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A note on a result by M. Oudghiri and K. Souilah

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Abstract. M. Oudghiri and K. Souilah recently proved that if a bounded linear operator $R \in L(X)$ is Riesz, then the following conditions are equivalent: (i) $R = Q + F$, where $Q \in L(X)$ is quasinilpotent, $F \in L(X)$ is finite rank and $QF = FQ$; (ii) $\sigma_{gD}(T) = \sigma_{gD}(T + R)$ for every $T \in L(X)$ commuting with R, where $\sigma_{gD}(\cdot)$ denotes the generalized Drazin spectrum. In this note we present two new conditions that are equivalent to the previously mentioned conditions (i) and (ii). Additionally, we offer an alternative proof for the implication (i) $\Rightarrow$ (ii) of the above result.

1. Introduction and preliminaries

Let X be an infinite-dimensional complex Banach space. We will denote by $L(X)$ the Banach algebra of all bounded linear operators on X. For $T \in L(X)$, denote by $\alpha(T)$, $\beta(T)$, $N(T)$ and $R(T)$, respectively, the *nullity*, the *deficiency*, the *kernel* and the *range* of T. The smallest integer n such that $N(T^n) = N(T^{n+1})$ is called the *ascent* of T and will be denoted by $\operatorname{asc}(T)$. If there is no such integer, then we set $\operatorname{asc}(T) = \infty$. The *descent* of T, $\operatorname{dsc}(T)$, is the smallest integer n such that $R(T^n) = R(T^{n+1})$. If always $R(T^n) \neq R(T^{n+1})$, then we set $\operatorname{dsc}(T) = \infty$.

Many classes of bounded linear operators can be defined in terms of nullity, deficiency, ascent and descent. For instance, an operator $T \in L(X)$ is *Fredholm* if $\alpha(T) < \infty$ and $\beta(T) < \infty$, while an operator $T \in L(X)$ is *Browder* if it is Fredholm, $\operatorname{asc}(T) < \infty$ and $\operatorname{dsc}(T) < \infty$. We note that the name "Browder operators" was first used by R. Harte [2]. It is said that $T \in L(X)$ is a *finite rank* operator if $\dim R(T) < \infty$. The set of all finite rank operators will be denoted by $F(X)$. We will say that $T \in L(X)$ is *power finite rank* if T^n is a finite rank operator for some $n \in \mathbb{N}$. Let $\sigma(T)$ be the *spectrum* of $T \in L(X)$. An operator $Q \in L(X)$ is *quasinilpotent* if $\sigma(Q) = \{0\}$. We write $Q(X)$ for the set of all quasinilpotent operators. In addition, an operator $T \in L(X)$ is *Riesz* if $T - \lambda I$ is Fredholm for all non-zero $\lambda \in \mathbb{C}$. The set of all Riesz operators will be denoted by $R(X)$. It is well known that $F(X) \subset R(X)$ and $Q(X) \subset R(X)$. Further, Browder operators are stable under commuting Riesz perturbations [7, Corollary 2]: Let $T \in L(X)$ be Browder. If $S \in R(X)$ commutes with T, then $T + S$ is Browder.

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Let $T \in L(X)$ and let $M \subset X$ be a closed subspace. It is said that M is *T-invariant* (or *invariant under T*) if $T(M) \subset M$. We define $T|M : M \to M$ as $T|Mx = Tx$, $x \in M$. Clearly, $T|M \in L(M)$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T|M \oplus T|N$.

The concept of generalized Drazin invertibility was introduced in 1996 by J. J. Koliha [4, Definition 4.1]: It is said that $T \in L(X)$ is *generalized Drazin invertible* (*Koliha-Drazin invertible*) if there exists $B \in L(X)$ such that

$$TB = BT, \quad B = TB^2, \quad T - T^2B \text{ is quasinilpotent.}$$

The sets of accumulation points and isolated points of $\sigma(T)$, $T \in L(X)$, will be denoted by $\operatorname{acc}\sigma(T)$ and $\operatorname{iso}\sigma(T)$, respectively. According to [4, Theorem 4.2], $T \in L(X)$ is Koliha-Drazin invertible if and only if $0 \notin \operatorname{acc}\sigma(T)$. This is equivalent to the existence of a pair $(X_1, X_2) \in Red(T)$ such that $T|X_1$ is invertible and $T|X_2$ is quasinilpotent [4, Theorems 3.1 and 7.1]. We will denote by $L(X)^{KD}$ the set of all Koliha-Drazin invertible operators on X. The *generalized Drazin spectrum* of $T \in L(X)$, $\sigma_{gD}(T)$, is defined as $\sigma_{gD}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin L(X)^{KD}\}$. It is well known that $\sigma_{gD}(T) = \operatorname{acc}\sigma(T)$.

The stability of Koliha-Drazin invertible operators under commuting perturbations has been studied by many authors [1, 4, 6, 8, 9]. In particular, if $T \in L(X)^{KD}$, $Q \in Q(X)$ and $TQ = QT$, then also $T + Q \in L(X)^{KD}$ [4, Theorem 5.6]. Consequently, if $Q \in Q(X)$ then

$$\sigma_{gD}(T) = \sigma_{gD}(T + Q) \text{ for every } T \in L(X) \text{ commuting with } Q.$$

In 2015, Q. Zeng, H. Zhong and K. Yan proved that the accumulation points of the spectrum of a bounded linear operator are invariant with respect to power finite rank perturbations [9, Theorem 2.2(3)]. More precisely, they showed that for a given $T \in L(X)$, if $F \in L(X)$ is a power finite rank operator commuting with T, then

$$\operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T + F).$$

This result immediately implies that Koliha-Drazin invertibility is stable under commuting power finite rank perturbations [9, Corollary 2.4]: if $F \in L(X)$ is a power finite rank operator then

$$\sigma_{gD}(T) = \sigma_{gD}(T + F) \text{ for every } T \in L(X) \text{ commuting with } F.$$

Further, in 2020, in their paper [6], M. Oudghiri and K. Souilah proved the following result:

Theorem 1.1. [6, Theorem 1] *Let $R \in L(X)$ be a Riesz operator. Then the following assertions are equivalent:*

(i) $R = Q + F$ *where* $Q \in Q(X)$, $F \in F(X)$ *and* $QF = FQ$;

(ii) $\sigma_{gD}(T) = \sigma_{gD}(T + R)$ *for every* $T \in L(X)$ *commuting with* R.

In this paper we first prove that if $T \in L(X)^{KD}$, $R \in R(X)$, $\sigma(R)$ is a finite set and $TR = RT$, then $T + R \in L(X)^{KD}$ (see Theorem 2.1). Then, using Theorem 2.1 and the spectral properties of power finite rank operators, we provide two new conditions that are equivalent to conditions (i) and (ii) of [6, Theorem 1] (see Theorem 2.3). We emphasize that our proof of the implication (i) $\Rightarrow$ (ii) of [6, Theorem 1] is more concise and direct.

2. Results

At the beginning of this section, we will prove that Koliha-Drazin invertibility is stable in the case when the commuting perturbation R is a Riesz operator of finite spectrum. Moreover, we also complement a result by V. Rakočević [8, Theorem 2.1], who proved the stability of Koliha-Drazin invertible operators with finite nullity under commuting Riesz perturbations. By considering only Riesz perturbations R such that $\sigma(R)$ is a finite set, our result shows that the condition of having finite nullity can be omitted.

Theorem 2.1. *Let $T \in L(X)$. Suppose that $R \in R(X)$ such that $\sigma(R)$ is a finite set and $TR = RT$. Then the following assertions hold:*

(i) *If $T \in L(X)^{KD}$, then also $T + R \in L(X)^{KD}$;*

(ii) $\sigma_{gD}(T) = \sigma_{gD}(T + R)$;

(iii) $\operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T + R)$.

Proof. (i). Let $T \in L(X)^{KD}$. Then T is invertible or $0 \in \operatorname{iso}\sigma(T)$. If T is invertible, then $T + R$ is Browder by [7, Corollary 2]. From [5, Corollary 20.20] it follows that $0 \notin \operatorname{acc}\sigma(T + R)$, i.e. $T + R \in L(X)^{KD}$.

Let $0 \in \operatorname{iso}\sigma(T)$ and let P be the spectral projection of T corresponding to 0. Set $X_1 = N(P)$ and $X_2 = R(P)$. According to [4, Theorems 3.1 and 7.1], $(X_1, X_2) \in Red(T)$, $T|X_1$ is invertible and $T|X_2$ is quasinilpotent. Since $TR = RT$, it follows that $PR = RP$, that is, $(X_1, X_2) \in Red(R)$. From [3, Proposition 52.7] we conclude that $R|X_1$ and $R|X_2$ are Riesz operators. Clearly, $T|X_i$ and $R|X_i$ commute ($i = 1, 2$).

From [7, Corollary 2], we have that $(T+R)|X_1 = T|X_1 + R|X_1$ is Browder. Invoking [5, Corollary 20.20], we deduce that $0 \notin \operatorname{acc}\sigma((T + R)|X_1)$. On the other hand, $\sigma(R) = \sigma(R|X_1) \cup \sigma(R|X_2)$ and the assumption that $\sigma(R)$ is finite implies that $\sigma(R|X_2)$ is finite. By the stability of the spectrum of a bounded linear operator under commuting quasinilpotent perturbations [10, Theorem 4.1], it follows that $\sigma((T + R)|X_2) = \sigma(T|X_2 + R|X_2) = \sigma(R|X_2)$ is a finite set. Consequently, $0 \notin \operatorname{acc}\sigma((T + R)|X_2)$.

Observe that $\sigma(T + R) = \sigma((T + R)|X_1) \cup \sigma((T + R)|X_2)$. Since $0 \notin \operatorname{acc}\sigma((T + R)|X_1)$ and since $0 \notin \operatorname{acc}\sigma((T + R)|X_2)$, we obtain $0 \notin \operatorname{acc}\sigma(T + R)$ and hence $T + R \in L(X)^{KD}$.

(ii). Let $\lambda \notin \sigma_{gD}(T)$. It follows that $T - \lambda I \in L(X)^{KD}$. According to (i), $(T + R) - \lambda I = (T - \lambda I) + R$ is Koliha-Drazin invertible. Consequently, $\lambda \notin \sigma_{gD}(T+R)$. To prove the reverse inclusion $\sigma_{gD}(T) \subset \sigma_{gD}(T+R)$, apply the above argument with T and R replaced by $T + R$ and $-R$, respectively.

(iii). Recall that $\sigma_{gD}(T) = \operatorname{acc}\sigma(T)$ and $\sigma_{gD}(T + R) = \operatorname{acc}\sigma(T + R)$. Then the result follows from (ii). □

In what follows, we summarize the important properties of power finite rank operators that are essential for our further work.

Remark 2.2. (i). *Let $F \in F(X)$. Then $\sigma(F)$ is a finite set.*

Proof. Choose $0 \neq \lambda \in \sigma(F)$. Since F is compact, λ is an eigenvalue of F, i.e. $Fx = \lambda x$ for some non-zero $x \in X$. From $x = \frac{1}{\lambda}Fx$ we conclude that $x \in R(F)$. Clearly, the finite-dimensional subspace $Y := R(F)$ is invariant under F. Let $F|Y$ be the restriction of F to its range Y. We have $(F|Y)x = Fx = \lambda x$, which means that λ is also an eigenvalue of $F|Y$. It follows that $\sigma(F) \setminus \{0\} \subset \sigma(F|Y)$. Since $\sigma(F|Y)$ is a finite set, we conclude that $\sigma(F)$ is also finite. □

(ii). *Suppose that $F \in L(X)$ is a power finite rank operator. Then $F \in R(X)$ and the spectrum $\sigma(F)$ is a finite set.*

Proof. There exists $n \in \mathbb{N}$ such that $F^n \in F(X)$. It follows that F^n is compact and we deduce that F is Riesz by [3, Exercises 1, p.221]. Moreover, $[\sigma(F)]^n = \sigma(F^n)$ is a finite set by (i). *Consequently, $\sigma(F)$ is finite.* □

We are now in a position to extend [6, Theorem 1]. To be more precise, we give new conditions (conditions (ii) and (iii) of Theorem 2.3) that are equivalent to conditions (i) and (ii) of [6, Theorem 1]. Moreover, we provide an alternative proof of the implication (i) ⇒ (ii) of [6, Theorem 1] (see the chain of implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) in the proof of Theorem 2.3), which is based on the application of Theorem 2.1.

Theorem 2.3. *Let $R \in R(X)$. Then the following assertions are equivalent:*

(i) *$R = Q + F$ where $Q \in Q(X)$, $F \in F(X)$ and $QF = FQ$;*

(ii) *$R = Q + F$ where $Q \in Q(X)$, $F^n \in F(X)$ for some $n \in \mathbb{N}$ and $QF = FQ$;*

(iii) *$\sigma(R)$ is a finite set;*

(iv) *$\sigma_{gD}(T) = \sigma_{gD}(T + R)$ for every $T \in L(X)$ commuting with R.*

Proof. (i) ⇒ (ii). Clear.

(ii) ⇒ (iii). It follows that $\sigma(F)$ is a finite set by Remark 2.2(ii). According to [10, Theorem 4.1], $\sigma(R) = \sigma(Q+F) = \sigma(F)$ and hence $\sigma(R)$ is a finite set.

(iii) ⇒ (iv). Follows from Theorem 2.1(ii).

(iv) ⇒ (i). See the proof of [6, Theorem 1]. □

As a consequence of Theorem 2.3, we obtain [9, Corollary 2.4].

Corollary 2.4. [9, Corollary 2.4] *Suppose that $F \in L(X)$ satisfies $F^n \in F(X)$ for some $n \in \mathbb{N}$. Then we have*

$$\sigma_{gD}(T) = \sigma_{gD}(T+F) \text{ for all } T \in L(X) \text{ commuting with } F.$$

Proof. Let $F \in L(X)$ be a power finite rank operator and let $T \in L(X)$ commute with F. As we know from before, F is a Riesz operator and $\sigma(F)$ is a finite set. The result follows from the implication (iii) ⇒ (iv) of Theorem 2.3. □

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