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GENERIC WARPED PRODUCT SUBMANIFOLDS IN A KAEHLER MANIFOLD

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Abstract

In this paper we have shown that there do not exist proper warped product submanifolds of the type $N \times_f N_T$ and $N_T \times_f N$ where N_T is an invariant and N is any real non-anti invariant submanifold of a Kaehler manifold. We thus generalize the results of B. Sahin [10] who projected same results for a restricted class, the class of warped product submanifolds $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$.

1. Introduction

Bishop and O'Niell [2] introduced the concept of warped product manifolds to study manifolds of negative curvature and applied the scheme to space-time. The geometrical aspect of these manifolds have attracted the attention of a lot of researchers recently [6], [8], [10]. Many research papers have appeared to see the existence of warped product submanifolds of manifolds under different settings after it was found that the space around a body with high gravitational field can be modeled on a warped product manifold.

B.Y.Chan [6] studied warped product CR-submanifolds of the type $N_\perp \times_f N_T$ and $N_T \times_f N_\perp$ of a Keahler manifold $\bar{M}$, where N_T is an invariant and $N_\perp$ is an anti invariant submanifold of $\bar{M}$. He has shown that there do not exist proper warped product submanifolds of the type $N_\perp \times_f N_T$, when as he and others found many examples of warped product submanifolds of type $N_T \times_f N_\perp$ in a Kaehler manifold. B. Sahin extended the study to slant warped product submanifolds of the type $M = N_T \times_f N_\theta$ and $M = N_\theta \times_f N_T$ of a Kaehler manifold $\bar{M}$, where N_T is an invariant and N_θ is a proper slant submanifolds of $\bar{M}$, and showed that they do not exist in either case.

In this paper, we have generalized the results of Chen [6] [7]and Sahin [10] and have shown that there are no proper warped product submanifolds of the type $M = N \times_f N_T$ and $M = N_T \times_f N$, where N_T is a invariant and N is any real non-anti invariant submanifold of a Kaehler manifold. We thus have extended this study to generic warped product submanifolds of Kaehler manifold.

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2. Some Basic Results

Let $\bar{M}$ be a Kaehler manifold with a complex structure J, Hermitian metric g and the Levi-Civita connection $\bar{\nabla}$. Then we have

$$J^2 = -I, \quad g(JU, JV) = g(U, V), \quad \bar{\nabla} J = 0 \tag{2.1}$$

for all vector fields U, V on $\bar{M}$.

Let $\bar{M}$ be a Kaehler manifold with a complex structure J, and M be a submanifold of $\bar{M}$. The induced Riemannian metric on M is denoted by the same symbol g whereas the induced connection on M is denoted by ∇. Then M is called holomorphic if $JT_pM \subset T_pM$, for every $p \in M$, where T_pM denotes the tangent space to M at the point p.

If $T\bar{M}$ and TM denote the Lie-algebra of vector fields on $\bar{M}$ and M respectively and $T^\perp M$, the set of all vector fields normal to M, then the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.2}$$

$$\bar{\nabla}_U \xi = -A_\xi U + \nabla^\perp_U \xi \tag{2.3}$$

for each $U, V \in TM$ and $\xi \in T^\perp M$, where $\nabla^\perp$ denotes the connection on the normal bundle $T^\perp M$. h and A_ξ are the second fundamental forms and the shape operator of the immersion of M into $\bar{M}$ corresponding to the normal vector field ξ. They are related as

$$g(A_\xi U, V) = g(h(U, V), \xi). \tag{2.4}$$

For any $U \in TM$ and $\xi \in T^\perp M$, we write

$$JU = PU + FU, \tag{2.5}$$

$$J\xi = t\xi + f\xi, \tag{2.6}$$

where PU and $t\xi$ are the tangential components of JU and $J\xi$ respectively whereas FU and $f\xi$ are the normal components of JU and $J\xi$ respectively. The covariant differentiation of the tensors P, F, t and f are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.7}$$

$$(\bar{\nabla}_U F)V = \nabla^\perp_U FV - F\nabla_U V, \tag{2.8}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla^\perp_U \xi, \tag{2.9}$$

$$(\bar{\nabla}_U f)\xi = \nabla^\perp_U f\xi - f\nabla^\perp_U \xi. \tag{2.10}$$

Let $\bar{M}$ be an almost Hermition manifold with an almost complex structure J, Hermitian metric g and M be a submanifold of $\bar{M}$. For each $x \in M$, let $D_x = T_xM \cap JT_xM$ i.e., a maximal holomorphic subspace of the tangent space T_xM at $x \in M$. If the dimension of D_x remains the same for each $x \in M$

and it defines a holomorphic distribution D on M, then M is called a generic submanifold [4].

A generic submanifold M of an almost Hermition manifold $\bar{M}$ is said to be *generic product submanifold* if it is locally a Riemannian product of the leaves of D and $D^{'}$, where $D^{'}$ is orthogonal complementry distribution to D in TM. In this case D and $D^{'}$ are parallel on M i.e., $\nabla_U X \in D$ or equivalently $\nabla_U Z \in D^{'}$ for all $U \in TM,\ X \in D$ and $Z \in D^{'}$.

Now we consider warped product of manifolds which are defined as follows

Definition 2.1. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and f be a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F. \tag{2.11}$$

The warped product manifold $(B \times F, g)$ is denoted by $B \times_f F$. If U is tangent to $M = B \times_f F$ at (p, q) then by equation (2.11),

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

where π_1 and π_2 are the canonical projections of M onto B and F respectively.

On a warped product manifold $B \times_f F$ one has

$$\nabla_U V = \nabla_V U = (Ulnf)V \tag{2.12}$$

for any vector fields U tangent to B and V tangent to F [2].

3. Generic Warped Product Submanifolds

In this section we study generic warped product submanifolds of a Kaehler manifold $\bar{M}$ of the form $M = N_T \times_f N$, $M = N \times_f N_T$ respectively, where N_T is a holomorphic submanifold and N is any real non anti-invariant submanifold of $\bar{M}$.

Theorem 3.1. *There do not exist proper generic warped product submanifold $M = N \times_f N_T$ of a Kaehler manifold $\bar{M}$, where N_T is an invariant submanifold and N is any real non anti-invariant submanifold of $\bar{M}$.*

Proof. For any $X \in TN_T$ and $U \in TM$ using (2.12) we obtain

$$\begin{aligned} g(\overline{\nabla}_X X, U) &= -g(\overline{\nabla}_X U, X) \\ &= -g(\nabla_X U, X) \\ &= -U \ln f \|X\|^2 \end{aligned} \tag{3.1}$$

But, we also have

$$
\begin{aligned}
g(\overline{\nabla}_X X, U) &= g(J\overline{\nabla}_X X, JU) \\
&= g(\overline{\nabla}_X JX, JU) \\
&= -g(\overline{\nabla}_X JU, JX) \\
&= -g(\overline{\nabla}_X PU, JX) - g(\overline{\nabla}_X FU, JX) \\
&= -PU \ln f g(X, JX) + g(A_{FU}X, JX) \\
&= g(h(X, JX), FU) \qquad (3.2)
\end{aligned}
$$

Thus from (3.1) and (3.2), we obtain

$$g(h(X, JX), FU) = -U \ln f \|X\|^2 \qquad (3.3)$$

Now replacing X by JX in (3.3), we obtain

$$g(h(JX, J^2X), FU) = -U \ln f \|X\|^2$$

$$-g(h(X, JX), FU) = -U \ln f \|X\|^2$$

$$g(h(X, JX), FU) = U \ln f \|X\|^2 \qquad (3.4)$$

Thus from (3.3) and (3.4), we get

$$U \ln f \|X\|^2 = 0$$

for all $U \in TM$.Which implies that f is constant or $X = 0$. Hence the theorem is proved.

We now interchange the factors N and N_T and prove the following:

Theorem 3.2. *There do not exist proper generic warped product submanifold $M = N_T \times_f N$ of a Kaehler manifold $\bar{M}$, where N_T is a holomorphic submanifold and N is any real non anti-invariant submanifold of $\bar{M}$.*

Proof. For any $U, V \in TM$ and using the fact that $\bar{M}$ is kaehler, we have

$$\bar{\nabla}_U JV = J\bar{\nabla}_U V,$$

therefore,

$$\bar{\nabla}_U PV + \bar{\nabla}_U FV = J(\nabla_U V + h(U, V)),$$

On using (2.2), (2.3), (2.5), we have

$$\nabla_U PV + h(U, PV) - A_{FV}U + \nabla^{\perp}_U FV = P\nabla_U V + F(\nabla_U V) + th(U, V) + fh(U, V).$$

Now, compairing tangential part and using (2.7), we obtain

$$(\bar{\nabla}_U P)V = A_{FV}U + th(U, V). \qquad (3.5)$$

Now, for $X \in TN_T$ and using (2.12),we get

$$
\begin{aligned}
(\bar{\nabla}_X P)U &= \nabla_X PU - P\nabla_X U \\
&= (Xlnf)PU - (Xlnf)PU \\
&= 0.
\end{aligned}
$$

Using it in (3.5), we get

$$A_{FU}X = -th(X, U). \tag{3.6}$$

On the other hand

$$(\bar{\nabla}_U P)X = (PX \ln f)U - (X \ln f)PU. \tag{3.7}$$

Also from (3.5), we have

$$(\bar{\nabla}_U P)X = th(X, U). \tag{3.8}$$

Thus from (3.7) and (3.8), we have

$$(PX \ln f)U - (X \ln f)PU = th(X, U). \tag{3.9}$$

From (3.6) and (3.9), it follows that

$$(PX \ln f)U - (X \ln f)PU = -A_{FU}X.$$

Now taking inner product with PU in above equation we get

$$g(h(X, PU), FU) = X \ln f\|PU\|^2. \tag{3.10}$$

Now, for $U \in TN$, $X \in TN_T$ we have

$$g(\bar{\nabla}_{PU}U, X) = 0, \tag{3.11}$$

Using the fact that $J\bar{\nabla}_{PU}U = \bar{\nabla}_{PU}JU$ in (3.11), we get

$$
\begin{aligned}
0 &= g(\overline{\nabla}_{PU}JU, JX) \\
&= g(\overline{\nabla}_{PU}PU, JX) + g(\overline{\nabla}_{PU}FU, JX) \\
&= g(\overline{\nabla}_{PU}PU, JX) - g(A_{FU}PU, JX) \\
&= -g(\overline{\nabla}_{PU}JX, PU) - g(h(PU, JX), FU) \\
&= -JX \ln f\|PU\|^2 - g(h(JX, PU), FU)
\end{aligned}
$$

$$-g(h(JX, PU), FU) = JX \ln f\|PU\|^2. \tag{3.12}$$

Replacing X by JX in (3.12), we get

$$-g(h(X, PU), FU) = X \ln f\|PU\|^2. \tag{3.13}$$

Now (3.10) and (3.13) implies that

$$X \ln f = 0.$$

Thus f is constant or $X = 0$, which proves the result.

References

[1] Bejancu. A., *Geometry of CR-Submanifolds,* Kluwer. Aead. Publ. Dordreeht. 1986.

[2] Bishop. R., and O'Niell., *Manifold of negative curvature,* Trans. Amer. Math. Soc. 145 (1969) 1-49.

[3] Chen B. Y., *Geometry of Submanifolds,* Marcell Dekker.inc, New York, 1973.

[4] Chen B. Y., *Differential geometry of Real submanifold in Kaehler manifold,* Monatsh. math. 91(1981), 257-274.

[5] Chen B. Y., *Geometry of Slant submanifold,* Katholieke University Leuven, Leuven 1990.

[6] Chen B. Y., *Geometry of warped product CR-submanifolds in Kaehler manifold I,* Monatsh. math. 133(2001), 177-195.

[7] Chen B. Y., *Geometry of warped product CR-submanifolds in Kaehler manifold II,* Monatsh. math. 134(2001), 103-119.

[8] Chen B. Y., *On isometric minimal immersions from wraped products into real space form,* Proc. Edinburgn. Math. Soc. 45 (2002), 579-587.

[9] Papaghiuc N., *Semi-slant submanifolds of Kaehler manifold,* An. St. Univ. AI. I. Cuza. Iasi, 40(1994), 55-61.

[10] Sahin B., *Nonexistence of warped product semi-slant submanifolds of Kaehler manifold,* Geom. Dedicata (2005),

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