FIXED POINT PROPERTY FOR HYPERSPACES OF ARBOROIDS

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Abstract

The main purpose of this paper is to study arboroids, a non-metric analogue of dendroids. It is proved that hyperspaces of some arboroids have the fixed point property.

1 Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space \( X \) is denoted by \( w(X) \). The cardinality of a set \( A \) is denoted by \( \text{card}(A) \). We shall use the notion of inverse system as in [5, pp. 135-142]. An inverse system is denoted by \( X = \{X_a, p_{ab}, A\} \).

A generalized arc is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval \( I = [0,1] \).

For a compact space \( X \) we denote by \( 2^X \) the hyperspace of all nonempty closed subsets of \( X \) equipped with the Vietoris topology. \( C(X) \) and \( X(n) \), where \( n \) is a positive integer, stand for the sets of all connected members of \( 2^X \) and of all nonempty subsets consisting of at most \( n \) points, respectively, both considered as subspaces of \( 2^X \), see [7].

For a mapping \( f : X \to Y \) define \( 2^f : 2^X \to 2^Y \) by \( 2^f(F) = f(F) \) for \( F \in 2^X \). By [12, 5.10] \( 2^f \) is continuous, \( 2^f(C(X)) \subset C(Y) \) and \( 2^f(X(n)) \subset Y(n) \). The restriction \( 2^f|C(X) \) is denoted by \( C(f) \).

An element \( \{x_a\} \) of the Cartesian product \( \prod \{X_a : a \in A\} \) is called a thread of \( X \) if \( p_{ab}(x_b) = x_a \) for any \( a, b \in A \) satisfying \( a \leq b \). The subspace of \( \prod \{X_a : a \in A\} \) consisting of all threads of \( X \) is called the limit of the inverse system \( X = \{X_a, p_{ab}, A\} \) and is denoted by \( \lim X \) or by \( \lim \{X_a, p_{ab}, A\} \) [5, p. 135].

Let \( X = \{X_a, p_{ab}, A\} \) be an inverse system of compact spaces with the natural projections \( p_a : \lim X \to X_a \), for \( a \in A \). Then \( 2^X = \{2^{X_a}, 2^{p_{ab}}, A\} \), \( C(X) = \{C(X_a), C(p_{ab}), A\} \) and \( X(n) = \{X_a(n), 2^{p_{ab}|X_b(n)}, A\} \) form inverse systems.

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systems. For each $F \in \lim_{\text{X}}$, i.e., for each closed $F \subseteq \lim X$ the set $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a} : \lim_{\text{X}} \rightarrow 2^{X_a}$ induced by $p_a$ for each $a \in A$. Define a mapping $M : \lim_{\text{X}} \rightarrow \lim_{\text{X}}$ by $M(F) = \{p_a(F) : a \in A\}$. Since $\{p_a(F) : a \in A\}$ is a thread of the system $2^X$, the mapping $M$ is continuous and one-to-one. It is also onto since for each thread $\{F_a : a \in A\}$ of the system $2^X$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Thus, $M$ is a homeomorphism. If $P_a : \lim 2^X \rightarrow 2^{X_a}, a \in A$, are the projections, then $P_aM = 2^{p_a}$. Identifying $F$ with $M(F)$ we have $P_a = 2^{p_a}$.

**Lemma 1.1.** [7, Lemma 2.]. Let $X = \lim X$. Then $2^X = \lim 2^X$, $C(X) = \lim C(X)$ and $X(n) = \lim X(n)$.

We say that an inverse system $X = \{X_a, p_{ab}, A\}$ is $\sigma$-directed if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of $A$ there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Let $A$ be a partially ordered directed set. We say that a subset $A_1 \subset A$ majorates [2, p. 9] another subset $A_2 \subset A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates $A$ is called cofinal in $A$. A subset of $A$ is said to be a chain if every two elements of it are comparable. The symbol $\sup B$, where $B \subset A$, denotes the lower upper bound of $B$ (if such an element exists in $A$). Let $\tau \geq \aleph_0$ be a cardinal number. A subset $B$ of $A$ is said to be $\tau$-closed in $A$ if for each chain $C \subset B$, with card$(B) \leq \tau$, we have sup$C \in B$, whenever the element sup$C$ exists in $A$. Finally, a directed set $A$ is said to be $\tau$-complete if for each chain $C$ of elements of $A$ with card$(C) \leq \tau$, there exists an element sup$C$ in $A$.

Suppose that we have two inverse systems $X = \{X_a, p_{ab}, A\}$ and $Y = \{Y_b, q_{bc}, B\}$. A morphism of the system $X$ into the system $Y$ [2, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \rightarrow A$ such that $\varphi(B)$ is cofinal in $A$, and of maps $f_b : X_{\varphi(b)} \rightarrow Y_b$ defined for all $b \in B$ such that the following diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : X \rightarrow Y$ induces a map, called the limit map of the morphism

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim X \rightarrow \lim Y$$

In the present paper we deal with the inverse systems defined on the same indexing set $A$. In this case, the map $\varphi : A \rightarrow A$ is taken to be the identity and we use the following notation \{\(f_a : X_a \rightarrow Y_a; a \in A\) : $X \rightarrow Y$.

We say that an inverse system $X = \{X_a, p_{ab}, A\}$ is factorizing [2, p. 17] if for each real-valued mapping $f : \lim X \rightarrow \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \rightarrow \mathbb{R}$ such that $f = f_ap_a$.

An inverse system $X = \{X_a, p_{ab}, A\}$ is said to be $\sigma$-directed if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of $A$ there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.
Lemma 1.2. [2, Corollary 1.3.2, p. 18]. If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a $\sigma$-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be $\tau$-continuous [2, p. 19] if for each chain $B$ in $A$ with $\text{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta \{p_{ab} : a \in B\}$ maps the space $X_b$ homeomorphically into the space $\lim \{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be $\tau$-system [2, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is $\tau$-continuous,

c) The indexing set $A$ is $\tau$-complete.

If $\tau = \aleph_0$, then $\tau$-system is called a $\sigma$-system. The following theorem is called the Spectral Theorem [2, p. 19].

Theorem 1.3. [2, Theorem 1.3.4, p. 19]. If a $\tau$-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another $\tau$-system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ is induced by a morphism of cofinal and $\tau$-closed subsystems. If two factorizing $\tau$-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and $\tau$-closed subsystems.

Let us remark that the requirement of surjectivity of limit projections of systems in Theorem 1.3 is essential [2, p. 21].

A fixed point of a function $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space $X$ is said to have the fixed point property provided that every surjective mapping $f : X \to X$ has a fixed point.

The following result is known.

Theorem 1.4. [10, Theorem 2, p. 17]. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a $\sigma$-system of compact spaces with the limit $X$ and onto projections $p_a : X \to X_a$. Let $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ be a morphism. Then the induced mapping $f = \lim \{f_a\} : X \to X$ has a fixed point if and only if each mapping $f_a : X_a \to X_a$, $a \in A$, $a \in A$, has a fixed point.

As an immediate consequence of this theorem and the Spectral theorem 1.3 we have the following result.

Theorem 1.5. Let a non-metric continuum $X$ be the inverse limit of an inverse $\sigma$-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each $X_a$ has the fixed point property and each bonding mapping $p_{ab}$ is onto. Then $X$ has the fixed point property.

Now we will prove some expanding theorems of non-metric compact spaces into $\sigma$-directed inverse systems of compact metric spaces.

Theorem 1.6. For each Cartesian product $X = \prod \{X_a : a \in A\}$ of spaces $X_a$ there exists a $\sigma$-directed inverse system $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ of the countable product $X^\mu$ such that $X$ is homeomorphic to $\lim \mathbf{X}$. Moreover, if each $X_a$ is metrizable continuum, then $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ is an inverse $\sigma$-system with monotone bonding mappings $P_{\mu\nu}$.
Proof. Let \( M \) be the set of all countable subsets \( \mu \) of \( A \) ordered by inclusion. If \( \mu \subseteq \nu \), then we write \( \mu \leq \nu \). It is clear that \( M \) is \( \sigma \)-directed. For each \( \mu \in M \) there exists \( X^\mu = \prod \{ \mathcal{X}_a : a \in \mu \} \). If \( \mu, \nu \in M \) and \( \mu \leq \nu \), then there exists the projection \( P_{\mu\nu} : X^\nu \to X^\mu \) which, as the projection, is monotone if \( X_\alpha \) are continua. Finally, we have the system \( \mathbf{X} = \{ X^\mu, P_{\mu\nu}, M \} \). Let us prove \( \mathbf{X} = \{ X^\mu, P_{\mu\nu}, M \} \) is an inverse \( \sigma \)-system. It is clear that \( M \) is \( \sigma \)-directed. Moreover, \( A \) is \( \sigma \)-complete. Namely, if \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n, \ldots \) is a countable chain in \( M \), then we have a countable chain \( \mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_n, \ldots \) of countable subsets of \( A \). It is clear that \( \mu = \bigcup \{ \mu_n : n \in \mathbb{N} \} \) is a countable subset of \( A \) and \( \mu = \sup \mu_n \).

It remains to prove that \( \mathbf{X} = \{ X^\mu, P_{\mu\nu}, M \} \) is continuous. Let \( B = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_\alpha, \ldots, \alpha < \tau, \mu_\alpha \in M \), be a chain with \( \sup \mu_\alpha = \gamma \in M \). We have transfinite inverse sequence \( \{ X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B \} \). Let us prove that the mappings \( P_{\mu_\alpha\gamma}, \alpha < \tau \) induce a homeomorphism of the spaces \( X^\gamma \) and \( \lim \{ X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B \} \). Let \( x \in X^\gamma \).

It is clear that \( P_{\mu_\alpha\gamma}(x) = x_{\mu_\alpha} \) is a point of \( X^{\mu_\alpha} \) and that \( P_{\mu_\alpha\mu_\beta}(x_{\mu_\alpha}) = x_{\mu_\beta} \) if \( \mu_\alpha \leq \mu_\beta \). This means that \( (x_{\mu_\alpha}) \) is a thread in \( \{ X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B \} \). Set \( H(x) = (x_{\mu_\alpha}). \) We have the mapping \( H : X^{\gamma} \to \lim \{ X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B \} \). It is clear that \( H \) is continuous, 1-1 and onto. Hence, \( H \) is a homeomorphism.

Theorem 1.7. For each Tychonoff cube \( I^m, m \geq \aleph_1, \) there exists an inverse \( \sigma \)-system \( \mathbf{I} = \{ I^a, P_{ab}, A \} \) of the Hilbert cubes \( I^a \) such that \( I^m \) is homeomorphic to \( \lim \mathbf{I} \). Equivalently, \( I^m \) has a \( \sigma \)-representation.

Proof. Let us recall that the Tychonoff cube \( I^m \) is the Cartesian product \( \prod \{ I_s : s \in S \} \), card(\( S \)) = \( m \), \( I_s = [0, 1] \) [5, p. 114]. If card(\( S \)) = \( \aleph_0 \), the Tychonoff cube \( I^m \) is called the Hilbert cube. Let \( A \) be the set of all countable subsets of \( S \) ordered by inclusion. If \( a \subseteq b \), then we write \( a \leq b \). It is clear that \( A \) is \( \sigma \)-directed. For each \( a \in A \) there exists the Hilbert cube \( I^a \). If \( a, b \in A \) and \( a \leq b \), then there exists the projection \( P_{ab} : I^b \to I^a \). Finally, we have the system \( \mathbf{I} = \{ I^a, P_{ab}, A \} \). The remaining part of the proof is the same as in the proof of Theorem 1.6.

Theorem 1.8. Let \( X \) be compact Hausdorff space such that \( w(X) \geq \aleph_1 \). There exists an inverse \( \sigma \)-system \( \mathbf{X} = \{ X_\alpha, P_{ab}, A \} \) such that \( X \) is homeomorphic to \( \lim \mathbf{X} \), i.e., every compact Hausdorff non-metric space has a \( \sigma \)-representation.

Proof. By [5, Theorem 2.3.23.] the space \( X \) is embeddable in \( I^{w(X)} \). From Theorem 1.7 it follows that \( I^{w(X)} \) is a limit of \( \mathbf{I} = \{ I^a, P_{ab}, A \} \), where every \( I^a \) is the Hilbert cube. Now, \( X \) is a closed subspace of \( \lim \mathbf{I} \). Let \( X_\alpha = P_m(X) \), where \( P_m : I^m \to I^a \) is a projection of the Tychonoff cube \( I^m \) onto the Hilbert cube \( I^a \). Let \( p_{ab} \) be the restriction of \( P_{ab} \) onto \( X_\alpha \). We have the inverse system \( \mathbf{X} = \{ X_\alpha, P_{ab}, A \} \) such that \( w(X_\alpha) \leq \aleph_0 \). It is obvious that \( X \) is homeomorphic to \( \lim \mathbf{X} \). Moreover, \( \mathbf{X} \) is an inverse \( \sigma \)-system since \( \mathbf{I} = \{ I^a, P_{ab}, A \} \) is an inverse \( \sigma \)-system.
2 Monotone-light factorization and inverse systems

A space $X$ is said to be rim-metrizable if it has a basis $\mathcal{B}$ such that $\text{Bd}(U)$ is metrizable for each $U \in \mathcal{B}$. Equivalently, a space $X$ is rim-metrizable if and only if for each pair $F, G$ of disjoint closed subsets of $X$ there exists a metrizable closed subset of $X$ which separates $F$ and $G$.

Lemma 2.1. [16, Theorem 1.2]. Let $X$ be a non-degenerate rim-metrizable continuum and let $Y$ be a continuous image of $X$ under a light mapping $f : X \to Y$. Then $w(X) = w(Y)$.

Lemma 2.2. [16, Theorem 3.2]. Let $X$ be a rim-metrizable continuum and let $f : X \to Y$ be a monotone mapping onto $Y$. Then $Y$ is rim-metrizable.

Let $\mathcal{M}$ be a class of continua such that $X$ is in $\mathcal{M}$ if and only if $X$ is the countable union of closed subsets $X_i$ which are either locally connected or rim-metrizable continua. Now we shall prove the following result.

Theorem 2.3. Let $X = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings $p_{ab}$. Then:

1) There exists an inverse system $M(X) = \{M_a, m_{ab}, A\}$ of compact spaces such that $m_{ab}$ are monotone surjections and $\lim X$ is homeomorphic to $\lim M(X)$,

2) If $X$ is $\sigma$-directed, then $M(X)$ is $\sigma$-directed,

3) If $X$ is $\sigma$-complete, then $M(X)$ is $\sigma$-complete,

4) If every $X_a$ is a metric space, $\lim X$ is in $\mathcal{M}$ and hereditarily unicoherent, then every $M_a$ is metrizable.

Proof. The statements 1)-3) are proved in [8, Theorem 3.12]. It remains to prove 4). Let $\lim X = \cup\{X_i : i \in \mathbb{N}\}$, where each $X_i$ is either a locally connected closed subset of $\lim X$ or a rim-metrizable subset of $\lim X$. From the proof of [8, Theorem 3.12] it follows that $M_a$ is a continuum such that there exist mappings $m_a : \lim X \to M_a$ and $\ell_a : M_a \to X_a$. Moreover, $m_a$ is monotone and $\ell_a$ is light. Firstly, suppose that $X_i$ is locally connected. Then $m_a(X_i) \subset M_a$ is locally connected [17, Lemma 1.5, p. 70]. Applying [11, Theorem 1] we conclude that $m_a(X_i)$ is metrizable. If $X_i$ is rim-metrizable, then $m_a(X_i)$ is rim-metrizable (Theorem 2.2) since from hereditarily unicoherence of $\lim X$ it follows that $m_a|X_i$ is monotone. Finally, from Theorem 2.1 it follows that $m_a(X_i)$ metrizable. Now, $M_a = \cup\{m_a(X_i) : i \in \mathbb{N}\}$. Using [5, Corollary 3.1.20, p. 171] we see that $M_a$ is metrizable. \qed

An arboroid is an hereditarily unicoherent continuum which is arcwise connected by generalized arcs. A metrizable arboroid is a dendroid. If $X$ is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in $X$ with endpoints $x$ and $y$. If $[x, y]$ is an arc, then $[x, y] \setminus \{x, y\}$ is denoted by $(x, y)$. 

A point \( t \) of an arboroid \( X \) is said to be a ramification point of \( X \) if \( t \) is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point \( e \) of an arboroid \( X \) is said to be end point of \( X \) if there exists no arc \([a, b]\) in \( X \) such that \( x \in [a, b] \setminus \{a, b\} \).

A continuum is a graph if it is the union of a finite number of metric free arcs. A tree is an acyclic graph. A continuum \( X \) is tree-like if for each open cover \( U \) of \( X \), there is a tree \( X_U \) and a \( U \)-mapping \( f_U : X \to X_U \) (the inverse image of each point is contained in a member of \( U \)).

Every tree-like continuum is hereditarily unicoherent. A dendroid is tree-like [3].

**Proposition 1.** If \( X = \{X_a, p_{ab}, A\} \) is an inverse system of tree-like continua and if \( p_{ab} \) are onto mappings, then the limit \( X = \lim X \) is a tree-like continuum.

**Proof.** Let \( U = \{U_1, ..., U_n\} \) be an open covering of \( X \). There exist an \( a \in A \) and an open covering \( U_a = \{U_1a, ..., U_ka\} \) such that \( \{p_{a}^{-1}(U_1a), ..., p_{a}^{-1}(U_ka)\} \) refines the covering \( U \). There exist a tree \( T_a \) and an \( U_a \)-mapping \( f_a : X_a \to T_a \) since \( X_a \) is tree-like. It is clear that \( f_a p_a : X \to T_a \) is an \( U \)-mapping. Hence, \( X \) is tree-like. \( \square \)

If an arboroid \( X \) has only one ramification point \( t \), it is called a generalized fan with the top \( t \). A metrizable generalized fan is called a fan.

The following result is known for the generalized fans.

**Theorem 2.4.** [9, Theorem 4.22, p. 410]. For every generalized fan \( X \) there exists a \( \sigma \)-directed inverse system \( X = \{X_a, p_{ab}, A\} \) of metric fans such that all the bonding mappings \( p_{ab} \) are surjective and the limit \( \lim X \) is homeomorphic to \( X \).

Now we shall prove that there is a \( \sigma \)-system with the property as in Theorem 2.4.

**Theorem 2.5.** For every generalized fan \( X \) there exists a \( \sigma \)-system \( X = \{X_a, p_{ab}, A\} \) of metric fans such that all the bonding mappings \( p_{ab} \) are surjective and the limit \( \lim X \) is homeomorphic to \( X \).

**Proof.** It remains to prove that there exists such \( \sigma \)-system. Let \( X = \{X_a, p_{ab}, A\} \) be as in Theorem 2.4. The proof is broken into several steps.

1. **Step 1.** For each subset \( \Delta_0 \) of \((A, \leq)\) we define sets \( \Delta_n, n = 0, 1, ..., \) by the inductive rule \( \Delta_{n+1} = \Delta_n \cup \{m(x, y) : x, y \in \Delta_n\} \), where \( m(x, y) \) is a member of \( A \) such that \( x, y \leq m(x, y) \). Let \( \Delta = \bigcup \{\Delta_n : n \in \mathbb{N}\} \). It is clear that \( \text{card}(\Delta) = \text{card}(\Delta_0) \). Moreover, \( \Delta \) is directed by \( \leq \). For each directed set \((A, \leq)\) we define

\[
A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \text{ and } \Delta \text{ is directed by } \leq\}.
\]

2. **Step 2.** If \( A \) is a directed set, then \( A_\sigma \) is \( \sigma \)-directed and \( \sigma \)-complete. Let \( \{\Delta^1, \Delta^2, ..., \Delta^n, ...\} \) be a countable subset of \( A_\sigma \). Then \( \Delta_0 = \cup \{\Delta^1, \Delta^2, ..., \Delta^n, ...\} \) is a countable subset of \( A_\sigma \). Define sets \( \Delta_n, n = 0, 1, ..., \) by the inductive rule \( \Delta_{n+1} = \Delta_n \cup \{m(x, y) : x, y \in \Delta_n\} \), where \( m(x, y) \) is a member of \( A \) such that
Let $\Delta = \bigcup \{\Delta_n: n \in \mathbb{N}\}$. It is clear that $\text{card}(\Delta) = \text{card}(\Delta_0)$. This means that $\Delta$ is countable. Moreover $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence $A_\sigma$ is $\sigma$-directed. Let us prove that $A_\sigma$ is $\sigma$-complete. Let $\Delta^1 \subset \Delta^2 \subset \ldots \subset \Delta^n \subset \ldots$ be a countable chain in $A_\sigma$. Then $\Delta = \bigcup \{\Delta^i : i \in \mathbb{N}\}$ is countable and directed subset of $A$, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup \{\Delta^i : i \in \mathbb{N}\}$. This means that $A_\sigma$ is $\sigma$-complete.

**Step 3.** If $\Delta \in A_\sigma$, let $X^\Delta = \{X_s, p_{\omega b}, \Delta\}$ and $X_\Delta = \lim X^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $P_{\Delta \Gamma} : X_\Gamma \to X_\Delta$ denote the map induced by the projections $p_{\delta}^\Gamma : X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system $X^\Gamma$.

**Step 4.** If $X = \{X_s, p_{ab}, A\}$ is an inverse system, then $X_\sigma = \{X_\Delta, P_{\Delta \Gamma}, A_\sigma\}$ is a $\sigma$-directed and $\sigma$-complete inverse system such that $\lim X$ and $\lim X_\sigma$ are homeomorphic. Each thread $x = (x_a : a \in A)$ induces the thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $x_\Delta \in X_\Delta$. This means that we have a mapping $H : \lim X \to \lim X_\sigma$ such that $H(x) = (x_\Delta : \Delta \in A_\sigma)$. It is obvious that $H$ is continuous and $1$-$1$. The mapping $H$ is onto since the collections of the threads $\{x_\Delta : \Delta \in A_\sigma\}$ induces the thread in $X$. We infer that $H$ is a homeomorphism since $\lim X$ is compact.

**Step 5.** Every $X_\Delta$ is a metric fan. Every $X_\Delta$ is a metric tree-like continuum. This follows from Proposition 1. This means that every $X_\Delta$ is hereditarily unicoherent. Let us prove that every $X_\Delta$ is arcwise connected. This follows from [15, Theorem]. As in the proof of Theorem 4.19. of [9] we conclude that every $X_\Delta$ is a fan.

**Step 6.** Every projection $P_{\Delta} : \lim X_\sigma \to X_\Delta$ is onto. This follows from the assumption that the bonding mappings $p_{ab}$ are surjective.

Finally, $X_\sigma = \{X_\Delta, P_{\Delta \Gamma}, A_\sigma\}$ is a desired $\sigma$-system. \hfill $\Box$

The following cardinal invariant is a ”connected” version of the cellularity. Let $X$ be a continuum and let $\tau(X) = \sup \{\text{card}(C) : C \text{ is a disjoint family of non-degenerate subcontinua in } X\}$.

Similarly, a ”connected” version of the density is defined as follows.

$\bar{d}(X) = \min \{\text{card}(D) : D \text{ is a subset of } X \text{ meeting each non-degenerate subcontinuum of } X\}$.

The main results of [1] are:

a) $w(X) \leq \min \{\bar{d}(X), \tau(X)^+\}$,

b) Under the generalized Suslin Hypothesis $w(X) \leq \bar{d}(X)$,

c) Each Suslinian continuum is hereditarily decomposable, has weight $\leq \omega_1$ (and is metrizable if the Suslin Hypothesis holds).

The main Theorem of [1] is

**Theorem 2.6.** Each compact space $X$ with $w(X) > \tau(X)$ is the limit of an inverse well-ordered spectrum of length $\tau(X)^+$ consisting of compacta with weight $\leq \tau(X)$ and monotone bonding mappings.
3 Fixed point property for $2^X$ and $C(X)$ if $X$ is a fan

In this section we shall prove the fixed point property for $2^X$ and $C(X)$ if $X$ is a fan. If $X$ is a metric fan, i.e., a fan then we have the following result.

**Theorem 3.1.** [6, Theorem 22.13, p. 194]. If $X$ is a fan, then $2^X$ and $C(X)$ have the fixed point property.

For generalized fans the proofs for $2^X$ and $C(X)$ are different. We start be the proof for $2^X$.

**Theorem 3.2.** If $X$ is a generalized fan, then $2^X$ have the fixed point property.

**Proof.** By Theorem 2.5 there exists a $\sigma$-system $X = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings $p_{ab}$ are surjective and the limit $\lim X$ is homeomorphic to $X$. Now we have the inverse system $2^X = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is $2^X$ (Lemma 1.1). It is clear that the mappings $2^{p_{ab}}$ are onto if the bonding mappings $p_{ab}$ are onto. Now we can apply Theorem 1.5 since, by Theorem 3.1, every $2^{X_a}$ has the fixed point property. Hence, $2^X$ has the fixed point property.

Let $X = \{X_a, p_{ab}, A\}$ be a $\sigma$-system. If we consider the inverse system $C(X) = \{C(X_a), C(p_{ab}), A\}$, then $C(p_{ab})$ are not always the surjections. This is the case only if $p_{ab}$ are weakly confluent mappings [13, Theorem (0.49.1), p. 24]. This means that we can apply Theorem 1.5 on the system $C(X) = \{C(X_a), C(p_{ab}), A\}$ only if $p_{ab}$ are weakly confluent mappings. Let us recall that a mapping $f : X \to Y$ is weakly confluent provided that for each subcontinuum $K$ of $Y$ there exists a component $A$ of $f^{-1}(K)$ such that $f(A) = K$ [13, (0.45.4), p. 22]. Each monotone mapping is weakly confluent. It follows that expanding Theorem 2.5 is not enough for proving the fixed point property of $C(X)$ when $X$ is a generalized fan. For this reason we shall consider the fixed point property for $2^X$ and $C(X)$ if $X$ is a generalized fan in class $\mathcal{M}$.

**Theorem 3.3.** If $X$ is a generalized fan in the class $\mathcal{M}$, then $C(X)$ have the fixed point property.

**Proof.** By Theorem 2.5 there exists a $\sigma$-system $X = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings $p_{ab}$ are surjective and the limit $\lim X$ is homeomorphic to $X$. Applying Theorem 2.3 we obtain an inverse system $M(X) = \{M_a, m_{ab}, A\}$ of compact metric spaces such that $m_{ab}$ are monotone surjections and $\lim X$ is homeomorphic to $\lim M(X)$, i.e., $X$ is homeomorphic to $\lim M(X)$. Moreover, from the fact that the projections $m_a : \lim M(X) \to M_a$ are monotone it follows that $M_a$ is a fan. Now we have the inverse system $C(M(X)) = \{C(M_a), C(m_{ab}), A\}$ whose limit is $C(X)$ (Lemma 1.1). It is clear that the mappings $C(p_{ab})$ are onto if the bonding mappings $m_{ab}$ are monotone. Now we can apply Theorem 1.5 since, by Theorem 3.1, every $C(M_a)$ has the fixed point property. Hence, $C(X)$ has the fixed point property.
4 Fixed point property for $2^X$ and $C(X)$ if $X$ is a smooth arboroid

An arboroid $X$ is said to be smooth if there exists a point $p \in X$, called an initial point of $X$, such that for every convergent net of points $\{a_n : n \in E\}$ of $X$ the condition

$$\lim_{n \in E} a_n = a$$

implies that the net of arcs $pa_n$ is convergent and

$$\lim_{n \in E} pa_n = pa.$$ 

The set of all points of $X$ each of them can be taken as an initial point will be called the initial set of $X$.

**Lemma 4.1.** [4, Corollary 10, p. 309]. If $f$ is a monotone mapping of a smooth arboroid $X$ onto $Y$, then $Y$ is a smooth arboroid and $f(P) \subset P^*$, where $P$ and $P^*$ denote the initial sets of $X$ and $Y$ respectively.

**Theorem 4.2.** [6, Theorem 22.12, p. 194]. If $X$ is a smooth dendroid, then $2^X$ and $C(X)$ have the fixed point property.

**Theorem 4.3.** If a non-metrizable arboroid $X$ is in class $M$, then there exists an inverse $\sigma$-system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a dendroid, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$.

**Proof.** From Corollary 2.3 it follows that there exists an inverse $\sigma$-system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a metric continuum, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$. From 4.1 we infer that every $X_a$ is an arboroid. Hence, every $X_a$ is a metrizable arboroid, i.e., a dendroid.

**Theorem 4.4.** If a non-metrizable smooth arboroid $X$ is in class $M$, then there exists an inverse $\sigma$-system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a smooth dendroid, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$.

**Proof.** Theorem follows from Theorems 2.3 and 4.1.

**Theorem 4.5.** If $X$ is a smooth arboroid in the class $M$, then $2^X$ and $C(X)$ have the fixed point property.

**Proof.** By Theorem 4.4 there exists an inverse $\sigma$-system $X = \{X_a, p_{ab}, A\}$ such that each $X_a$ is a smooth dendroid, every $p_{ab}$ is monotone and $X$ is homeomorphic to $\lim X$. Now the systems $2^X = \{2^{X_a}, 2^{p_{ab}}, A\}$, and $C(X) = \{C(X_a), C(p_{ab}), A\}$ satisfy the conditions of Theorem 1.5. Hence, $2^X$ and $C(X)$ have the fixed point property.
5 Fixed point property for \(2^X\) and \(C(X)\) if \(X\) is a cone over a generalized fan or a smooth arboroid

Let \(Y\) be a topological space. The suspension over \(Y\), which we denote by \(\Sigma(Y)\), is the quotient space obtained from \(Y \times [-1,1]\) by shrinking \(Y \times \{-1\}\) and \(Y \times \{1\}\) to (different) points.

Theorem 5.1. [6, Theorem 22.15, p. 195]. Let \(X = \text{Cone}(Y)\), where \(Y\) is a fan or a smooth dendroid. Then, \(2^X\) and \(C(X)\) have the fixed point property.

Proof. If \(Y \in \mathcal{M}\) is a generalized fan or a smooth arboroid, then there exists an inverse \(\sigma\)-system \(Y = \{Y_a, p_{ab}, A\}\) such that each \(Y_a\) is a smooth dendroid, every \(p_{ab}\) is monotone and \(Y\) is homeomorphic to \(\lim Y\). Furthermore, \(X = \text{Cone}(Y) = \lim \{\text{Cone}(Y_a), q_{ab}, A\}\) [14, 3.15, p. 41 and Exercise 3.30, p. 49]. Let us observe that \(q_{ab}\) are monotone. This means that the inverse systems \(\{2\text{Cone}(Y_a), 2q_{ab}, A\}\) and \(\{C(\text{Cone}(Y_a)), C(q_{ab}), A\}\) satisfy the conditions of Theorem 1.5. Hence, \(2^X\) and \(C(X)\) have the fixed point property. \(\square\)

For suspension \(\Sigma(Y)\) over \(Y\) we have the following result.

Theorem 5.3. [6, Theorem 22.16, p. 196]. Let \(X = \Sigma(Y)\), where \(Y\) is a fan or a smooth dendroid. Then, \(2^X\) and \(C(X)\) have the fixed point property.

Analogue result for non-metric settings is as follows.

Theorem 5.4. Let \(X = \Sigma(Y)\), where \(Y \in \mathcal{M}\) is a generalized fan or a smooth arboroid. Then, \(2^X\) and \(C(X)\) have the fixed point property.

6 Fixed point property for \(2^X\) and \(C(X)\) if \(X\) is a product of generalized fans or smooth arboroids

In this section we shall generalize the following result.

Theorem 6.1. [6, Theorem 22.14, p. 195]. Let \(X\) be a finite or countably infinite Cartesian product, where each coordinate space is a fan or a smooth dendroid. Then \(2^X\) and \(C(X)\) have the fixed point property.

Theorem 6.2. Let \(X\) be a Cartesian product, where each coordinate space is a fan or a smooth dendroid. Then \(2^X\) and \(C(X)\) have the fixed point property.

Proof. If \(X\) is a finite or countably infinite Cartesian product, then apply Theorem 6.1. Suppose now that \(X\) is the Cartesian product \(X = \prod \{X_a : a \in A\}\), where \(\text{card}(A) > \aleph_0\). From Theorem 1.6 it follows that for product \(X = \prod \{X_a : a \in A\}\) of spaces \(X_a\) there exists a \(\sigma\)-directed inverse system \(X = \{X^\mu, P_{\mu\nu}, M\}\) of the
countable product \( X^\mu \) such that \( X \) is homeomorphic to \( \lim X \). Moreover, if each \( X_n \) is metrizable continuum, then \( X = \{ X^\mu, P_{\mu\nu}, M \} \) is an inverse \( \sigma \)-system with monotone bonding mappings \( P_{\mu\nu} \). The inverse systems \( 2^X = \{ 2^{X^\nu}, 2^{P_{\mu\nu}}, M \} \) and \( C(X) = \{ C(X^\mu), C(P_{\mu\nu}), M \} \) satisfy the assumptions of Theorem 1.5. Hence, \( 2^X \) and \( C(X) \) have the fixed point property. \( \square \)

We close this section with the following result.

**Theorem 6.3.** Let \( X \) be a Cartesian product, where each coordinate space is a generalized fan or a smooth arboroid of the same weight. Then \( 2^X \) and \( C(X) \) have the fixed point property.

**Proof.** Now we have \( X = \prod \{ X_m : m \in M \} \) and \( w(X_m) = k \) for every \( m \in M \), where \( k \) is an uncountable cardinal. This means that for every \( m \in M \) we have an inverse \( \sigma \)-system \( X_m = \{ X_{m,a}, p_{m,ab}, A \} \) whose limit is \( X_m \). Now \( X \) is homeomorphic to \( \lim \{ \Pi X_{m,a}, \Pi p_{m,ab}, A \} \) [5, Exercise 2.5.D.(b), p. 143]. Finally the systems \( \{ 2^{\Pi X_{m,a}}, 2^{\Pi p_{m,ab}}, A \} \) and \( \{ C(\Pi X_{m,a}), C(\Pi p_{m,ab}), A \} \) satisfy the conditions of Theorem 1.5 since, by Theorem 6.2, the continua \( 2^{\Pi X_{m,a}} \) and \( C(\Pi X_{m,a}) \) have the fixed point property. Hence, \( 2^X \) and \( C(X) \) have the fixed point property. \( \square \)

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