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INEQUALITIES FOR THE VOLUME OF THE UNIT BALL IN ℓ_p^{n*}

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Abstract

Let $B_p^n = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}$ be the unit ball in ℓ_p^n. We prove the inequalities for the volume of the B_p^n:

$$V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}$$

$$2\Gamma(\frac{1}{p}+1)\sqrt[p]{\frac{p}{n+p}}V_{B_p^n} \leq V_{B_p^{n+1}}$$

for all $n \geq 1$ and $p \geq 1$, where $V_{B_p^n}$ denotes the volumes of B_p^n. Furthermore, we obtain the upper and lower bounds of $V_{B_p^{n+1}}^{\frac{n}{n+1}}/V_{B_p^n}$ and $V_{B_p^{n+1}}/V_{B_p^n}$. Our results are generalizations for inequalities in R^n proved and refined by G.D. Anderson et al., K.H. Borgwardt, D.A.Klain and G.-C. Rota and H. Alzer.

1. Introduction

Let $B_p^n = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}$ be the unit ball in ℓ_p^n, and $V_{B_p^n}$ denotes the volume of unit ball in ℓ_p^n. Then $V_{B_2^n}$ means the volume of unit ball in $\mathbb{R}^n$. In past several years, there have been many works about the inequalities for $V_{B_2^n}$. According to the results of G.D.Anderson, M.K.Vamanamurthy and M.Vuorinen in [3] and of D.A.Klain and G.-C. Rota in [7], we have

$$V_{B_2^{n+1}}^{\frac{1}{n+1}} < V_{B_2^n}^{\frac{1}{n}}, (n = 1, 2, ...). \tag{1.1}$$

Another inequality regarding the upper and lower bounds for the ratio of $V_{B_2^{n+1}}/V_{B_2^n}$ was obtained by Brogwardt in [5]:

$$\sqrt{\frac{2\pi}{n+2}} \leq \frac{V_{B_2^{n+1}}}{V_{B_2^n}} \leq \sqrt{\frac{2\pi}{n+1}}, (n = 1, 2, ...), \tag{1.2}$$

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which leads to

$$1 < \frac{V^2_{B_2^n}}{V_{B_2^{n-1}}V_{B_2^{n+1}}} < \sqrt{\frac{n+2}{n}}, (n = 2, 3, ...). \tag{1.3}$$

The next inequality about $V_{B_2^n}$ is proved by H. Alzer in [2], and he pointed out that

$$1 < \frac{V^2_{B_2^n}}{V_{B_2^{n-1}}V_{B_2^{n+1}}} < 1 + \frac{1}{n}, (n = 2, 3, ...). \tag{1.4}$$

This inequality can also be deduced from the results in [3]. However, the right-hand side inequality of (1.4) is weaker than that of (1.3) as $\sqrt{\frac{n+2}{n}} \leq 1 + \frac{1}{n}$.

Inequalities (1.1), (1.2) and (1.4) have been refined by Horst Alzer in [2]. His results are:

$$\frac{2}{\sqrt{\pi}} V^{\frac{n}{n+1}}_{B_2^{n+1}} \leq V_{B_2^n} < \sqrt{e} V^{\frac{n}{n+1}}_{B_2^{n+1}}, (n = 1, 2, ...); \tag{1.5}$$

$$\sqrt{\frac{2\pi}{n + \frac{8}{\pi} - 1}} \leq \frac{V_{B_2^{n+1}}}{V_{B_2^n}} < \sqrt{\frac{2\pi}{n + \frac{3}{2}}}, (n = 1, 2, ...); \tag{1.6}$$

$$\left(1 + \frac{1}{n}\right)^{2 - \frac{\log \pi}{\log 2}} \leq \frac{V^2_{B_2^n}}{V_{B_2^{n-1}}V_{B_2^{n+1}}} < \left(1 + \frac{1}{n}\right)^{\frac{1}{2}}, (n = 2, 3, ...). \tag{1.7}$$

On the other hand, there are also lots of results about the volume of B_p^n, such as in M.Meyer and A.Pajor [8], M.Schmuckenschlager [10], Jesus Bastero etc [4] and Peng Gao [6]. From these results and those inequalities for the volumes of unit ball in $\mathbb{R}^n$, it is natural to ask whether there exist similar inequalities for the volumes of unit ball in ℓ_p^n ? In this paper, we give the answer to this question by proving Theorem 1 and 2 and Corollary 1, which are similar to (1.1), (1.2) and (1.3). Moreover, we prove Theorem 3 and 4, whose results are similar to (1.5) and (1.6). Our results are:

$$V^{\frac{1}{n+1}}_{B_p^{n+1}} < V^{\frac{1}{n}}_{B_p^n}, (n = 1, 2, ...);$$

$$2\Gamma(\frac{1}{p} + 1)\sqrt[p]{\frac{p}{n+p}} V_{B_p^n} \leq V_{B_p^{n+1}}, (n = 1, 2, ...);$$

$$2\Gamma(\frac{1}{p} + 1)\sqrt[p]{\frac{p}{n+p}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} \leq 2\Gamma(\frac{1}{p} + 1)\sqrt[p]{\frac{p}{n+1}}, (n = p, p+1, ...);$$

$$\sqrt[p]{\frac{n+1}{p+n-1}} \leq \frac{V^2_{B_p^n}}{V_{B_p^{n-1}}V_{B_p^{n+1}}} \leq \sqrt[p]{\frac{n+p}{n}}, (n = p, p+1, ...);$$

$$a V^{\frac{n}{n+1}}_{B_p^{n+1}} \leq V_{B_p^n} < b V^{\frac{n}{n+1}}_{B_p^{n+1}}, (n = p-1, p, ...);$$

$$2\Gamma(\frac{1}{p} + 1)\sqrt[p]{\frac{p}{n+A}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} < 2\Gamma(\frac{1}{p} + 1)\sqrt[p]{\frac{p}{n+B}}, (n = 1, 2, ...),$$

where $a = \frac{\sqrt[\frac{p-1}{p}]{\Gamma(2)}}{\Gamma(\frac{p-1}{p}+1)}$, $b = \sqrt[p]{e}$, $A = p\left(\frac{\Gamma(\frac{2}{p}+1)}{\Gamma(\frac{1}{p}+1)}\right)^p - 1$ and $B = \frac{p+1}{2}$.

2. Volume of the unit ball in ℓ_p^n and some inequalities of $\Gamma(x)$ and $\Psi(x)$

Before we start our proof, it is necessary for us to introduce the formula of the volumes of unit ball in ℓ_p^n spaces and some properties of gamma function and psi function (the logarithmic derivative of the gamma function).

Lemma 1. *Let $B_p^n = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}$, then*

$$V_{B_p^n} = \frac{(2\Gamma(\frac{1}{p}+1))^n}{\Gamma(\frac{n}{p}+1)}, \tag{2.1}$$

where $V_{B_p^n}$ is the volume of unit ball in ℓ_p^n.

The proof of Lemma 1 can be found in [12].

Lemma 2. *For all $x > 0$ we have*

$$\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + \log\sqrt{2\pi} + \frac{1}{12x} + O\left(\frac{1}{x^3}\right), \tag{2.2}$$

$$\log\Gamma(x) > \left(x - \frac{1}{2}\right)\log x - x + \log\sqrt{2\pi}, \tag{2.3}$$

and

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), (x \to \infty). \tag{2.4}$$

Lemma 2 is provided by Horst Alzer in [2] , Another paper of him [1] gives us the proof of (2.3), which is also found in [11], and the proofs of (2.2) and (2.4) can is given in [9].

Lemma 3. *For $x > 0$, let*

$$\Psi(x) = \frac{d\log(\Gamma(x))}{dx} = \frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx}.$$

We also have the integral representations

$$\Psi(x) = -C + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}dt, \tag{2.5}$$

where $C = Euler'sconstant$,

$$\Psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty e^{-xt}\frac{t^n}{1 - e^{-t}}dt, \tag{2.6}$$

and the asymptotic formula

$$\Psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \tag{2.7}$$

Then,

$$\Psi(x) < \log x - \frac{1}{2x}. \tag{2.8}$$

Lemma 3 is also mentioned in [2] and [11]. The integral representation and asymptotic formula of $\Psi(x)$ is given in [9]. Actually, (2.6) follows from (2.5) by differentiation, and the proof of (2.8) is proved in [1.3], which can be deduced from (2.7) easily.

Lemma 4. *Let $n \geq 0$ be an integer and let $x > 0$ and $s \in (0,1)$ be real numbers. Then*

$$A_n(s;x) < \Psi(x+1) - \Psi(x+s), \tag{2.9}$$

where

$$A_n(s;x) = (1-s)\left(\frac{1}{x+s+n} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+s)}\right). \tag{2.10}$$

We can find the proof of Lemma 4 in [1]. Horst Alzer proved this Lemma by Jensen's inequality,

$$h(su+(1-s)v) < sh(u) + (1-s)h(v), (u, v > 0; u \neq v; 0 < s < 1).$$

3. Inequalities for $V_{B_p^n}$

Theorem 1. *For all integers $n \geq 1$, we have*

$$V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}. \tag{3.1}$$

Proof. We define for positive real numbers x

$$f(x) = \frac{2\Gamma(\frac{1}{p}+1)}{\left(\Gamma(\frac{x}{p}+1)\right)^{\frac{1}{x}}}.$$

Differentiation yields

$$\frac{df(x)}{dx} = \frac{2\Gamma(\frac{1}{p}+1)}{x^2\left(\Gamma(\frac{x}{p}+1)\right)^{\frac{1}{x}}}\left(\log\Gamma(\frac{x}{p}+1) - \frac{x}{p}\Psi(\frac{x}{p}+1)\right).$$

Then, we define for $y > 1$

$$g(y) = \log\Gamma(y) - (y-1)\Psi(y).$$

Differentiation yields

$$\frac{dg(y)}{dy} = -(y-1)\frac{d\Psi(y)}{dy}.$$

By (2.6), we know

$$\frac{d\Psi(y)}{dy} = \int_0^\infty e^{-yt}\frac{t}{1-e^{-t}}dt > 0. \tag{3.2}$$

According to (3.2), $\frac{dg(y)}{dy} \leq 0$ for $y \geq 1$. Thus, for $y > 1$

$$g(y) < g(1) = 0,$$

which implies

$$\frac{df(x)}{dx} < 0.$$

Hence, we obtain that $V_{B_p^{n+1}}^{\frac{1}{n+1}} < V_{B_p^n}^{\frac{1}{n}}$.

Theorem 2. *For all integers $n \geq 1$, we have*

$$2\Gamma(\frac{1}{p}+1)\sqrt[p]{\frac{p}{n+p}}V_{B_p^n} \leq V_{B_p^{n+1}}. \tag{3.3}$$

Proof. We define for positive real numbers x

$$f(x) = 2\Gamma(\frac{1}{p}+1)\frac{\Gamma(\frac{x}{p}+1)}{\Gamma(\frac{x+1}{p}+1)}.$$

Differentiation yields

$$\frac{df(x)}{dx} = 2\Gamma(\frac{1}{p}+1)\frac{\Gamma(\frac{x}{p}+1)}{\Gamma(\frac{x+1}{p}+1)}\left(\Psi(\frac{x}{p}+1) - \Psi(\frac{x+1}{p}+1)\right) < 0.$$

As $\Psi(x)$ is an increasing function by (3.5). Hence, we obtain

$$\left(2\Gamma(\frac{1}{p}+1)\right)^p \frac{p}{n+p} = \prod_{n}^{n+p-1} f(i) \leq f^p(n).$$

Hence, the theorem is proved. It may be noted that the equality sign holds, if and only if $p = 1$.

Corollary 1. *For all integers $n \geq p$, we have*

$$2\Gamma(\frac{1}{p}+1)\sqrt[p]{\frac{p}{n+p}} \leq \frac{V_{B_p^{n+1}}}{V_{B_p^n}} \leq 2\Gamma(\frac{1}{p}+1)\sqrt[p]{\frac{p}{n+1}}, \tag{3.4}$$

and

$$\sqrt[p]{\frac{n+1}{p+n-1}} \leq \frac{V_{B_p^n}^2}{V_{B_p^{n-1}} V_{B_p^{n+1}}} \leq \sqrt[p]{\frac{n+p}{n}}. \tag{3.5}$$

Proof. The above leads to (3.3). Proceeding precisely in the same way as what we have done in the previous proof, we obtain that, for all $n \geq p$,

$$\left(2\Gamma(\frac{1}{p}+1)\right)^p \frac{p}{n+p} = \prod_{n}^{n+p-1} f(i) \leq f^p(n) \leq \prod_{n-p+1}^{n} f(i) = \left(2\Gamma(\frac{1}{p}+1)\right)^p \frac{p}{n+1}. \tag{3.6}$$

Then, applying (3.3), we can obtain (3.4) easily. All equality sign hold, if and only if $p = 1$.

4. Bounds of $V_{B_p^n}$

Theorem 3. *For all integers $n \geq p-1$, we have*

$$aV_{B_p^{n+1}}^{\frac{n}{n+1}} \leq V_{B_p^n} < bV_{B_p^{n+1}}^{\frac{n}{n+1}}, \tag{4.1}$$

with $a = \frac{\sqrt[\frac{p-1}{p}]{\Gamma(2)}}{\Gamma(\frac{p-1}{p}+1)}$ and $b = \sqrt[p]{e}$.

Proof. First, we define the sequence

$$\begin{aligned} x_n &= \log V_{B_p^n} - \frac{n}{n+1} \log V_{B_p^{n+1}} \\ &= \frac{n}{n+1} \log \Gamma(\frac{n+1}{p}+1) - \log \Gamma(\frac{n}{p}+1), (n = p-1, p, ...), \end{aligned}$$

and for positive real number x, let

$$f(x) = \frac{x}{x+\frac{1}{p}} \log \Gamma(x+\frac{1}{p}+1) - \log \Gamma(x+1),$$

then,

$$p(x+\frac{1}{p})^2 \frac{df(x)}{dx} = \log \Gamma(x+\frac{1}{p}+1) + px(x+\frac{1}{p})\Psi(x+\frac{1}{p}+1) - p(x+\frac{1}{p})^2 \Psi(x+1).$$

We define for $y = x+1+\frac{1}{p} \geq 2$

$$g(y) = \log \Gamma(y) + (py-p-1)(y-1)\Psi(y) - p(y-1)^2 \Psi(y-\frac{1}{p}).$$

Applying (2.3),(2.8) and Lemma 4, we consider that

$$\begin{aligned} g(y) &\geq \log\sqrt{2\pi}+\frac{1}{2}+\frac{1}{2}\log y-y \\ &\quad -\frac{1}{2y}+(y-1)^2\left(\frac{1}{y-\frac{1}{p}+2}+\frac{1}{y(y-\frac{1}{p})}+\frac{1}{(y+1)(y+1-\frac{1}{p})}\right) \\ &\geq \log\sqrt{2\pi}+\frac{1}{2}+\frac{1}{2}\log y-y-\frac{1}{2y}+(y-1)^2\left(\frac{1}{y+2}+\frac{1}{y^2}+\frac{1}{(y+1)^2}\right). \end{aligned}$$

A simple calculation reveals for $y \geq 2$,

$$(y-1)^2\left(\frac{1}{y+2}+\frac{1}{y^2}\right)+\frac{1}{(y+1)^2}-y-\frac{1}{2y}\geq -2,$$

which means

$$g(y)\geq \log\sqrt{2\pi}+\frac{1}{2}+\frac{1}{2}\log 2-2>0$$

Thus, $\frac{df(y)}{dy}>0$, so that $x_n (n=1,2,...)$ is strictly increasing. Applying (2.2),

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left(\frac{2n+p}{2p}\log\frac{n+p+1}{n+p}-\frac{1}{2(n+1)}\log(n+p+1)+O\left(\frac{1}{n}\right)\right) \\ &= \frac{1}{p}. \end{aligned}$$

Hence, for all $n \geq p-1$

$$\frac{\sqrt[\frac{p-1}{p}]{\Gamma(2)}}{\Gamma(\frac{p-1}{p}+1)}V_{B_p^{n+1}}^{\frac{n}{n+1}}\leq V_{B_p^n}<\sqrt[p]{e}V_{B_p^{n+1}}^{\frac{n}{n+1}}.$$

Theorem 4. *For all integers $n \geq 1$, we have*

$$2\Gamma(\frac{1}{p}+1)\sqrt[p]{\frac{p}{n+A}}\leq\frac{V_{B_p^{n+1}}}{V_{B_p^n}}<2\Gamma(\frac{1}{p}+1)\sqrt[p]{\frac{p}{n+B}} \qquad (4.2)$$

with $A=p\left(\frac{\Gamma(\frac{2}{p}+1)}{\Gamma(\frac{1}{p}+1)}\right)^p-1$ and $B=\frac{p+1}{2}$.

Proof. Double-inequality (4.2) is equivalent to

$$B<ph(\frac{n}{p})\leq A,$$

where

$$h(x)=\left(\frac{\Gamma(x+1+\frac{1}{p})}{\Gamma(x+1)}\right)^p-x,(x>0).$$

Define $r = p\left(\frac{\Gamma(x+1+\frac{1}{p})}{\Gamma(x+1)}\right)^p \left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1)\right)$ and $L(r,s) = \frac{r-s}{\log r - \log s}$. Let $s = 1$. Differentiation yields

$$\begin{aligned}\frac{1}{L(r,s)}\frac{dh(x)}{dx} &= p\log\Gamma(x+1+\frac{1}{p}) - p\log\Gamma(x+1) \\ &\quad + \log\left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1)\right) + \log p.\end{aligned}$$

Define $q(x) = \frac{1}{L(r,s)}\frac{dh(x)}{dx}$, and from (2.5) and (2.6), we obtain

$$\begin{aligned}\left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1)\right)\frac{dq(x)}{dx} &= \frac{d\Psi(x+1+\frac{1}{p})}{dx} - \frac{d\Psi(x+1)}{dx} \\ &\quad + p\left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1)\right)^2 \\ &= -\int_0^\infty e^{-xt}t\delta(t)dt + p\left(\int_0^\infty e^{-xt}\delta(t)dt\right)^2,\end{aligned}$$

where

$$\delta(t) = \frac{-e^{-(1+\frac{1}{p})t} + e^{-t}}{1-e^{-t}}.$$

Applying the convolution theorem for Laplace transforms, we get

$$\left(\Psi(x+1+\frac{1}{p}) - \Psi(x+1)\right)\frac{dq(x)}{dx} = \int_0^\infty e^{-xt}\int_0^t (p\delta(s)\delta(t-s) - \delta(t))dsdt.$$

Let $0 < s < t$, we have

$$\begin{aligned}& p\delta(s)\delta(t-s) - \delta(t) \\ = \ & \frac{p(1-e^{-\frac{s}{p}})(1-e^{-\frac{t-s}{p}})(1-e^{-t}) - (1-e^{-\frac{t}{p}})(1-e^{-s})(1-e^{-(t-s)})}{(e^s-1)(e^{t-s}-1)(e^t-1)} \\ = \ & \frac{(1-e^{-\frac{s}{p}})(1-e^{-\frac{t-s}{p}})(1-e^{-\frac{t}{p}})(p\sum_{i=0}^{p-1}e^{-\frac{i}{p}t} - \sum_{i=0}^{p-1}e^{-\frac{i}{p}s}\sum_{i=0}^{p-1}e^{-\frac{i}{p}(t-s)})}{(e^s-1)(e^{t-s}-1)(e^t-1)} \\ > \ & 0.\end{aligned}$$

Thus, for $x > 0, \frac{dq(x)}{dx} > 0$.

Applying (2.4) and (2.7), we get

$$\begin{aligned}\lim_{z\to\infty} e^{q(z)} &= \lim_{z\to\infty} p\left(\frac{\Gamma(z+1+\frac{1}{p})}{\Gamma(z+1)}z^{-\frac{1}{p}}\right)^p z\left(\Psi(z+1+\frac{1}{p}) - \Psi(z+1)\right) \\ &= 1,\end{aligned}$$

which means $q(x) < 0$.

We conclude that $h(x)$ is a decreasing function. Hence, for $n \geq 1$

$$p \lim_{n\to\infty} h(\frac{n}{p}) < ph(\frac{n}{p}) \leq ph(\frac{1}{p}) = p\left(\frac{\Gamma(\frac{2}{p}+1)}{\Gamma(\frac{1}{p}+1)}\right)^p - 1.$$

From (2.4),

$$\lim_{n\to\infty} h(n) = \frac{p+1}{2p}.$$

This is the end of the proof.

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References

[1] H. Alzer, On some inequalities for the gamma and psi functions, *Math. Comput.* 1997: 373-389.

[2] H. Alzer, Inequalities for the volume of the unit ball in R^n, *J. Math. Anal. Appl.* 2000, 136:353-363.

[3] G.D.Anderson, M.K.Vamanamurthy, and M.Vuorinen, Special function of quasiconformal theory, *Exposition. Math* 1989: 97-136.

[4] J.Bastero, F.Galve, A.Pena, M.Romance, Inequalities for the gamma function and estimates for the volume of section B_p^n, *Proc. Amer. Math. Soc.* 2002, 130 183-192.

[5] K.H.Borgwardt, The simplex method, Berlin: Spring-Verlag Press, 1987.

[6] P. Gao, A note on the volume of section of B_p^n, *J. Math. Anal. Appl.* 2007, 326:632-640.

[7] D.A.Klain and G.-C.Rota, A continuous analogue of Sperner's theorem, *Comm. Pure Appl.* 1997, 50:118-123.

[8] M.Meyer, A.Pajor, Sections of the unit ball of L_p^n, *J. Funct. Anal.* 1988, 80:109-123.

[9] A.F.Nikiforov and V.B.Uvarov, Special Functions of Mathematical Physics, Birkhauser, Basel, 1988.

[10] M. Schmuckenschlager, Volume of intersection and sections of the unit ball of ℓ_p^n, *Proc. Amer. Math. Soc* 1998, 126: 1527-1530.

[11] Z. Wang, D. Guo, Introduction to Special Function , Bejing: Peking University Press, 2000.

[12] G. M. Fihtengolc, Kurs differencialnogo i integralnogo ischislenija, Peking: Higher Education Press, 2005.

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