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MULTIPLE INTEGRAL REPRESENTATION OF BINOMIAL COEFFICIENTS

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Abstract

We investigate the integral representation of infinite sums involving the ratio of multiple binomial coefficients. We also recover some well-known properties of the Zeta function.

1 Introduction

In this paper we investigate the summation of multiple products of combinatorial coefficients. In particular, we develop integral representations for

$$\sum_{n=0}^{\infty} \frac{t^n \ n^p}{\prod_{i=1}^{k} (\alpha_i n + \beta_i \beta_i)} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{t^n \ n^p \ \ (n+m-1n)}{\prod_{i=1}^{k} (\alpha_i n + \beta_i \beta_i)}.$$

For the representation of sums of reciprocals of single and double binomial coefficients, in integral form, one may refer to some results in the papers [3] and [5], see also the book [4].

For designated cases of the parameter values α_i, β_i, m, p and t, various particular sums may be expressed in terms of Zeta functions. For many interesting properties of the Zeta function the interested reader is referred to the internet site [8].

Yang and Zhao [7] have recently obtained some nice results on sums of ratios of double binomial coefficients. Also in a recent paper Muzaffar [1] obtained some results of the combinatorial type

$$\sum_{n=0}^{\infty} \frac{(2nn)}{(2n+11)\,(2n+k+11)\,(2n+2kn+k)} = \alpha_k \pi^2 + \beta_k$$

by utilising the power series expansion of $(\sin^{-1} x)^q$ and (α_k, β_k) are constants depending on $k \geq 0$.

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Recently Rhin and Viola [2] introduced the integral

$$\int_0^1 \int_0^1 \int_0^1 \frac{u^h (1-u)^l \ v^k (1-v)^s \ w^j (1-w)^q}{(1-(1-uv)\ w)^{q+h-r+1}} du\ dv\ dw \in \ Q_1\zeta(3) + Q_2$$

in their study of an irrationality measure for $\zeta(3)$, where Q_1 and Q_2 are rational numbers based on the constants (h, l, k, s, j, q, r).

The representation of sums in terms of integrals is extremely useful because it allows one to estimate bounds on the sums in cases they cannot be written in closed form. Convexity properties for sums may also be investigated, see [6].

2 The Main Results

In this section we develop integral identities for reciprocals of multiple products of binomial coefficients.

The following lemma is given

Lemma 1 *For α_i and m positive real numbers, $i \in \{1, ..., k\}$, $k \geq 1$ and $t \in \mathbb{R}$ let*

$$\lambda_m(f_i) = \sum_{n=0}^{\infty} \binom{n+m-1}{n} {f_i}^n = \frac{1}{(1-f_i)^m}$$

where

$$f_i = t x_i^{\alpha_i}. \tag{2.1}$$

The consecutive partial derivative operator of the continuous function $(1-f_i)^{-m}$ for $(x_1, x_2, ..., x_k) \in [0,1]^k$ is defined as

$$[\lambda_m(f_i)]^{(0)} = \frac{1}{(1-t x_i^{\alpha_i})^m}$$

$$\vdots$$

$$[\lambda_m(f_i)]^{(p)} = \underbrace{x_1 \frac{\partial}{\partial x_1}\left(x_1 \frac{\partial}{\partial x_1}\left(\cdots x_1 \frac{\partial}{\partial x_1}\left(\frac{1}{(1-f_i)^m}\right)\right)\right)}_{p-times}$$

so that

$$[\lambda_m(f_i)]^{(p)} = \alpha_1^p \sum_{n=0}^{\infty} n^p \ {f_i}^n = \frac{\alpha_1^p}{(1-f_i)^{m+p}} \sum_{r=1}^{p} (-1)^{p+r+1} \ C_{p,m}(r) \ f_i^r, \tag{2.2}$$

where the convolution coefficient

$$C_{p,m}(r) = \sum_{\nu=1}^{r} (-1)^{\nu} (m)_{\nu} \binom{p-\nu}{r-\nu} S(p,\nu), \tag{2.3}$$

$$S\,(p,\nu)=\{p\nu\}=\frac{1}{\nu!}\sum_{\mu=0}^{r}(-1)^{\mu}\binom{r}{\mu}(r-\mu)^{p}$$

are Stirling numbers of the second kind, and

$$(m)_{\nu}=m\,(m+1)\cdots(m+\nu-1)=\frac{\Gamma\,(m+\nu)}{\Gamma\,(m)}$$

is known as Pochhammer's symbol.

Proof. *We note that* $x_1\frac{\partial f_i}{\partial x_1}=x_1 f_i$ *and*

$$[\lambda_m\,(f_i)]^{(1)}=\alpha_1\sum_{n=0}^{\infty}n\binom{n+m-1}{n}{f_i}^{n}=\frac{m\alpha_1 f_i}{(1-f_i)^{m+1}}$$

$$[\lambda_m\,(f_i)]^{(2)}=\alpha_1^2\sum_{n=0}^{\infty}n^2\binom{n+m-1}{n}{f_i}^{n}=\frac{\alpha_1^2}{(1-f_i)^{m+2}}\left\{mf_i(1-f_i)+m(m+1)f_i^2\right\}$$

$$[\lambda_m\,(f_i)]^{(3)}=\alpha_1^3\sum_{n=0}^{\infty}n^3\binom{n+m-1}{n}{f_i}^{n}=\frac{\alpha_1^3}{(1-f_i)^{m+3}}\{mf_i(1-f_i)^2+3\cdot m(m+1){f_i}^2(1-f_i)+m(m+1)(m+2){f_i}^3\}$$

....

$$\begin{aligned}[\lambda_m\,(f_i)]^{(p)}&=\alpha_1^p\sum_{n=0}^{\infty}n^p\binom{n+m-1}{n}{f_i}^{n}\\&=\frac{\alpha_1^p}{(1-f_i)^{m+p}}\sum_{r=1}^{p}S\,(p,r)\,(m)_r\quad {f_i}^{r}\,(1-f_i)^{p-r}\\&=\frac{\alpha_1^p}{(1-f_i)^{m+p}}\sum_{r=1}^{p}S\,(p,r)\,(m)_r\quad {f_i}^{r}\sum_{j=0}^{p-r}(-1)^j\binom{p-r}{j}{f_i}^{j}.\end{aligned}$$

Collecting powers of f_i *we have that*

$$[\lambda_m\,(f_i)]^{(p)}\;=\frac{\alpha_1^p}{(1-f_i)^{m+p}}\sum_{r=1}^{p}(-1)^{p+r+1}\;C_{p,m}(r)\;{f_i}^{r}$$

where $C_{p,m}(r)$ *is given by (2.3).*

By induction we see that

$$\begin{aligned}[\lambda_m\,(f_i)]^{(p+1)}\;&=\;\alpha_1^p\frac{\partial}{\partial x_1}\,[\lambda_m\,(f_i)]^{(p)}\\&=\alpha_1^p\;x_1\frac{\partial}{\partial x_1}\left[\sum_{r=1}^{p}S\,(p,r)\,(m)_r\quad {f_i}^{r}\,(1-f_i)^{-m-r}\right]\\&=\alpha_1^{p+1}\Big[S\,(p,1)\,mf_i\;\;(1-f_i)^{-m-1}+\cdots+{f_i}^{\;p}\,(1-f_i)^{-m-p}\\&\qquad\times\{(m+p-1)(m)_{p-1}S\,(p,p-1)+p(m)_pS\,(p,p)\}\\&\qquad+(m+p)\,(m)_pS(p,p){f_i}^{\;p+1}\,(1-f_i)^{-m-p-1}\Big].\end{aligned}$$

We may write

$$\begin{array}{cc} (m+p-1)(m)_{p-1} = (m)_p, & (m+p)\,(m)_p = (m)_{p+1} \\ S\,(p,1) = S\,(p+1,1) = 1, & S\,(p,p) = S\,(p+1,p+1) = 1 \end{array}$$

and by the recurrence of Stirling numbers of the second kind, $S\,(p,p-1)+pS\,(p,p) = S\,(p+1,p)$ *we have that*

$$\begin{aligned}
&\alpha_1^{p+1}\Big[S\,(p,1)\,mf_i\ \ (1-f_i)^{-m-1} + \cdots + f_i\ ^p\,(1-f_i)^{-m-p} \\
&\qquad \times \{(m+p-1)(m)_{p-1}S\,(p,p-1) + p(m)_pS\,(p,p)\} \\
&\qquad + (m+p)\,(m)_pS(p,p)f_i\ ^{p+1}\,(1-f_i)^{-m-p-1}\Big] \\
&= \frac{\alpha_1^{p+1}}{(1-f_i)^{m+p+1}}\Big[S\,(p+1,1)\,(m)_1\ f_i\ \ (1-f_i)^p + \cdots \\
&\qquad + S\,(p+1,p)\,(m)_p\ f_i\ ^p\ \ (1-f_i) + (m)_{p+1}S(p+1,p+1)f_i\ ^{p+1}\Big] \\
&= \frac{\alpha_1^{p+1}}{(1-f_i)^{m+p+1}}\sum_{r=1}^{p+1} S\,(p+1,r)\,(m)_r\ f_i\ ^r\,(1-f_i)^{p+1-r}
\end{aligned}$$

so that (2.2) follows. ■

Now we investigate the following theorem

Theorem 1 *For* α_i, β_i *and* m *positive real numbers,* $i \in \{1,...,k\}$,$k \geq 1$ *and* $t \in \mathbb{R}$, *with* $\alpha_i+\ \beta_i \geq m+p$, *and* $p \geq 0$ *then:*

$$Q = \sum_{n=0}^{\infty} \frac{t^n n^p\ \ (n+m-1n)}{\prod\limits_{i=1}^{k}(\alpha_i n + \beta_i\beta_i)} \tag{2.4}$$

$$= \sum_{n=0}^{\infty} \frac{t^n n^p\ \ \ (n+1)_{m-1}}{(m-1)!}\prod_{i=1}^{k}\frac{(\beta_i)!}{(\alpha_i n+1)_{\beta_i}} \tag{2.5}$$

$$= \prod_{i=1}^{k}\beta_i \int\limits_{x_i\in[0,1]^k} \frac{(1-x_i)^{\beta_i-1}}{(1-f_i)^{m+p}}\sum_{r=1}^{p}(-1)^{p+r+1}\ C_{p,m}(r)\ f_i{}^r dx_i \tag{2.6}$$

where $(w)_\alpha$ *is Pochhammer's symbol, and* $C_{p,m}(r)$ *is given by (2.3).*

Proof. Consider (2.4). To arrive at the result (2.6) consider

$$
\begin{aligned}
Q &= \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{\prod\limits_{i=1}^{k} (\alpha_i \ n + \beta_i \beta_i)} \\
&= \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} \prod_{i=1}^{k} \frac{\Gamma(\beta_i + 1)\,\Gamma(\alpha_i \ n + 1)}{\Gamma(\alpha_i \ n + \beta_i + 1)} \\
&= \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} \prod_{i=1}^{k} \beta_i \ B(\alpha_i \ n + 1, \ \beta_i)
\end{aligned}
$$

where $\Gamma(\cdot)$ is the classical Gamma function and $B(\cdot,\cdot)$ is the Beta function.

$$
\begin{aligned}
Q &= \sum_{n=0}^{\infty} t^n \binom{n+m-1}{n} \prod_{i=1}^{k} \beta_i \int\limits_{x_i \in [0,1]^k} x_i^{\alpha_i \ n} (1 - x_i^{\alpha_i})^{\beta_i - 1} dx_i \\
&= \prod_{i=1}^{k} \beta_i \int\limits_{x_i \in [0,1]^k} (1 - x_i^{\alpha_i})^{\beta_i - 1} \sum_{n=0}^{\infty} \binom{n+m-1}{n} (t x_i^{\alpha_i})^n \ dx_i
\end{aligned}
$$

by an allowable change of integral and sum and on the assumption of convergence.

Now consider

$$
\begin{aligned}
[\lambda_m(f_i)]^{(0)} &= \frac{1}{(1 - t x_i^{\alpha_i})^m} \\
&\ \ \vdots \\
[\lambda_m(f_i)]^{(p)} &= \underbrace{x_1 \frac{\partial}{\partial x_1} \left(x_1 \frac{\partial}{\partial x_1} \left(\cdots x_1 \frac{\partial}{\partial x_1} \left(\frac{1}{(1 - f_i)^m} \right) \right) \right)}_{p-\text{times}}
\end{aligned}
$$

and by Lemma 1

$$
\begin{aligned}
[\lambda_m(f_i)]^{(p)} &= \alpha_1^p \sum_{n=0}^{\infty} n^p \binom{n+m-1}{n} f_i{}^n \\
&= \frac{\alpha_1^p}{(1 - f_i)^{m+p}} \sum_{r=1}^{p} (-1)^{p+r+1} \ C_{p,m}(r) \ f_i{}^r,
\end{aligned}
$$

so that

$$
Q = \prod_{i=1}^{k} \beta_i \int\limits_{x_i \in [0,1]^k} \frac{(1 - x_i)^{\beta_i - 1}}{(1 - f_i)^{m+p}} \sum_{r=1}^{p} (-1)^{p+r+1} \ C_{p,m}(r) \ f_i{}^r dx_i \ .
$$

which is the result (2.6). ■

Example: Consider the case $p=3, t=1, m=2$ and $k=5$ so that $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\} = \{4,6,2,4,1\}$ and $\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\} = \{4,5,6,2,2\}$ then

$$\begin{aligned}
Q &= \sum_{n=0}^{\infty} \frac{n^3(n+11)}{\binom{4n+4}{4}\binom{6n+5}{5}\binom{2n+6}{6}\binom{4n+2}{2}\binom{n+2}{2}} \\
&= 960\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(1-x)^3(1-y)^4(1-z)^5(1-w)(1-u)}{(1-x^4y^6z^2w^4u)^4} \\
&\quad \times x^4y^6z^2w^4u(1+7x^4y^6z^2w^4u+4\left(x^4y^6z^2w^4u\right)^2)dxdydzdwdu \\
&= \frac{1}{20374200}\ {}_{16}F_{15}\left[\left.\begin{array}{c} \frac{11}{6},\frac{7}{6},\frac{7}{4},\frac{5}{4},\frac{5}{4},\frac{5}{3},\frac{4}{3},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},2,2,2,2,2 \\ \frac{17}{6},\frac{13}{6},\frac{11}{4},\frac{9}{4},\frac{9}{4},\frac{8}{3},\frac{7}{3},\frac{9}{2},\frac{5}{2},\frac{5}{2},\frac{5}{2},5,4,1,1 \end{array}\right| 1\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1939}{8}\zeta(3) - \frac{469235848}{1688511825} - \left(\frac{14387544064}{7844067} - \frac{16784095613517\sqrt{3}}{10272822560}\right)\pi \\
&\quad + \frac{131072}{11319}G + \frac{53774744084224}{3939860925}\ln(2) - \frac{101084132563389}{10272822560}\ln(3) \\
&\quad - \frac{1111574741}{905520}\zeta(2) - \frac{225}{4}\zeta(4).
\end{aligned}$$

where G is Catalans constant.

Now consider the following corollaries, which are specific cases of Lemma 1, and Theorem 1.

Corollary 1 *For α_i positive real numbers, $i \in \{1,...,k\}$, $k \geq 1$ and $t \in \mathbb{R}$ let*

$$\lambda(f_i) = \sum_{n=0}^{\infty} f_i{}^n = \frac{1}{1-f_i},$$

where f_i is given by (2.1). The consecutive partial derivative operator of the continuous function $(1-f_i)^{-1}$ for $(x_1,x_2,...,x_k) \in [0,1]^k$ is defined as

$$\begin{aligned}
[\lambda(f_i)]^{(0)} &= \frac{1}{1-tx_i^{\alpha_i}} \\
&\vdots \\
[\lambda(f_i)]^{(p)} &= \underbrace{x_1\frac{\partial}{\partial x_1}\left(x_1\frac{\partial}{\partial x_1}\left(\cdots x_1\frac{\partial}{\partial x_1}\left(\frac{1}{1-f_i}\right)\right)\right)}_{p-times}
\end{aligned}$$

so that

$$[\lambda(f_i)]^{(p)} = \alpha_1^p \sum_{n=0}^{\infty} n^p\ f_i{}^n = \frac{x_1^p}{(1-f_i)^{1+p}}\sum_{r=1}^{p}(-1)^{p+r+1}\ C_p(r)\ f_i^r, \tag{2.7}$$

where the convolution coefficient

$$C_p(r) = \sum_{\nu=1}^{r} (-1)^{\nu} \nu! \binom{p-\nu}{r-\nu} S(p,\nu) \tag{2.8}$$

and

$$S(p,\nu) = \{p\nu\} = \frac{1}{\nu!} \sum_{\mu=0}^{r} (-1)^{\mu} (r\mu)(r-\mu)^p$$

are Stirling numbers of the second kind.

Proof. The proof follows by that putting $m = 1$, into Lemma 1, hence we obtain the desired result (2.7). ■

Now we give the following corollary.

Corollary 2 *For α_i and β_i positive real numbers, $i \in \{1, ..., k\}$, $k \geq 1$ and $t \in \mathbb{R}$, $p \geq 0$ then*

$$W = \sum_{n=0}^{\infty} \frac{t^n \ n^p}{\prod_{i=1}^{k} (\alpha_i n + \beta_i \beta_i)} \tag{2.9}$$

$$= \prod_{i=1}^{k} \beta_i \int\limits_{x_i \in [0,1]^k} \frac{(1-x_i)^{\beta_i - 1}}{(1-f_i)^{1+p}} \sum_{r=1}^{p} (-1)^{p+r+1} \ C_p(r) \ f_i^r dx_i \tag{2.10}$$

$$= \ T_0 \ \ {}_{p+B_k}F_{p-1+B_k} \left[\begin{array}{c} \frac{\alpha_1+\beta_1}{\alpha_1}, ..., \frac{\alpha_1+1}{\alpha_1}, ..., \frac{\alpha_k+\beta_k}{\alpha_k}, ..., \frac{\alpha_k+1}{\alpha_k}, 2, ..., 2 \\ \frac{2\alpha_1+\beta_1}{\alpha_1}, ..., \frac{2\alpha_1+1}{\alpha_1}, ..., \frac{2\alpha_k+\beta_k}{\alpha_k}, ..., \frac{2\alpha_k+1}{\alpha_k}, 1, ..., 1 \end{array} \middle| t \right], \tag{2.11}$$

where

$$T_0 = \frac{t}{\prod_{i=1}^{k} (\alpha_i + \beta_i \beta_i)}, \qquad B_k = \sum_{i=1}^{k} \beta_i$$

and $C_p(r)$ is given by (2.8).

Proof. The proof follows by putting $m = 1$ into Theorem 1, hence we obtain the desired result (2.10). The generalised hypergeometric representation ${}_pF_q \ [\cdot, \cdot]$, is defined as

$${}_pF_q \left[\begin{array}{c} a_1, a_2, ..., a_p \\ b_1, b_2, ..., b_q \end{array} \middle| t \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ (a_2)_n ... (a_p)_n \ \ t^n}{(b_1)_n \ (b_2)_n ... (b_q)_n \ \ n!}$$

where $(w)_\alpha$ is Pochhammer's symbol. By the consideration of the ratio of successive terms $\frac{T_{n+1}}{T_n}$ where

$$T_n = \frac{t^n \ n^p}{\prod_{i=1}^{k} (\alpha_i n + \beta_i \beta_i)}$$

from (2.9) we can obtain the result (2.11). ■

Example: For $\alpha_i = \beta_i = 1$ for $i = 2, 3..., k,\ p = 0,$ and $t = 1$ we have the result

$$W = \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} = \zeta(k) = \prod_{i=1}^{k} \int_{x_i \in [0,1]^k} \frac{1}{(1-x_i)} dx_i.$$

When $k = 4, \alpha_i = \beta_i = 1,\ p = 1$ and $t = -1$ we have

$$W = \sum_{n=0}^{\infty} \frac{(-1)^n\ n}{(n+1)^4} = \frac{3}{4}\zeta(3) - \frac{7}{8}\zeta(4) = -\int_0^1\int_0^1\int_0^1\int_0^1 \frac{xyzw}{(1+xyzw)^2} dxdydzdw$$
$$= -\frac{1}{16}\ {}_5F_4\left[\begin{array}{c} 2,2,2,2,2 \\ 3,3,3,3 \end{array}\middle| -1\right].$$

3 Conclusion

We have provided multiple integral identities for sums of the reciprocal of multiple binomial coefficients. In doing so we have recovered the standard representation for the Zeta function and have generalised and extended some results published in the literature.

In another forum we shall extend our results to consider more general sums of binomial coefficients.

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