GENERIC WARPED PRODUCT SUBMANIFOLDS IN A KAHLER MANIFOLD

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Abstract

In this paper we have shown that there do not exist proper warped product submanifolds of the type $N \times f N_T$ and $N_T \times f N$ where $N_T$ is an invariant and $N$ is any real non-anti invariant submanifold of a Kaehler manifold. We thus generalize the results of B. Sahin [10] who projected same results for a restricted class, the class of warped product submanifolds $N_\theta \times f N_T$ and $N_T \times f N_\theta$.

1. Introduction

Bishop and O’Neill [2] introduced the concept of warped product manifolds to study manifolds of negative curvature and applied the scheme to space-time. The geometrical aspect of these manifolds have attracted the attention of a lot of researchers recently [6], [8], [10]. Many research papers have appeared to see the existence of warped product submanifolds of manifolds under different settings after it was found that the space around a body with high gravitational field can be modeled on a warped product manifold.

B.Y.Chan [6] studied warped product CR-submanifolds of the type $N_L \times f N_T$ and $N_T \times f N_L$ of a Kaehler manifold $\bar{M}$, where $N_T$ is an invariant and $N_L$ is an anti invariant submanifold of $\bar{M}$. He has shown that there do not exist proper warped product submanifolds of the type $N_L \times f N_T$, when as he and others found many examples of warped product submanifolds of type $N_T \times f N_L$ in a Kaehler manifold. B. Sahin extended the study to slant warped product submanifolds of the type $M = N_T \times f N_\theta$ and $M = N_\theta \times f N_T$ of a Kaehler manifold $\bar{M}$, where $N_T$ is an invariant and $N_\theta$ is a proper slant submanifolds of $\bar{M}$, and showed that they do not exist in either case.

In this paper, we have generalized the results of Chen [6] [7] and Sahin [10] and have shown that there are no proper warped product submanifolds of the type $M = N \times f N_T$ and $M = N_T \times f N$, where $N_T$ is a invariant and $N$ is any real non-anti invariant submanifold of a Kaehler manifold. We thus have extended this study to generic warped product submanifolds of Kaehler manifold.

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2. Some Basic Results

Let $\tilde{M}$ be a Kähler manifold with a complex structure $J$, Hermitian metric $g$ and the Levi-Civita connection $\nabla$. Then we have

$$J^2 = -I, \quad g(JU, JV) = g(U, V), \quad \nabla J = 0$$

(2.1)

for all vector fields $U, V$ on $\tilde{M}$.

Let $\tilde{M}$ be a Kähler manifold with a complex structure $J$, and $M$ be a submanifold of $\tilde{M}$. The induced Riemannian metric on $M$ is denoted by the same symbol $g$ whereas the induced connection on $M$ is denoted by $\nabla$. Then $M$ is called holomorphic if $JT_pM \subset T_pM$, for every $p \in M$, where $T_pM$ denotes the tangent space to $M$ at the point $p$.

If $T\tilde{M}$ and $TM$ denote the Lie-algebra of vector fields on $\tilde{M}$ and $M$ respectively and $T^\bot M$, the set of all vector fields normal to $M$, then the Gauss and Weingarten formulae are respectively given by

$$\nabla_UV = \nabla_UV + h(U, V),$$

$$\nabla_U\xi = -A_\xi U + \nabla^\bot_U\xi,$$

(2.2)

(2.3)

for each $U, V \in TM$ and $\xi \in T^\bot M$, where $\nabla^\bot$ denotes the connection on the normal bundle $T^\bot M$. $h$ and $A_\xi$ are the second fundamental forms and the shape operator of the immersion of $M$ into $\tilde{M}$ corresponding to the normal vector field $\xi$. They are related as

$$g(A_\xi U, V) = g(h(U, V), \xi).$$

(2.4)

For any $U \in TM$ and $\xi \in T^\bot M$, we write

$$JU = PU + FU,$$

$$J\xi = t\xi + f\xi,$$

(2.5)

(2.6)

where $PU$ and $t\xi$ are the tangential components of $JU$ and $J\xi$ respectively whereas $FU$ and $f\xi$ are the normal components of $JU$ and $J\xi$ respectively. The covariant differentiation of the tensors $P, F, t$ and $f$ are defined respectively as

$$(\nabla_U P)V = \nabla_U PV - P\nabla_U V,$$

$$\nabla_U F = \nabla_U^\bot FV - F\nabla_U V,$$

$$\nabla_U t\xi = \nabla_U t\xi - t\nabla_U^\bot\xi,$$

$$\nabla_U f\xi = \nabla_U^\bot f\xi - f\nabla_U^\bot\xi,$$

(2.7)

(2.8)

(2.9)

(2.10)

Let $\tilde{M}$ be an almost Hermition manifold with an almost complex structure $J$, Hermitian metric $g$ and $M$ be a submanifold of $\tilde{M}$. For each $x \in M$, let $D_x = T_xM \cap JT_xM$ i.e., a maximal holomorphc subspace of the tangent space $T_xM$ at $x \in M$. If the dimension of $D_x$ remains the same for each $x \in M$
and it defines a holomorphic distribution $D$ on $M$, then $M$ is called a generic submanifold [4].

A generic submanifold $M$ of an almost Hermitian manifold $\bar{M}$ is said to be generic product submanifold if it is locally a Riemannian product of the leaves of $D$ and $D'$, where $D'$ is orthogonal complementry distribution to $D$ in $TM$. In this case $D$ and $D'$ are parallel on $M$ i.e., $\nabla_U X \in D$ or equivalently $\nabla_U Z \in D'$ for all $U \in TM$, $X \in D$ and $Z \in D'$.

Now we consider warped product of manifolds which are defined as follows

**Definition 2.1.** Let $(B, g_B)$ and $(F, g_F)$ be two Riemannian manifolds with Riemannian metrics $g_B$ and $g_F$ respectively and $f$ be a positive differentiable function on $B$. The warped product of $B$ and $F$ is the Riemannian manifold $(B \times f F, g)$, where

$$g = g_B + f^2 g_F. \quad (2.11)$$

The warped product manifold $(B \times f F, g)$ is denoted by $B \times f F$. If $U$ is tangent to $M = B \times f F$ at $(p, q)$ then by equation (2.11),

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p) \|d\pi_2 U\|^2$$

where $\pi_1$ and $\pi_2$ are the canonical projections of $M$ onto $B$ and $F$ respectively.

On a warped product manifold $B \times f F$ one has

$$\nabla_U V = \nabla_V U = (U \ln f)V \quad (2.12)$$

for any vector fields $U$ tangent to $B$ and $V$ tangent to $F$ [2].

### 3. Generic Warped Product Submanifolds

In this section we study generic warped product submanifolds of a Kaehler manifold $\bar{M}$ of the form $M = N_T \times f N$, $M = N \times f N_T$ respectively, where $N_T$ is a holomorphic submanifold and $N$ is any real non anti-invariant submanifold of $\bar{M}$.

**Theorem 3.1.** There do not exist proper generic warped product submanifold $M = N \times f N_T$ of a Kaehler manifold $M$, where $N_T$ is an invariant submanifold and $N$ is any real non anti-invariant submanifold of $\bar{M}$.

**Proof.** For any $X \in TN_T$ and $U \in TM$ using (2.12) we obtain

$$g(\nabla_X X, U) = -g(\nabla X U, X)$$

$$= -g(\nabla X U, X)$$

$$= -U \ln f \|X\|^2 \quad (3.1)$$

But, we also have
\[ g(\nabla_X X, U) = g(J\nabla_X X, JU) \]
\[ = g(\nabla_X JX, JU) \]
\[ = -g(\nabla_X JU, JX) \]
\[ = -g(\nabla_X PU, JX) - g(\nabla_X FU, JX) \]
\[ = -PU \ln f g(X, JX) + g(A_{FU} X, JX) \]
\[ = g(h(X, JX), FU) \] (3.2)

Thus from (3.1) and (3.2), we obtain

\[ g(h(X, JX), FU) = -U \ln f\|X\|^2 \] (3.3)

Now replacing \( X \) by \( JX \) in (3.3), we obtain

\[ g(h(JX, J^2X), FU) = -U \ln f\|X\|^2 \]
\[ -g(h(X, JX), FU) = -U \ln f\|X\|^2 \]
\[ g(h(JX, JX), FU) = U \ln f\|X\|^2 \] (3.4)

Thus from (3.3) and (3.4), we get

\[ U \ln f\|X\|^2 = 0 \]

for all \( U \in TM \). Which implies that \( f \) is constant or \( X = 0 \). Hence the theorem is proved.

Theorem 3.2. There do not exist proper generic warped product submanifold \( M = N_T \times fN \) of a Kähler manifold \( \bar{M} \), where \( N_T \) is a holomorphic submanifold and \( N \) is any real non anti-invariant submanifold of \( \bar{M} \).

Proof. For any \( U, V \in TM \) and using the fact that \( \bar{M} \) is kähler, we have

\[ \nabla_U JV = J\nabla_U V, \]

therefore,

\[ \nabla_U PV + \nabla_U FV = J(\nabla_U V + h(U, V)), \]

On using (2.2), (2.3), (2.5), we have

\[ \nabla_U PV + h(U, PV) - A_{FU} U + \nabla_U^1 FV = P\nabla_U V + F(\nabla_U V) + th(U, V) + fh(U, V). \]

Now, comparing tangential part and using (2.7), we obtain

\[ (\nabla_U P)V = A_{FU} U + th(U, V). \] (3.5)

Now, for \( X \in TN_T \) and using (2.12), we get
\[
(\bar{\nabla}_X P)U = \nabla_X PU - P\nabla_X U
= (X \ln f)PU - (X \ln f)PU
= 0.
\]

Using it in (3.5), we get
\[
A_{FU}X = -th(X, U). \tag{3.6}
\]

On the other hand
\[
(\bar{\nabla}_U P)X = (PX \ln f)U - (X \ln f)PU. \tag{3.7}
\]

Also from (3.5), we have
\[
(\bar{\nabla}_U P)X = th(X, U). \tag{3.8}
\]

Thus from (3.7) and (3.8), we have
\[
(PX \ln f)U - (X \ln f)PU = th(X, U). \tag{3.9}
\]

From (3.6) and (3.9), it follows that
\[
(PX \ln f)U - (X \ln f)PU = -A_{FU}X.
\]

Now taking inner product with \(PU\) in above equation we get
\[
g(h(X, PU), FU) = X \ln f \|PU\|^2. \tag{3.10}
\]

Now, for \(U \in TN, X \in TN_T\) we have
\[
g(\bar{\nabla}_{PU}U, X) = 0, \tag{3.11}
\]

Using the fact that \(J\bar{\nabla}_{PU}U = \bar{\nabla}_{PU}JU\) in (3.11), we get
\[
0 = g(\bar{\nabla}_{PU}JU, JX)
= g(\nabla_{PU}PU, JX) + g(\nabla_{PU}FU, JX)
= g(\nabla_{PU}PU, JX) - g(A_{FU}PU, JX)
= -g(\nabla_{PU}JX, PU) - g(h(NU, JX), FU)
= -JX \ln f \|PU\|^2 - g(h(JX, PU), FU)
\]

\[
-g(h(JX, PU), FU) = JX \ln f \|PU\|^2. \tag{3.12}
\]

Replacing \(X\) by \(JX\) in (3.12), we get
\[
-g(h(X, PU), FU) = X \ln f \|PU\|^2. \tag{3.13}
\]
Now (3.10) and (3.13) implies that

\[ X \ln f = 0. \]

Thus \( f \) is constant or \( X = 0 \), which proves the result.

References


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