{T} = \begin{pmatrix}\widetilde{B} & \widetilde{C}\\ \widetilde{D} & \widetilde{E}\end{pmatrix} \in \mathcal{B}\left(\mathbf{R}(A^{\frac{1}{2}}) \oplus \mathbf{R}(A^{\frac{1}{2}})\right).$$

Hence, by applying Lemma 3.2 to the operator $\widetilde{T}$, we get

$$w_q(\widetilde{T}) \le \max\left\{\|\widetilde{B}\|_{\mathbf{R}(A^{\frac{1}{2}})}, \|\widetilde{E}\|_{\mathbf{R}(A^{\frac{1}{2}})}\right\} + \left(1 - \frac{3q^2}{4} + q\sqrt{1-q^2}\right)^{\frac{1}{2}} \times \left(\|\widetilde{C}\|_{\mathbf{R}(A^{\frac{1}{2}})} + \|\widetilde{D}\|_{\mathbf{R}(A^{\frac{1}{2}})}\right).$$

Now, by combining Proposition 2.12 and Theorem 2.13, we derive that

$$w_{q,\mathbb{A}}(T) \le \max\{\|B\|_A, \|E\|_A\} + \left(1 - \frac{3q^2}{4} + q\sqrt{1-q^2}\right)^{\frac{1}{2}} \left(\|C\|_A + \|D\|_A\right).$$

By using a similar reasoning, we also get the other inequalities. □

To present our next result, we need the following lemmas.

Lemma 3.4. *Let* $T = \begin{pmatrix} T_1 & T_2 & \dots & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{B}_{\mathbb{A}}(\oplus_{i=1}^n H)$. *If* $T_1, T_2, \dots, T_n \in \mathcal{B}_A(H)$, *then*

$$\widetilde{T} = \begin{pmatrix} \widetilde{T_1} & \widetilde{T_2} & \dots & \widetilde{T_n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Proof. Since for every i, we have $AT_i = \widetilde{T_i}A$, it follows that

$$\mathbb{A}T = \begin{pmatrix} \widetilde{T_1} & \widetilde{T_2} & \cdots & \widetilde{T_n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \mathbb{A}.$$

Therefore, the required equality follows from the uniqueness of the operator $\widetilde{T}$. □

Lemma 3.5. *[15] Let* $T = \begin{pmatrix} T_1 & T_2 & \dots & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{B}(\oplus_{i=1}^n H)$. *Then*

$$w(T) \le \frac{1}{2}\left(w(T_1) + \sqrt{w^2(T_1) + \sum_{j=2}^n \|T_j\|^2}\right).$$

The next theorem is a generalization of [15, Theorem 2.5]

Theorem 3.6. *Let* $T = \begin{pmatrix} T_1 & T_2 & ... & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{B}(\oplus_{i=1}^n H)$. *If* $T_1, T_2, \dots, T_n \in \mathcal{B}_A(H)$ *and* $\frac{1}{\sqrt{2}} \le q \le 1$, *then*

$$w_{q,\mathbb{A}}^2(T) \le \frac{2q^2-1}{4}\left(w_A(T_1) + \sqrt{w_A(T_1) + \sum_{j=2}^{n} \|T_j\|_A^2} \right)^2 + \left(1 - q^2 + q\sqrt{1-q^2}\right)\sum_{j=1}^{n} \|T_j\|_A^2. \tag{3.5}$$

Proof. By Theorem 2.18, we have

$$w_{q,\mathbb{A}}^2(T) \le (2q^2-1)w_{\mathbb{A}}^2(T) + \left(1 - q^2 + q\sqrt{1-q^2}\right)\|T\|_{\mathbb{A}}^2 . \tag{3.6}$$

Combining Corollary 2.14 and Lemma 3.4, we get

$$w_{\mathbb{A}}(T) = w(\widetilde{T}) = w\left(\begin{pmatrix} \widetilde{T_1} & \widetilde{T_2} & ... & \widetilde{T_n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}\right).$$

Hence

$$w_{\mathbb{A}}(T) \le \frac{1}{2}\left(w(\widetilde{T_1}) + \sqrt{w^2(\widetilde{T_1}) + \sum_{j=2}^{n} \|\widetilde{T_j}\|^2_{\mathbf{R}(A^{\frac{1}{2}})}} \right) \tag{3.7}$$

$$= \frac{1}{2}\left(w_A(T_1) + \sqrt{w_A^2(T_1) + \sum_{j=2}^{n} \|T_j\|_A} \right), \tag{3.8}$$

where the inequality in (3.7) follows from Lemma 3.5, and the equality in (3.8) follows by applying again Corollary 2.14.

On the other hand, we have

$$\begin{aligned} \|T\|_{\mathbb{A}}^2 &= \left\|\widetilde{T}\right\|^2_{\mathbf{R}(\mathbb{A}^{\frac{1}{2}})} \\ &= \left\|\begin{pmatrix} \widetilde{T_1} & \widetilde{T_2} & ... & \widetilde{T_n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}\right\|^2_{\mathbf{R}(\mathbb{A}^{\frac{1}{2}})} \\ &= \sum_{j=1}^{n} \|\widetilde{T_j}\|^2_{\mathbf{R}(A^{\frac{1}{2}})} \\ &= \sum_{j=1}^{n} \|T_j\|_A^2. \end{aligned} \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), we deduce (3.5) which completes the proof. □

Lemma 3.7. *[15] Let* $T = \begin{pmatrix} T_1 & T_2 & \dots & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{B}(\oplus_{i=1}^n H)$. *Then*

$$w(T) \le \frac{1}{2}\left(\|T_1\| + \sqrt{\|\sum_{j=2}^n T_j T_j^*\|^2} \right).$$

Lemma 3.8. *[21] Let* $T \in \mathcal{B}_{A^2}(H)$. *Then*

$$\widetilde{T_A} = (\widetilde{T})^* \text{ and } \widetilde{T} = [\widetilde{(T_A)_A}].$$

Theorem 3.9. *Let* $T = \begin{pmatrix} T_1 & T_2 & ... & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{B}(\oplus_{i=1}^n H)$. *If* $T_1, T_2, \dots, T_n \in \mathcal{B}_{A^2}(H)$, *then*

$$w_{\mathbb{A}^2}(T) \le \frac{1}{2}\left(\|T_1\|_{A^2} + \sqrt{\|\sum_{j=2}^n T_j T_{jA}\|_{A^2}^2} \right).$$

Proof. From the proof of Theorem 3.6, we have

$$w_{\mathbb{A}^2}(T) = w(\widetilde{T}) = w\left(\begin{pmatrix} \widetilde{T_1} & \widetilde{T_2} & ... & \widetilde{T_n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right).$$

Then by Lemma 3.7, we infer that

$$w_{\mathbb{A}^2}(T) \le \frac{1}{2}\left(\|\widetilde{T_1}\|_{\mathbf{R}(A)} + \sqrt{\|\sum_{j=2}^n \widetilde{T_j}(\widetilde{T_j})^*\|_{\mathbf{R}(A)}^2} \right).$$

By Lemma 3.8, we have

$$\widetilde{T_j}(\widetilde{T_j})^* = \widetilde{T_j}\,\widetilde{(T_j)_A}.$$

Since, by [13, Lemma 2.1], $\widetilde{XY} = \widetilde{X}\widetilde{Y}$ and $\widetilde{X+Y} = \widetilde{X} + \widetilde{Y}$ for every $X, Y \in \mathcal{B}_{A^2}(H)$, we derive that $\widetilde{T_j}(\widetilde{T_j})^* = \widetilde{T_j(T_j)_A}$ and

$$\begin{aligned} \|\sum_{j=2}^n \widetilde{T_j}(\widetilde{T_j})^*\|_{\mathbf{R}(A)} &= \|\sum_{j=2}^n \widetilde{T_j(T_j)_A}\|_{\mathbf{R}(A)} \\ &= \|\widetilde{\sum_{j=2}^n T_j(T_j)_A}\|_{\mathbf{R}(A)} \\ &= \|\sum_{j=2}^n T_j(T_j)_A\|_{A^2}. \end{aligned}$$

Then

$$w(\widetilde{T}) \le \frac{1}{2}\left(\|T_1\|_{A^2} + \sqrt{\|\sum_{j=2}^n T_j(T_j)_A\|_{A^2}^2} \right).$$

Therefore

$$w_{\mathbb{A}^2}(T) \leq \frac{1}{2}\left(\|T_1\|_{A^2} + \sqrt{\|\sum_{j=2}^{n} T_j(T_j)_A\|_{A^2}^2}\right).$$

□

Proposition 3.10. *Let* $T = \begin{pmatrix} T_1 & T_2 & ... & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{B}(\oplus_{i=1}^n H)$*. If* $T_1, T_2, \dots, T_n \in \mathcal{B}_{A^2}(H)$ *and* $\frac{1}{\sqrt{2}} \leq q \leq 1$*, then*

$$w^2_{q,\mathbb{A}^2}(T) \leq \frac{2q^2-1}{4}\left(\|T_1\|_{A^2} + \sqrt{\|\sum_{j=2}^{n} T_j(T_j)_A\|_{A^2}^2}\right)^2 + \left(1-q^2+q\sqrt{1-q^2}\right)\sum_{j=1}^{n}\|T_j\|_{A^2}^2. \tag{3.10}$$

Proof. By Theorem 2.18, we have

$$w^2_{q,\mathbb{A}^2}(T) \leq (2q^2-1)w^2_{\mathbb{A}^2}(T) + \left(1-q^2+q\sqrt{1-q^2}\right)\|T\|^2_{\mathbb{A}^2},$$

and by (3.9), we have

$$\|T\|^2_{\mathbb{A}^2} = \sum_{j=1}^{n}\|T_j\|^2_{A^2}.$$

Thus, the inequality in (3.10) follows from Theorem 3.9. □

To present our next result, we need the following estimate established by Guelfen and Kittaneh [15].

Lemma 3.11. *[15] Let* $T = \left(T_{ij}\right)$ *be an* $n \times n$ *operator matrix with* $T_{ij} \in \mathcal{B}(H)$*. Then*

$$w(T) \leq \max\left\{w(T_{ii}) : 1 \leq i \leq n\right\} + \frac{1}{2}\sum_{i=1}^{n}\sqrt{\|\sum_{\substack{j=1\\ j\neq i}}^{n} T_{ij}T_{ij}^*\|}.$$

Proposition 3.12. *Let* $T = \left(T_{ij}\right)$ *be an* $n \times n$ *operator matrix with* $T_{ij} \in \mathcal{B}_{A^2}(H)$*. Then*

$$w_{\mathbb{A}^2}(T) \leq \max\left\{w_{A^2}(T_{ii}) : 1 \leq i \leq n\right\} + \frac{1}{2}\sum_{i=1}^{n}\sqrt{\|\sum_{\substack{j=1\\ j\neq i}}^{n} T_{ij}(T_{ij})_A\|_{A^2}}.$$

Proof. The proof follows by applying Lemma 3.11 and using an argument similar to that used in the proof of Theorem 3.9. □

Theorem 3.13. *Let* $T = \left(T_{ij}\right)$ *be an* $n \times n$ *operator matrix with* $T_{ij} \in \mathcal{B}_{A^2}(H)$*. If* $\frac{1}{\sqrt{2}} \leq q \leq 1$*, then*

$$\begin{aligned} w_{q,\mathbb{A}^2}(T) \leq\ & 2\left(2q^2-1\right)\max\left\{w^2_{A^2}(T_{ii}) : 1 \leq i \leq n\right\} \\ & + \frac{n}{2}\left(2q^2-1\right)\sum_{i=1}^{n}\|\sum_{j=1_{j\neq i}}^{n} T_{ij}(T_{ij})_A\|_{A^2} \\ & + \left(1-q^2+q\sqrt{1-q^2}\right)\sum_{j=1}^{n}\|T_j\|^2_{A^2}. \end{aligned}$$

Proof. By Proposition 3.12, we have

$$w^2_{\mathbb{A}^2}(T) \leq \left(\max\left\{ w_A(T_{ii}) : 1 \leq i \leq n \right\} + \frac{1}{2}\sum_{i=1}^{n} \sqrt{\|\sum_{\substack{j=1\\ j\neq i}}^{n} T_{ij}(T_{ij})_A\|_{A^2}} \right)^2.$$

Hence

$$w^2_{\mathbb{A}^2}(T) \leq 2\max\left\{ w^2_A(T_{ii}) : 1 \leq i \leq n \right\} + \frac{1}{2}\left(\sum_{i=1}^{n} \sqrt{\|\sum_{\substack{j=1\\ j\neq i}}^{n} T_{ij}(T_{ij})_A\|_{A^2}} \right)^2.$$

Since the function $h(t) = t^2$ is convex on $(0, \infty)$, we easily check that

$$w^2_{\mathbb{A}^2}(T) \leq 2\max\left\{ w^2_A(T_{ii}) : 1 \leq i \leq n \right\} + \frac{n}{2}\sum_{i=1}^{n} \|\sum_{\substack{j=1\\ j\neq i}}^{n} T_{ij}(T_{ij})_A\|_{A^2}.$$

Thus the proof follows by using an argument similar to that used in proof of Proposition 3.10. □

4. *A*-*q*-center

In this section we generalize some results of Kaadoud and Moulaharabbi [18]. Let K be a compact subset of $\mathbb{C}$. By R_K and c_K denote the radius and the center of the smallest disk D_K containing K, respectively. Set $|K| = \sup\{|\lambda| : \lambda \in K\}$ and $\mathbb{S}_A = \{x \in H : \|x\|_A = 1\}$. The identity operator on H is denoted by I.

We begin with the following lemma.

Lemma 4.1. *[17] Let K be a compact subset of $\mathbb{C}$. Then*

$$R_K = |K - c_K| = \sup_{\lambda \in K} |c_K - \lambda| = \inf_{\alpha\in\mathbb{C}} \sup_{\lambda \in K} |\alpha - \lambda|.$$

Proposition 4.2. *Let $T \in \mathcal{B}_A(H)$ and $0 < |q| \leq 1$. Then*

$$R_{W_{q,A}(T)} = \left| W_{q,A}(T) - c_{W_{q,A}(T)} \right| = w_{q,A}(T - \frac{1}{q} c_{W_{q,A}(T)} I).$$

Proof. Since $T \in \mathcal{B}_A(H)$, then $w_{q,A}(T) \leq \|T\|_A$. This shows that $\overline{W_{q,A}(T)}$ is a compact subset of$\mathbb{C}$. Using Lemma 4.1, we get

$$\begin{aligned} R_{W_{q,A}(T)} &= \sup_{\lambda\in W_{q,A}(T)} \left|\lambda - c_{W_{q,A}(T)}\right| \\ &= \sup\left\{ |\langle Tx, y\rangle_A - c_{W_{q,A}(T)}| : x, y \in \mathbb{S}_A, \langle x, y\rangle_A = q \right\} \\ &= \sup\{ |\langle (T - \frac{1}{q} c_{W_{q,A}(T)} I)x, y\rangle_A| : x, y \in \mathbb{S}_A, \langle x, y\rangle_A = q \} \\ &= w_{q,A}(T - \frac{1}{q} c_{W_{q,A}(T)} I). \end{aligned}$$

□

Definition 4.3. Let $T \in \mathcal{B}_A(H)$ and $0 < |q| \leq 1$. The scalar λ that satisfies

$$\inf_{\alpha\in\mathbb{C}} w_{q,A}(T - \alpha I) = w_{q,A}(T - \lambda I),$$

is called q-A-center of T; we denote it by $c_{q,A}(T)$.

For $A = I$, we have $c_{q,A}(T) = c_q(T)$ the q-center defined in [18].

Proposition 4.4. *Let $T \in \mathcal{B}_A(H)$ and $0 < |q| \leq 1$. Then*

$$R_{W_{q,A}(T)} = \inf_{\alpha\in\mathbb{C}} w_{q,A}(T - \alpha I) = w_{q,A}(T - c_{q,A}(T)I),$$

and

$$c_{q,A}(T) = \frac{1}{q} c_{W_{q,A}(T)}.$$

Proof. By the definition of a q-A-center, we have

$$\begin{aligned} w_{q,A}(T - c_{q,A}(T)I) &= \inf_{\alpha\in\mathbb{C}} w_{q,A}(T - \alpha I) \\ &= \inf_{\alpha\in\mathbb{C}} w_{q,A}(T - \frac{\alpha}{q} I) \\ &= \inf_{\alpha\in\mathbb{C}} \sup \left\{ |\langle Tx, y\rangle_A - \alpha| : x, y \in \mathbb{S}_A, \langle x, y\rangle_A = q \right\} \\ &= \left| W_{q,A}(T) - c_{W_{q,A}(T)} \right| \\ &= R_{W_{q,A}(T)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left| W_{q,A}(T) - qc_{q,A}(T) \right| &= \sup \left\{ |\langle Tx, y\rangle_A - qc_{q,A}(T)| : x, y \in \mathbb{S}_A, \langle x, y\rangle_A = q \right\} \\ &= \sup \left\{ |\langle (T - c_{q,A}(T))x, y\rangle_A| : x, y \in \mathbb{S}_A, \langle x, y\rangle_A = q \right\} \\ &= w_{q,A}(T - c_{q,A}(T)I). \end{aligned}$$

It follows that

$$|W_{q,A}(T) - qc_{q,A}(T)| = |W_{q,A}(T) - c_{W_{q,A}(T)}|.$$

By the unicity of the center $c_{W_{q,A}(T)}$ of $W_{q,A}(T)$, we deduce that

$$c_{W_{q,A}(T)} = qc_{q,A}(T).$$

□

Proposition 4.5. *If a sequence $T_n \in \mathcal{B}_A(H)$ converges to $T \in \mathcal{B}_A(H)$ and $0 < |q| \leq 1$, then the sequence $c_{q,A}(T_n)$ converges to $c_{q,A}(T)$.*

Proof. By [17], the sequence $c_{W_{q,A}(T_n)}$ converges to $c_{W_{q,A}(T)}$. Using Proposition 4.4, we get $c_{W_{q,A}(T_n)} = qc_{q,A}(T_n)$. Hence the sequence $c_{q,A}(T_n)$ converges to $c_{q,A}(T)$. □

Proposition 4.6. *[17] Let K be a compact subset of $\mathbb{C}$ and $c \in \mathbb{C}$. The following assertions are equivalent:*

1. $c = c_K$,
2. $|K - c| < |K - (c + \alpha)|$ *for all nonzero* $\alpha \in \mathbb{C}$,
3. $|K - c|^2 + |\alpha|^2 \leq |K - (c + \alpha)|^2$ *for all* $\alpha \in \mathbb{C}$.

Proposition 4.7. *If $T \in \mathcal{B}_A(H)$ and $0 < |q| \leq 1$, then the following assertions are equivalent:*

1. $c = c_{q,A}(T)$,
2. $w_{q,A}(T - cI) < w_{q,A}(T - (c + \frac{\alpha}{q})I)$ *for all nonzero* $\alpha \in \mathbb{C}$,
3. $w_{q,A}(T - cI)^2 + |\alpha|^2 \leq w_{q,A}(T - (c + \frac{\alpha}{q})I)^2$ *for all* $\alpha \in \mathbb{C}$.

Proof. Since $w_{q,A}(T - \frac{\alpha}{q}) = |W_{q,A}(T) - \alpha|$ for all $\alpha \in \mathbb{C}$, then by Proposition 4.4, we get

$$c_{W_{q,A}(T)} = qc_{q,A}(T).$$

Thus Proposition 4.6 implies that the three assertions are equivalent. □

As a consequence of the above proposition, we have the following inequality.

Corollary 4.8. *Let $T \in \mathcal{B}_A(H)$ and $0 < |q| \leq 1$. If $0 \in W_{q,A}(T)$, then*

$$|c_{q,A}(T)| \leq \frac{1}{\sqrt{2}|q|} w_{q,A}(T).$$

Proof. By [17], we have

$$|c_{W_{q,A}(T)}| \leq \frac{1}{\sqrt{2}} w_{q,A}(T).$$

By Proposition 4.4, we have $c_{W_{q,A}(T)} = qc_{q,A}(T)$; thus

$$|c_{q,A}(T)| \leq \frac{1}{\sqrt{2}|q|} w_{q,A}(T).$$

□

For a subset $\Omega \subseteq \mathbb{C}$, we denote by diam$\Omega$ its diameter defined by

$$\text{diam}\Omega = \sup\{|\alpha - \beta| : \alpha, \beta \in \Omega\}\cdot$$

Proposition 4.9. *[18] Let $T \in \mathcal{B}(H)$ and $0 \leq q \leq 1$. Then*

$$w_q(T) \leq qw(T) + (1 - \frac{q}{2}) diam W(T).$$

Proposition 4.10. *If $T \in \mathcal{B}_A(H)$ and $0 \leq q \leq 1$, then*

$$w_{q,A}(T) \leq qw_A(T) + (1 - \frac{q}{2}) diam W_A(T).$$

Proof. By applying Proposition 4.9 to the operator $\widetilde{T}$ induced by T, we get

$$w_q(\widetilde{T}) \leq qw(\widetilde{T}) + \text{diam}W(\widetilde{T}).$$

Now by Theorem 2.13, we have $w_{q,A}(T) = w_q(\widetilde{T})$. In addition, by Corollary 2.14, we have $\overline{W(\widetilde{T})} = \overline{W_A(T)}$. Hence

$$w_{q,A}(T) \leq qw_A(T) + \text{diam}W_A(T)$$

which completes the proof. □

Conflict of interest The authors have no conflict of interest to disclose.

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