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FIXED POINT PROPERTY FOR HYPERSPACES OF ARBOROIDS

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Abstract

The main purpose of this paper is to study arboroids, a non-metric analogue of dendroids. It is proved that hyperspaces of some arboroids have the fixed point property.

1 Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $\mathrm{w}(X)$. The cardinality of a set A is denoted by $\mathrm{card}(A)$. We shall use the notion of inverse system as in [5, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

For a compact space X we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology. $C(X)$ and $X(n)$, where n is a positive integer, stand for the sets of all connected members of 2^X and of all nonempty subsets consisting of at most n points, respectively, both considered as subspaces of 2^X, see [7].

For a mapping $f : X \to Y$ define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [12, 5.10] 2^f is continuous, $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [5, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}|\ X_b(n), A\}$ form inverse

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systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$ the set $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a} : 2^{\lim \mathbf{X}} \to 2^{X_a}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$. Since $\{p_a(F) : a \in A\}$ is a thread of the system $2^{\mathbf{X}}$, the mapping M is continuous and one-to-one. It is also onto since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Thus, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{X_a}, a \in A$, are the projections, then $P_a M = 2^{p_a}$. Identifying F with $M(F)$ we have $P_a = 2^{p_a}$.

Lemma 1.1. *[7, Lemma 2.]. Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ *and* $X(n) = \lim \mathbf{X}(n)$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ*-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Let A be a partially ordered directed set. We say that a subset $A_1 \subset A$ *majorates* [2, p. 9] another subset $A_2 \subset A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subset A$, denotes the lower upper bound of B (if such an element exists in A). Let $\tau \geq \aleph_0$ be a cardinal number. A subset B of A is said to be τ*-closed* in A if for each chain $C \subset B$, with $\mathrm{card}(B) \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ*-complete* if for each chain C of A of elements of A with $\mathrm{card}(C) \leq \tau$, there exists an element $\sup C$ in A.

Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system* X *into the system* $\mathbf{Y}$ [2, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array} \tag{1.1}$$

diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [2, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ*-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1.2. *[2, Corollary 1.3.2, p. 18]. If* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *is a* σ*-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be τ*-continuous* [2, p. 19] if for each chain B in A with $\operatorname{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta\,\{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be τ*-system* [2, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

If $\tau = \aleph_0$, then τ-system is called a σ*-system*. The following theorem is called the *Spectral Theorem* [2, p. 19].

Theorem 1.3. *[2, Theorem 1.3.4, p. 19]. If a* τ*-system* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *with surjective limit projections is factorizing, then each map of its limit space into the limit space of another* τ*-system* $\mathbf{Y} = \{Y_a,\ q_{ab},\ A\}$ *is induced by a morphism of cofinal and* τ*-closed subsystems. If two factorizing* τ*-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and* τ*-closed subsystems.*

Let us remark that the requirement of surjectivity of limit projections of systems in Theorem 1.3 is essential [2, p. 21].

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every surjective mapping $f : X \to X$ has a fixed point.

The following result is known.

Theorem 1.4. *[10, Theorem 2, p. 17]. Let* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *be a* σ*-system of compact spaces with the limit* X *and onto projections* $p_a : X \to X_a$*. Let* $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ *be a morphism. Then the induced mapping* $f = \lim\{f_a\} : X \to X$ *has a fixed point if and only if each mapping* $f_a : X_a \to X_a$, $a \in A$, *has a fixed point.*

As an immediate consequence of this theorem and the Spectral theorem 1.3 we have the following result.

Theorem 1.5. *Let a non-metric continuum* X *be the inverse limit of an inverse* σ*-system* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *such that each* X_a *has the fixed point property and each bonding mapping* p_{ab} *is onto. Then* X *has the fixed point property.*

Now we will prove some expanding theorems of non-metric compact spaces into σ-directed inverse systems of compact metric spaces.

Theorem 1.6. *For each Cartesian product* $X = \prod\{X_a : a \in A\}$ *of spaces* X_a *there exists a* σ*-directed inverse system* $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ *of the countable product* X^μ *such that* X *is homeomorphic to* $\lim \mathbf{X}$*. Moreover, if each* X_a *is metrizable continuum, then* $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ *is an inverse* σ*-system with monotone bonding mappings* $P_{\mu\nu}$*.*

Proof. Let M be the set of all countable subsets μ of A ordered by inclusion. If $\mu \subseteq \nu$, then we write $\mu \leq \nu$. It is clear that M is σ-directed. For each $\mu \in M$ there exists $X^\mu = \prod\{X_a : a \in \mu\}$. If $\mu, \nu \in M$ and $\mu \leq \nu$, then there exists the projection $P_{\mu\nu} : X^\nu \to X^\mu$ which, as the projection, is monotone if X_a are continua. Finally, we have the system $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$. Let us prove $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ is an inverse σ-system. It is clear that M is σ-directed. Moreover, A is σ-complete. Namely, if $\mu_1 \leq \mu_2 \leq ... \leq \mu_n, ...$ is a countable chain in M, then we have a countable chain $\mu_1 \subseteq \mu_2 \subseteq ... \subseteq \mu_n, ...$ of countable subsets of A. It is clear that $\mu = \bigcup\{\mu_n : n \in \mathbb{N}\}$ is a countable subset of A and $\mu = \sup \mu_n$. It remains to prove that $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ is continuous. Let $B = \mu_1 \leq \mu_2 \leq ... \leq \mu_\alpha, ..., \alpha < \tau, \mu_\alpha \in M$, be a chain with $\sup \mu_\alpha = \gamma \in M$. We have transfinite inverse sequence $\{X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B\}$. Let us prove that the mappings $P_{\mu_\alpha\gamma}, \alpha < \tau$ induce a homeomorphism of the spaces X^γ and $\lim\{X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B\}$. Let $x \in X^\gamma$. It is clear that $P_{\mu_\alpha\gamma}(x) = x_{\mu_\alpha}$ is a point of X^{μ_α} and that $P_{\mu_\alpha\mu_\beta}(x_{\mu_\beta}) = x_{\mu_\alpha}$ if $\mu_\alpha \leq \mu_\beta$. This means that (x_{μ_α}) is a thread in $\{X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B\}$. Set $H(x) = (x_{\mu_\alpha})$. We have the mapping $H : X^\gamma \to \lim\{X^{\mu_\alpha}, P_{\mu_\alpha\mu_\beta}, B\}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. If each X_a is metrizable, then $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ is an inverse σ-system since $w(X^\mu) \leq \aleph_0$. Let us prove that X is homeomorphic to $\lim \mathbf{X}$. Let $x \in X$. It is clear that $P_\mu(x) = x_\mu$ is a point of X^μ and that $P_{\mu\nu}(x_\nu) = x_\mu$ if $\mu \leq \nu$. This means that (x_μ) is a thread in $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$. Set $H(x) = (x_\mu)$. We have the mapping $H : X \to \lim \mathbf{X}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. □

Theorem 1.7. *For each Tychonoff cube I^m, $m \geq \aleph_1$, there exists an inverse σ-system $\mathbf{I} = \{I^a, P_{ab}, A\}$ of the Hilbert cubes I^a such that I^m is homeomorphic to* $\lim \mathbf{I}$. *Equivalently, I^m has a σ-representation.*

Proof. Let us recall that the *Tychonoff cube* I^m is the Cartesian product $\prod\{I_s : s \in S\}$, $\text{card}(S) = m$, $I_s = [0, 1]$ [5, p. 114]. If $\text{card}(S) = \aleph_0$, the Tychonoff cube I^m is called the *Hilbert cube*. Let A be the set of all countable subsets of S ordered by inclusion. If $a \subseteq b$, then we write $a \leq b$. It is clear that A is σ-directed. For each $a \in A$ there exists the Hilbert cube I^a. If $a, b \in A$ and $a \leq b$, then there exists the projection $P_{ab} : I^b \to I^a$. Finally, we have the system $\mathbf{I} = \{I^a, P_{ab}, A\}$. The remaining part of the proof is the same as in the proof of Theorem 1.6 □

Theorem 1.8. *Let X be compact Hausdorff space such that $w(X) \geq \aleph_1$. There exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to* $\lim \mathbf{X}$, *i.e., every compact Hausdorff non-metric space has a σ-representation.*

Proof. By [5, Theorem 2.3.23.] the space X is embeddable in $I^{w(X)}$. From Theorem 1.7 it follows that $I^{w(X)}$ is a limit of $\mathbf{I} = \{I^a, P_{ab}, A\}$, where every I^a is the Hilbert cube. Now, X is a closed subspace of $\lim \mathbf{I}$. Let $X_a = P_m(X)$, where $P_m : I^m \to I^a$ is a projection of the Tychonoff cube I^m onto the Hilbert cube I^a. Let p_{ab} be the restriction of P_{ab} onto X_b. We have the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \aleph_0$. It is obvious that X is homeomorphic to $\lim \mathbf{X}$. Moreover, $\mathbf{X}$ is an inverse σ-system since $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse σ-system. □

2 Monotone-light factorization and inverse systems

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $Bd(U))$ is metrizable for each $U \in \mathcal{B}$. Equivalently, a space X is rim-metrizable if and only if for each pair F, G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

Lemma 2.1. *[16, Theorem 1.2]. Let X be a non-degenerate rim-metrizable continuum and let Y be a continuous image of X under a light mapping $f : X \to Y$. Then $w(X) = w(Y)$.*

Lemma 2.2. *[16, Theorem 3.2]. Let X be a rim-metrizable continuum and let $f : X \to Y$ be a monotone mapping onto Y. Then Y is rim-metrizable.*

Let $\mathcal{M}$ be a class of continua such that X is in $\mathcal{M}$ if and only if X is the countable union of closed subsets X_i which are either locally connected or rim-metrizable continua. Now we shall to prove the following result.

Theorem 2.3. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

1) *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and* $\lim X$ *is homeomorphic to* $\lim M(X)$*,*

2) *If $\mathbf{X}$ is σ-directed, then $M(\mathbf{X})$ is σ-directed,*

3) *If $\mathbf{X}$ is σ-complete, then $M(\mathbf{X})$ is σ-complete,*

4) *If every X_a is a metric space,* $\lim X$ *is in $\mathcal{M}$ and hereditarily unicoherent, then every M_a is metrizable.*

Proof. The statements 1)-3) are proved in [8, Theorem 3.12]. It remains to prove 4). Let $\lim \mathbf{X} = \cup\{\mathbf{X}_i : i \in \mathbb{N}\}$, where each X_i is either a locally connected closed subsets of $\lim \mathbf{X}$ or a rim-metrizable subsets of $\lim \mathbf{X}$. From the proof of [8, Theorem 3.12] it follows that M_a is a continuum such that there exists the mappings $m_a : \lim \mathbf{X} \to M_a$ and $\ell_a : M_a \to X_a$. Moreover, m_a is monotone and ℓ_a is light. Firstly, suppose that X_i is locally connected. Then $m_a(X_i) \subset M_a$ is locally connected [17, Lemma 1.5, p. 70]. Applying [11, Theorem 1] we conclude that $m_a(X_i)$ is metrizable. If X_i is rim-metrizable, then $m_a(X_i)$ is rim-metrizable (Theorem 2.2) since from hereditarily unicoherence of $\lim \mathbf{X}$ it follows that $m_a|X_i$ is monotone. Finally, from Theorem 2.1 it follows that $m_a(X_i)$ metrizable. Now, $M_a = \cup\{m_a(\mathbf{X}_i) : i \in \mathbb{N}\}$. Using [5, Corollary 3.1.20, p. 171] we see that M_a is metrizable. □

An *arboroid* is an hereditarily unicoherent continuum which is arcwise connected by generalized arcs. A metrizable arboroid is a *dendroid.* If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y. If $[x, y]$ is an arc, then $[x, y] \setminus \{x, y\}$ is denoted by (x, y).

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point e of an arboroid X is said to be *end point* of X if there exists no arc $[a, b]$ in X such that $x \in [a, b] \diagdown \{a, b\}$.

A continuum is a *graph* if it is the union of a finite number of metric free arcs. A *tree* is an acyclic graph. A continuum X is *tree-like* if for each open cover $\mathcal{U}$ of X, there is a tree $X_{\mathcal{U}}$ and a $\mathcal{U}$-mapping $f_{\mathcal{U}} : X \to X_{\mathcal{U}}$ (the inverse image of each point is contained in a member of $\mathcal{U}$).

Every tree-like continuum is hereditarily unicoherent. A dendroid is tree-like [3].

Proposition 1. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of tree-like continua and if p_{ab} are onto mappings, then the limit $X = \lim \mathbf{X}$ is a tree-like continuum.*

Proof. Let $\mathcal{U} = \{U_1, ..., U_n\}$ be an open covering of X. There exist an $a \in A$ and an open covering $\mathcal{U}_a = \{U_{1a}, ..., U_{ka}\}$ such that $\{p_a^{-1}(U_{1a}), ..., p_a^{-1}(U_{ka})\}$ refines the covering $\mathcal{U}$. There exist a tree T_a and an $\mathcal{U}_a$-mapping $f_u : X_a \to T_a$ since X_a is tree-like. It is clear that $f_u p_a : X \to T_a$ is an $\mathcal{U}$-mapping. Hence, X is tree-like. □

If an arboroid X has only one ramification point t, it is called a *generalized fan* with the top t. A metrizable generalized fan is called a *fan*.

The following result is known for the generalized fans.

Theorem 2.4. *[9, Theorem 4.22, p. 410]. For every generalized fan X there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X.*

Now we shall prove that there is a σ-system with the property as in Theorem 2.4.

Theorem 2.5. *For every generalized fan X there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X.*

Proof. It remains to prove that there exists such σ-system. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be as in Theorem 2.4. The proof is broken into several steps.

Step 1. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, ...,$ by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x, y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\text{card}(\Delta) = \text{card}(\Delta_0)$. Moreover, Δ is directed by $\leq$. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \text{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Step 2. *If A is a directed set, then A_σ is σ-directed and σ-complete.* Let $\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ be a countable subset of A_σ. Then $\Delta_0 = \cup\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ is a countable subset of A_σ. Define sets Δ_n, $n = 0, 1, ...,$ by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x, y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that

$x, y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n \colon n \in \mathbb{N}\}$. It is clear that $\mathrm{card}(\Delta) = \mathrm{card}(\Delta_0)$. This means that Δ is countable. Moreover $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence A_σ is σ-directed. Let us prove that A_σ is σ-complete. Let $\Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ...$ be a countable chain in A_σ. Then $\Delta = \cup\{\Delta^i : i \in \mathbb{N}\}$ is countable and directed subset of A, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that A_σ is σ-complete.

Step 3. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b,\ p_{bb'},\ \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $P_{\Delta\Gamma} \colon X_\Gamma \to X_\Delta$ denote the map induced by the projections $p^\Gamma_\delta : X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system $\mathbf{X}^\Gamma$.

Step 4. *If* $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ *is an inverse system, then* $\mathbf{X}_\sigma = \{X_\Delta,\ P_{\Delta\Gamma},\ A_\sigma\}$ *is a σ-directed and σ-complete inverse system such that* $\lim \mathbf{X}$ *and* $\lim \mathbf{X}_\sigma$ *are homeomorphic.* Each thread $x = (x_a : a \in A)$ induces the thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $x_\Delta \in X_\Delta$. This means that we have a mapping $H : \lim \mathbf{X} \to \lim \mathbf{X}_\sigma$ such that $H(x) = (x_\Delta : \Delta \in A_\sigma)$. It is obvious that H is continuous and 1-1. The mapping H is onto since the collections of the threads $\{x_\Delta : \Delta \in A_\sigma\}$ induces the thread in $\mathbf{X}$. We infer that H is a homeomorphism since $\lim \mathbf{X}$ is compact.

Step 5. *Every* X_Δ *is a metric fan.* Every X_Δ is a metric tree-like continuum. This follows from Proposition 1. This means that every X_Δ is hereditarily unicoherent. Let us prove that every X_Δ is arcwise connected. This follows from [15, Theorem]. As in the proof of Theorem 4.19. of [9] we conclude that every X_Δ is a fan.

Step 6. *Every projection* $P_\Delta : \lim \mathbf{X}_\sigma \to X_\Delta$ *is onto.* This follows from the assumption that the bonding mappings p_{ab} are surjective.

Finally, $\mathbf{X}_\sigma = \{X_\Delta,\ P_{\Delta\Gamma},\ A_\sigma\}$ is a desired σ-system. □

The following cardinal invariant is a "connected" version of the cellularity. Let X be a continuum and let

$$\overline{c}(X) = \sup\{\mathrm{card}(\mathcal{C}) : \mathcal{C} \quad \text{is a disjoint family of non-degenerate subcontinua in } X\}.$$

Similarly, a "connected" version of the density is defined as follows.

$$\overline{d}(X) = \min\{\mathrm{card}(D) : D \text{ is a subset of } X \text{ meeting each non-degenerate subcontinuum of } X\}.$$

The main results of [1] are:

a) $w(X) \leq \min\{\overline{d}(X), \overline{c}(X)^+\}$,

b) Under the generalized Suslin Hypothesis $w(X) \leq \overline{c}(X)$,

c) Each Suslinian continuum is hereditarily decomposable, has weight $\leq \omega_1$ (and is metrizable if the Suslin Hypothesis holds).

The main Theorem of [1] is

Theorem 2.6. *Each compact space* X *with* $w(X) > \overline{c}(X)$ *is the limit of an inverse well-ordered spectrum of lenght* $\overline{c}(X)^+$ *consisting of compacta with weight* $\leq \overline{c}(X)$ *and monotone bonding mappings.*

3 Fixed point property for 2^X and $C(X)$ if X is a fan

In this section we shall prove the fixed point property for 2^X and $C(X)$ if X is a fan. If X is a metric fan, i.e., a fan then we have the following result.

Theorem 3.1. *[6, Theorem 22.13, p. 194]. If X is a fan, then 2^X and $C(X)$ have the fixed point property.*

For generalized fans the proofs for 2^X and $C(X)$ are different. We start be the proof for 2^X.

Theorem 3.2. *If X is a generalized fan, then 2^X have the fixed point property.*

Proof. By Theorem 2.5 there exists a σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X. Now we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X (Lemma 1.1). It is clear that the mappings $2^{p_{ab}}$ are onto if the bonding mappings p_{ab} are onto. Now we can apply Theorem 1.5 since, by Theorem 3.1, every 2^{X_a} has the fixed point property. Hence, 2^X has the fixed point property. □

Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be a σ-system. If we consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$, then $C(p_{ab})$ are not always the surjections. This is the case only if p_{ab} are weakly confluent mappings [13, Theorem (0.49.1), p. 24]. This means that we can apply Theorem 1.5 on the system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ only if p_{ab} are weakly confluent mappings. Let us recall that a mapping $f : X \to Y$ is *weakly confluent* provided that for each subcontinuum K of Y there exists a component A of $f^{-1}(K)$ such that $f(A) = K$ [13, (0.45.4), p. 22]. Each monotone mapping is weakly confluent. It follows that expanding Theorem 2.5 is not enough for proving the fixed point property of $C(X)$ when X is a generalized fan. For this reason we shall consider the fixed point property for 2^X and $C(X)$ if X is a generalized fan in class $\mathcal{M}$.

Theorem 3.3. *If X is a generalized fan in the class $\mathcal{M}$, then $C(X)$ have the fixed point property.*

Proof. By Theorem 2.5 there exists a σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X. Applying Theorem 2.3 we obtain an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact metric spaces such that m_{ab} are monotone surjections and $\lim X$ is homeomorphic to $\lim M(\mathbf{X})$, i.e., X is homeomorphic to $\lim M(\mathbf{X})$. Moreover, from the fact that the projections $m_a : \lim M(\mathbf{X}) \to M_a$ are monotone it follows that M_a is a fan. Now we have the inverse system $C(M(\mathbf{X})) = \{C(M_a), C(m_{ab}), A\}$ whose limit is $C(X)$ (Lemma 1.1). It is clear that the mappings $C(p_{ab})$ are onto if the bonding mappings m_{ab} are monotone. Now we can apply Theorem 1.5 since, by Theorem 3.1, every $C(M_a)$ has the fixed point property. Hence, $C(X)$ has the fixed point property. □

4 Fixed point property for 2^X and $C(X)$ if X is a smooth arboroid

An arboroid X is said to be *smooth* if there exists a point $p \in X$, called an *initial point* of X, such that for every convergent net of points $\{a_n : n \in E\}$ of X the condition

$$\lim_{n\in E} a_n = a$$

implies that the *net of arcs* pa_n *is convergent* and

$$\operatorname{Lim}_{n\in E}\ pa_n = pa.$$

The set of all points of X each of them can be taken as an initial point will be called the *initial set* of X.

Lemma 4.1. *[4, Corollary 10, p. 309]. If f is a monotone mapping of a smooth arboroid X onto Y, then Y is a smooth arboroid and $f(P) \subset P^*$, where P and P^* denote the initial sets of X and Y respectively.*

Theorem 4.2. *[6, Theorem 22.12, p. 194]. If X is a smooth dendroid, then 2^X and $C(X)$ have the fixed point property.*

Theorem 4.3. *If a non-metrizable arboroid X is in class $\mathcal{M}$, then there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a dendroid, every p_{ab} is monotone and X is homeomorphic to* $\lim \mathbf{X}$.

Proof. ¿From Corollary 2.3 it follows that there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric continuum, every p_{ab} is monotone and X is homeomorphic to $\lim \mathbf{X}$. From 4.1 we infer that every X_a is an arboroid. Hence, every X_a is a metrizable arboroid, i.e., a dendroid. □

Theorem 4.4. *If a non-metrizable smooth arboroid X is in class $\mathcal{M}$, then there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a smooth dendroid, every p_{ab} is monotone and X is homeomorphic to* $\lim \mathbf{X}$.

Proof. Theorem follows from Theorems 2.3 and 4.1. □

Theorem 4.5. *If X is a smooth arboroid in the class $\mathcal{M}$, then 2^X and $C(X)$ have the fixed point property.*

Proof. By Theorem 4.4 there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a smooth dendroid, every p_{ab} is monotone and X is homeomorphic to $\lim \mathbf{X}$. Now the systems $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ satisfy the conditions of Theorem 1.5. Hence, 2^X and $C(X)$ have the fixed point property. □

5 Fixed point property for 2^X and $C(X)$ if X is a cone over a generalized fan or a smooth arboroid

Let Y be a topological space. The suspension over Y, which we denote by $\Sigma(Y)$, is the quotient space obtained from $Y \times [-1, 1]$ by shrinking $Y \times \{-1\}$ and $Y \times \{1\}$ to (different) points.

Theorem 5.1. *[6, Theorem 22.15, p. 195]. Let $X = Cone(Y)$, where Y is a fan or a smooth dendroid. Then, 2^X and $C(X)$ have the fixed point property.*

Theorem 5.2. *Let $X = Cone(Y)$, where $Y \in \mathcal{M}$ is a generalized fan or a smooth arboroid. Then, 2^X and $C(X)$ have the fixed point property.*

Proof. If $Y \in \mathcal{M}$ is a generalized fan or a smooth arboroid, then there exists an inverse σ-system $\mathbf{Y} = \{Y_a, p_{ab}, A\}$ such that each Y_a is a smooth dendroid, every p_{ab} is monotone and Y is homeomorphic to $\lim \mathbf{Y}$. Furthermore, $X = Cone(Y) = \lim\{Cone(Y_a), q_{ab}, A\}$ [14, 3.15, p. 41 and Exercise 3.30, p. 49]. Let us observe that q_{ab} are monotone. This means that the inverse systems $\{2^{Cone(Y_a)}, 2^{q_{ab}}, A\}$ and $\{C(Cone(Y_a)), C(q_{ab}), A\}$ satisfy the conditions of Theorem 1.5. Hence, 2^X and $C(X)$ have the fixed point property. □

For suspension $\Sigma(Y)$ over Y we have the following result.

Theorem 5.3. *[6, Theorem 22.16, p. 196]. Let $X = \Sigma(Y)$, where Y is a fan or a smooth dendroid. Then, 2^X and $C(X)$ have the fixed point property.*

Analogue result for non-metric settings is as follows.

Theorem 5.4. *Let $X = \Sigma(Y)$, where $Y \in \mathcal{M}$ is a generalized fan or a smooth arboroid. Then, 2^X and $C(X)$ have the fixed point property.*

6 Fixed point property for 2^X and $C(X)$ if X is a product of generalized fans or smooth arboroids

In this section we shall generalize the following result.

Theorem 6.1. *[6, Theorem 22.14, p. 195]. Let X be a finite or countably infinite Cartesian product, where each coordinate space is a fan or a smooth dendroid. Then 2^X and $C(X)$ have the fixed point property.*

Theorem 6.2. *Let X be a Cartesian product, where each coordinate space is a fan or a smooth dendroid. Then 2^X and $C(X)$ have the fixed point property.*

Proof. If X is a finite or countably infinite Cartesian product, then apply Theorem 6.1. Suppose now that X is the Cartesian product $X = \prod\{X_a : a \in A\}$, where $\text{card}(A) > \aleph_0$. From Theorem 1.6 it follows that for product $X = \prod\{X_a : a \in A\}$ of spaces X_a there exists a σ-directed inverse system $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ of the countable product X^μ such that X is homeomorphic to $\lim \mathbf{X}$. Moreover, if each X_a is metrizable continuum, then $\mathbf{X} = \{X^\mu, P_{\mu\nu}, M\}$ is an inverse σ-system with monotone bonding mappings $P_{\mu\nu}$. The inverse systems $2^{\mathbf{X}} = \{2^{X^\mu}, 2^{P_{\mu\nu}}, M\}$ and $C(\mathbf{X}) = \{C(X^\mu), C(P_{\mu\nu}), M\}$ satisfy the assumptions of Theorem 1.5. Hence, 2^X and $C(X)$ have the fixed point property. □

We close this section with the following result.

Theorem 6.3. *Let X be a Cartesian product, where each coordinate space is a generalized fan or a smooth arboroid of the same weight. Then 2^X and $C(X)$ have the fixed point property.*

Proof. Now we have $X = \prod\{X_m : m \in M\}$ and $\mathrm{w}(X_m) = k$ for every $m \in M$, where k is an uncountable cardinal. This means that for every $m \in M$ we have an inverse σ-system $\mathbf{X}_m = \{X_{m,a}, p_{m,ab}, A\}$ whose limit is X_m. Now X is homeomorphic to $\lim\{\Pi X_{m,a}, \Pi p_{m,ab}, A\}$ [5, Exercise 2.5.D.(b), p. 143]. Finally the systems $\{2^{\Pi X_{m,a}}, 2^{\Pi p_{m,ab}}, A\}$ and $\{C(\Pi X_{m,a}), C(\Pi p_{m,ab}), A\}$ satisfy the conditions of Theorem 1.5 since, by Theorem 6.2, the continua $2^{\Pi X_{m,a}}$ and $C(\Pi X_{m,a})$ have the fixed point property. Hence, 2^X and $C(X)$ have the fixed point property. □

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