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AN ISOMORPHISM THEOREM FOR ANTI-ORDERED SETS

Daniel Abraham Romano*

Abstract

In this paper we show some kind of isomorphism theorem for ordered sets under antiorders. Let $(X, =_X, \neq_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders, where the apartness $\neq_Y$ is tight. If $\varphi : X \longrightarrow Y$ is reverse isotone strongly extensional mapping, then there exists a strongly extensional and embedding reverse isotone bijection

$$((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \longrightarrow (Im(\varphi), =_Y, \neq_Y, \beta)$$

where $c(R)$ is the biggest quasi-antiorder relation on X under $R = \alpha \cap Coker(\varphi)$, $q = c(R) \cup c(R)^{-1}$ and γ is an antiorder induced by the quasi-antiorder $c(R)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the above bijection is isomorphism.

1 Introduction

1.1 Setting. The arguments in this paper conform to Constructive mathematics in the sense of Bishop ([2]). So, our setting is Bishop's constructive mathematics, mathematics developed with Constructive logic (or Intuitionistic logic ([24])) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold also. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'. As Intuitionistic logic is a fragment of Classical logic, our arguments should be valid from a classical point of view.

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1.2 Set with apartness. Let $(X,=,\neq)$ be a set, where "$\neq$" is a apartness ([2],[4],[8],[12],[14],[23],[24]). *Apartness* is a binary relation on X which satisfies the following properties:

$$\neg(x \neq x),\ \ x \neq y \Longrightarrow y \neq x,\ \ x \neq z \Longrightarrow (\forall y \in X)(x \neq y \vee y \neq z)$$

for every x, y and z in X. The apartness is *compatible* with the equality in the following sense $(\forall x,y,z \in X)(x = y \wedge y \neq z \Longrightarrow x \neq z)$. The apartness $\neq$ is *tight* ([23], [24]) if and only if $\neg(x \neq y) \Longrightarrow x = y$ for any element x and y in X. Let x be an element of X and A subset of X. We write $x \bowtie A$ if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$.
A relation $q \subseteq X \times X$ is *coequality relation* on X ([10],[12]) if

$$q \subseteq \neq,\ \ q^{-1} = q,\ \ (\forall x,z \in X)((x,z) \in q \Longrightarrow (\forall y \in X)((x,y) \in q \vee (y,z) \in q)).$$

The relation $q^C = \{(x,y) \in X \times X : (x,y) \bowtie q\}$ is an equality on X compatible with q, in the following sense

$$(\forall a,b,c \in X)((a,b) \in q^C \wedge (b,c) \in q \Longrightarrow (a,c) \in q)$$

([18], Theorem 1). We can ([10], [12]) construct factor-sets

$$(X/(q^C,q),=_1,\neq_1) = \{aq^C : a \in X\} \quad \text{and} \quad (X/q,=_1,\neq_1) = \{aq : a \in X\},$$

where

$$aq^C =_1 bq^C \Longleftrightarrow (a,b) \bowtie q,\ \ aq^C \neq_1 bq^C \Longleftrightarrow (a,b) \in q,$$
$$aq =_1 bq \Longleftrightarrow (a,b) \bowtie q,\ \ aq \neq_1 bq \Longleftrightarrow (a,b) \in q.$$

It is easy to check that $X/(q^C,q) \cong X/q$.

Examples I: (1) The relation $\neg(=)$ is an apartness on the set $\mathbf{Z}$ of integers.
(2) ([8]) The relation q, defined on the set $\mathbf{Q}^N$ by

$$(f,g) \in q \Longleftrightarrow (\exists k \in N)(\exists n \in N)(m \geq n \Longrightarrow |f(m) - g(m)| > k^{-1}),$$

is a coequality relation.
(3) ([8]) A ring R is a local ring if for each $r \in R$, either r or $1 - r$ is a unit. Let M be a module over R. Then the relation q on M, defined by $(x,y) \in q$ if there exists a homomorphism $f : M \longrightarrow R$ such that $f(x - y)$ is a unit, is a coequality relation on M.
(4) ([12]) Let T be a set and $\boldsymbol{J}$ be a subfamily of $\wp(T)$ such that $\emptyset \in \boldsymbol{J}$, $A \subseteq B \wedge B \in \boldsymbol{J} \Longrightarrow A \in \boldsymbol{J}$, $A \cap B \in \boldsymbol{J} \Longrightarrow A \in \boldsymbol{J} \vee B \in \boldsymbol{J}$. If $(X_t)_{t\in T}$ is a family of sets, then the relation q on $\prod X_t\ (\neq \emptyset\ )$, defined by $(f,g) \in q \Longleftrightarrow \{s \in T : f(s) = g(s)\} \in \boldsymbol{J}$, is a coequality relation on the Cartesian product $\prod X_t$. ♦

1.3 Algebraic structures with apartness. For a function

$$f : (X, =, \neq) \longrightarrow (Y, =, \neq)$$

we say that it is:
(a) ([8]) *strongly extensional* if and only if $(\forall a, b \in X)(f(a) \neq f(b) \Longrightarrow a \neq b)$;
(b) ([7]) an *embedding* if and only if $(\forall a, b \in X)(a \neq b \Longrightarrow f(a) \neq f(b))$.
In general, all functions in this text are strongly extensional functions. For example, if $\omega : X \times X \longrightarrow X$ is an internal binary operation on X, then must be:

$$(\forall a, b, x, y \in X)(\omega(a, b) \neq \omega(x, y) \Longrightarrow (a, b) \neq (x, y)).$$

Examples II: Let $(S, =, \neq, \cdot)$ be a semigroup with apartness. Let us note that the internal operation "$\cdot$" is a strongly extensional function in the following sense:

$$(\forall x, a, b \in S)((ax \neq bx \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \neq b)).$$

A subset T of semigroup S is a *right consistent subset* of S ([3]) of S if and only if

$$(\forall x, y \in S)(xy \in T \Longrightarrow y \in T);$$

a subset T of S is a *left consistent subset* of S ([3]) of S if and only if

$$(\forall x, y \in S)(xy \in T \Longrightarrow x \in T);$$

a subset T of S is a *consistent subset* of S ([3]) of S if and only if

$$(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T).$$

Let q be a coequality relation on a semigroup S such that

$$(\forall a, b, y \in S)((ay, by) \in q \Longrightarrow (a, b) \in q).$$

Then we say that it is a *left anticongruence* on S. If for q holds

$$(\forall a, b, x \in S)((xa, xb) \in q \Longrightarrow (a, b) \in q)$$

then q is a *right anticongruence* on S. The coequality relation q on S is an *anticongruence* on S, or relation compatible with semigroup operation on S, if and only if it is a left and right anticongruence.
(1) ([17]) Let e and f be idempotents of a semigroup S with apartness. Then:
(a) the set $X(e) = \{a \in S : ae \neq a\}$ is a strongly extensional right consistent subset of S;
(b) the set $Y(e) = \{b \in S : eb \neq b\}$ is a strongly extensional left consistent subset of S;
(c) the set $P(e) = \{a \in S : e \bowtie Sa\}$ is a strongly extensional left ideal of S;
(d) the set $Q(e) = \{a \in S : e \bowtie aS\}$ is a strongly extensional right ideal of S;
(e) the set $R(e) = \{a \in S : e \bowtie SaS\}$ is a strongly extensional ideal of S such

that $e \bowtie R(e)$;
(f) the set $M(e) = X(e) \cup Y(e) \cup P(e) \cup R(e)$ is a strongly extensional completely prime subset of S such that $e \bowtie M(e)$. Besides, if $e \neq f$, then $M(e) \cup M(f) = S$.
(2) Let $S = \{0\} \times [0,1]$ ($\subset \mathbf{R} \times \mathbf{R}$, where $\mathbf{R}$ is the set of reals). The multiplication in S is the coordinatewise usual multiplication. Then S is a semigroup with apartness. The set $\{0\} \times [0, >$ is an ideal of S and the set $\{0\} \times [1/2, 1]$ is a consistent subset of S.
(3) The set $S = \{(x,y) \in \mathbf{R} \times \mathbf{R} : x \geq 0 \wedge y \geq 0\}$ with the multiplication on S defined by $(x,y)(a,b) = (xa, xb + y)$ is a semigroup with apartness. The subset $Q = \{(x,y) \in S : x > 0\}$ is a consistent subset of S and filter in S.
(4) Let T be a strongly extensional consistent subset of semigroup S, i.e. let $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$. Then relation q on semigroup S, defined by $(a,b) \in q$ if and only if $a \neq b \wedge (a \in T \vee b \in T)$, is a coequality relation on S.
(5) ([18], Theorem 5) Let q be a coequality relation on a semigroup S with apartness. Then the relation $q^+ = \{(x,y) \in S \times S : (\exists a, b \in S^1)((axb, ayb) \in q)\}$ is an anticongruence on S such that $q \subseteq q^+$. If ρ is an anticongruence on S such that $q \subseteq \rho$, then $q^+ \subseteq \rho$. ♦

Examples III: Let $(R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring. A subset Q of R is a *coideal* of R if and only if

$$0 \bowtie Q,$$
$$-x \in Q \Longrightarrow x \in Q,$$
$$x + y \in Q \Longrightarrow x \in Q \vee y \in Q,$$
$$xy \in Q \Longrightarrow x \in Q \wedge y \in Q.$$

Coideals of commutative ring with apartness where studied by Ruitenburg 1982 ([23]). After that, coideals (anti-ideals) studied by A.S. Troelstra and D. van Dalen in their monograph [24]. The author proved, in his paper [9], if Q is a coideal of a ring R, then the relation q on R, defined by $(x,y) \in q \Longleftrightarrow x - y \in Q$, satisfies the following properties:
(a) q is a coequality relation on R;
(b) $(\forall x, y, u, v \in R)((x+u, y+v) \in q \Longrightarrow (x,y) \in q \vee (u,v) \in q)$;
(c) $(\forall x, y, u, v \in R)((xu, yv) \in q \Longrightarrow (x,y) \in q \vee (u,v) \in q)$.
A relation q on R, which satisfies the properties (a)-(c), is called *anticongruence* on R ([9]) or relation compatible with ring operations. If q is an anticongruence on a ring R, then the set $Q = \{x \in R : (x,0) \in q\}$ is a coideal of R ([9]). Let J be an ideal of R and if Q is a coideal of R. Wim Ruitenburg, in his dissertation ([23], page 33) first stated a demanded that $J \subseteq \neg Q$. This condition is equivalent with the following condition

$$(\forall x, y \in R)(x \in J \wedge y \in Q \Longrightarrow x + y \in Q).$$

In this case we say that they are *compatible* ([11]) and we can construct the quotient-ring $R/(J,Q)$. W.Ruitenburg, in his dissertation, first stated the question on the existence an ideal J of R compatible with a given coideal Q and the question on the existence of a coideal Q of R compatible with a given ideal J. If e is a congruence on R, determined by the ideal J and if q is an anticongruence on R, determined by Q, then J and Q are compatible if and only if

$$(\forall x, y, z \in R)((x, y) \in e \wedge (y, z) \in q \Longrightarrow (x, z) \in q).$$

In this case we say that e and q are compatible.
(1) Let $R = (R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring with apartness. Then the sets $\emptyset$ and $R = \{a \in R : a \neq 0\}$ are coideals of R. Let a be an element of the ring R. Then the ideal $Ann(a)$ and the coideal $Cann(a) = \{x \in R : ax \neq 0\}$ are compatible.
(2) Let m and $i \in \{1, 2, ..., m-1\}$ be integers. We set $m\mathbf{Z}+i = \{mz+i : z \in \mathbf{Z}\}$. Then the set $\cup\{m\mathbf{Z} + i : i \in \{1, ..., m-1\}\}$ is a coideal of the ring $\mathbf{Z}$.
(3) Let K be a Richman field and x be an unknown variable under K. Then the set $C = \{f \in K[x] : f(0) \neq 0\}$ is a coideal of the ring $K[x]$.
(4) Let R be a commutative ring. Then the set $B = R^N$ is a ring. For $n \in N$, the set $M = \{f \in B : f(n) \neq 0\}$ is a coideal of B.
(5) Let R be a local ring. Then the set $M = \{a \in R : (\exists x \in R)(ax = 1)\}$ is a coideal of R.
(6) Let S be a coideal of a ring R and let X be a subset of R. Then the set $[S : X] = \{a \in R : (\exists x \in X)(ax \in S)\}$ is a coideal of R.
(7) Let H be a nonempty family of inhabited subsets of T. Then the set $S(H) = \{r \in \prod R_t : (\exists A \in H)(A \cap Z(r) \neq \emptyset)\}$, where $Z(r) = \{t \in T : r(t) \neq 0\}$, is a coideal of the ring $\prod R_t$ ($\neq \emptyset$). ♦

1.3 Filed product. Let $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ be relations. As in [15], we define

$$\beta * \alpha = \{(x, z) \in X \in Z : (\forall y \in Y)((x, y) \in \alpha \vee (y, z) \in \beta)\}.$$

For a relation $R \subseteq X \times X$ we put ${}^1R = R$, ${}^nR = R * R * ... * R$ $(n \geq 2)$ and $c(R) = \bigcap_{n \in N} {}^nR$. In [14] and [15] this author proved that the relation $c(R)$ is a cotransitive relation under R. This relation is called the *cotransitive fulfillment* of R.

1.4 Goal of this paper. We will briefly recall the constructive definition of linear order and we will use a generalization of J. von Plato ([9]) and M.A. Baroni's ([1]) excess relation for the definition of a partially ordered set. Let X be a nonempty set. A binary relation $<$ (less than) on X is called a *linear* order if the following axioms are satisfied for all elements x and y:

$$\neg(x < y \wedge y < x),$$
$$x < y \Longrightarrow (\forall z \in S)(x < z \vee z < y).$$

An example is the standard strict order relation $<$ on R, as described in [2], [4], [8] and [9]. For an axiomatic definition of the real number line as a constructive ordered field, the reader is referred to [2], [4], [9]. A detailed investigation of linear orders in lattices can be found in [9]. The binary relation $\nleqslant$ on X is called an excess relation if it satisfies the following axioms:

$$\neg(x \nleqslant x),$$
$$x \nleqslant y \Longrightarrow (\forall z \in S)(x \nleqslant z \vee z \nleqslant y).$$

Clearly, each linear order is an excess relation. As shown in [9], we obtain an apartness relation $\neq$ and a partial order $\leq$ on X by the following definitions:

$$x \neq y \Longleftrightarrow (x \nleqslant y \vee y \nleqslant x) \ ,$$
$$x \leq y \Longleftrightarrow \neg(x \nleqslant y).$$

Note that the statement $\neg(x \leq y) \Longrightarrow x \nleqslant y$ does not hold in general.

Let $(X, =, \neq)$ be a set with apartness. A relation $\alpha \subseteq X \times X$ is an *antiorder* relation on X if and only if

$$\alpha \subseteq \neq \ ,$$
$$(\forall x, y, z \in X)((x, z) \in \alpha \Longrightarrow (x, y) \in \alpha \vee (y, z) \in \alpha),$$
$$(\forall x, y \in X)(x \neq y \Longrightarrow (x, y) \in \alpha \vee (y, x) \in \alpha).$$

A ordered set under an antiorder α is a structure $(X, =, \neq, \alpha)$ where α is an antiorder relation on X. Antiorder relation on a set was first defined by author in paper [14] and [16].

Example IV: Let $S = \{a, b, c, d, e\}$ with multiplication defined by schema

	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

The relation $\alpha \subseteq S \times S$, defined by $\alpha = \{(a, b),(a, c),(a, e),(b, a),(b, c),(b, d),$ $(b, e),(c, a),(c, b),(c, d),(d, a),(d, b), (d, c), (d, e),(e, a),(e, b),(e, c), (e, d)\}$ is an antiorder relation on semigroup S. ♦

A relation σ on $(X, =, \neq)$ is a *quasi-antiorder* relation on X if and only if

$$\sigma \subseteq \neq \ ,$$
$$(\forall x, y, z \in X)((x, z) \in \sigma \Longrightarrow (x, y) \in \sigma \vee (y, z) \in \sigma).$$

If there exists an antiorder α on the set $(X, =, \neq)$, different from $\neq$, then we have to put a stronger demand in the definition of quasi-antiorder: $\sigma \subseteq \alpha$ instead of $\sigma \subseteq \neq$. A quasi-antiordered set is a structure $(X, =, \neq, \sigma)$ where σ is a quasi-antiorder relation on X. Note that if σ is a quasi-antiorder on X, then σ^{-1} is a quasi-antiorder in X too. Indeed:

(a) $\sigma \subseteq \neq \Longrightarrow \sigma^{-1} \subseteq \neq^{-1} = \neq$(because the relation $\neq$ is symmetric);
(b) $(x,z) \in \sigma^{-1} \Longleftrightarrow (z,x) \in \sigma$

$$\Longrightarrow (\forall y \in X)((z,y) \in \sigma \vee (y,x) \in \sigma)$$
$$\Longrightarrow (\forall y \in X)((y,z) \in \sigma^{-1} \vee (x,y) \in \sigma^{-1})$$
$$\Longleftrightarrow (\forall y \in X)((x,y) \in \sigma^{-1} \vee (y,z) \in \sigma^{-1}).$$

There is a theory of quasi-order relation in ordered semigroup. See, for example, papers [5] and [6]. In this paper we continue the research parallel relations of antiorder and quasi-antiorder.

Example V: Let $S = \{a,b,c,d,e\}$ with multiplication defined by schema

	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Relation α , defined by $\alpha = \{(a,c),(a,d),(a,e),(b,a),(b,c),(b,d),(b,e),(c,a),(c,b),(c,d),(c,e),(d,a),(d,e),(e,a),(e,b),(e,d)\}$, is an antiorder relation on semigroup S. The relation $\sigma = \{(a,e),(b,e),(c,a),(c,b),(c,d),(c,e),(d,e),(e,a),(e,b),(e,d)\}$ is a quasi-antiorder relation on semigroup S. ♦

The notion of quasi-antiorder relation in set with apartness was introduced for first time is in paper [14]. After that, quasi-antiorders are studied by this author in his paper [18], [19], [20] [21], [22]. Sometime, in the definition of antiorder relation on a set $(X,=,\neq)$, we add the condition $\alpha \cap \alpha^{-1} = \emptyset$. In that case, in the definition of quasi-antiorder relation on the ordered set $(X,=,\neq,\alpha)$ under the antiorder α, we must add the following condition $\sigma \cap \sigma^{-1} = \emptyset$. What is different between anti-order relation and excess relation? Clearly, an anti-order relation on set with tight apartness is an excess relation, and, opposite, an excess relation is an anti-order relation.

In this note we proved some kind of isomorphism theorem for ordered sets under antiorders. Let $(X,=_X,\neq_X,\alpha)$ and $(Y,=_Y,\neq_Y,\beta)$ be ordered sets under antiorders, where the apartness $\neq_Y$ is tight. If $\varphi : X \longrightarrow Y$ is reverse isotone function, then there exists a strongly extensional, injective and embedding reverse isotone bijection

$$((X,=_X,\neq_X,\alpha,c(R))/q,=_1,\neq_1,\gamma) \longrightarrow (Im(\varphi),=_Y,\neq_Y,\beta),$$

where $c(R)$ is the biggest quasi-antiorder relation on X under $R = \alpha \cap Coker(\varphi)$, $q = c(R) \cup c(R)^{-1}$ and γ is the antiorder induced by the quasi-antiorder $c(R)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the above bijection is an isomorphism.

1.5 References. For undefined notions and notations for Constructive mathematics we refer to the books [2], [4], [8], [23] and [24], and to papers [7], [11]-[20]. For classical and constructive semigroup theory we refer to [3], [5], [6] and [7], [17]-[20].

2 Preliminaries

In this section we start with the following explanations:

Remarks A.
(0) A relation q on a set $(X,=,\neq)$ is a coequality relation on X if and only if

$$q\subseteq\neq,\ q^{-1}=q,\ \ q\ \subseteq\ q*q.$$

(1) A relation α is an antiorder relation on a set $(X,=,)$ if and only if

$$\alpha\ \subseteq\neq,\ \alpha\ \subseteq\ \alpha*\alpha,\ \ \neq\subseteq\ \alpha\cup\alpha^{-1}.$$

(2) A relation σ on a ordered set $(X,=,\neq,\alpha)$ under an antiorder α is a quasi-antiorder relation on X iff

$$\sigma\ \subseteq\ \alpha,\ \sigma\ \subseteq\ \sigma*\sigma.$$

(3) Sometimes, in the definition of antiorder relation on set $(X,=,\neq)$, we add another condition

$$(\forall x,y\in X)((x,y)\in\alpha\Longrightarrow\neg((y,x)\in\alpha)),$$

which is equivalent with the condition

$$\alpha\cap\alpha^{-1}=\emptyset.$$

In that case, in the definition of quasi-antiorder relation on the ordered set $(X,=,\neq,\alpha)$ under the antiorder α , we must add the following condition

$$(\forall x,y\in X)((x,y)\in\sigma\Longrightarrow\neg((y,x)\in\sigma)),$$

i.e. the demand

$$\sigma\cap\sigma^{-1}=\emptyset.$$

Let $(X,=,\neq,\alpha)$, $(Y,=,\neq,\beta)$ be ordered sets under antiorders α and β respective, $f:X\longrightarrow Y$ a mapping from X into Y. f is called *isotone* if

$$(\forall x,y\in S)((x,y)\in\alpha\Longrightarrow(f(x),f(y))\in\beta).$$

f is called *reverse isotone* if and only if

$$(\forall x,y\in S)((f(x),f(y))\in\beta\Longrightarrow(x,y)\in\alpha).$$

The mapping f is called an *isomorphism* if it is injective and embedding, onto, isotone and reverse isotone. X and Y are called isomorphic, symbolically $X \cong Y$, if exists an isomorphism between them.

Remarks B.
B.1. Every isotone mapping $f : X \longrightarrow Y$ satisfies the following condition:
(1) Let $x, y \in X$ and $x \neq_X y$. Then $(x, y) \in \alpha$ or $(y, x) \in \alpha$ by linearity of α and we have $(f(x), f(y)) \in \beta \subseteq \neq_Y$ or $(f(y), f(x)) \in \beta \subseteq \neq_Y$. So, the mapping f is an embedding.
(2) Let $x, y \in X$ and $f(x) = f(y)$. Then $\neg(f(x) \neq_Y f(y))$, and from this we conclude $\neg((f(x), f(y)) \in \beta)$ and $\neg((f(y), f(x)) \in \beta)$. Hence $\neg((x, y) \in \alpha)$ and $\neg((y, x) \in \alpha)$. Therefore $\neg(x \neq_X y)$. If the apartness $\neq_X$ on set X is tight, then $x = y$. So, in that case when the apartness is tight, the mapping f is an injective.

B.2. Every reverse isotone mapping $f : X \longrightarrow Y$ satisfies the following condition:
(3) Let $x, y \in X$ such that $f(x) \neq_Y f(y)$. Then $(f(x), f(y)) \in \beta$ or $(f(y), f(x)) \in \beta$ by linearity of β and we have $(x, y) \in \neq_X$ or $(y, x) \in \neq_X$. So, the mapping f is strongly extensional.
(4) Let $x =_X y$. Then $\neg(x \neq_X y)$, i.e. then $\neg((x, y) \in \alpha \cup \alpha^{-1})$. Suppose that $f(x) \neq_Y f(x)$, i.e. suppose that $(f(x), f(y)) \in \beta \cup \beta^{-1}$. Thus we conclude $(x, y) \in \alpha \cup \alpha^{-1}$ which is impossible. So, our proposition $f(x) \neq_Y f(x)$ is wrong, i.e. must holds $\neg(f(x) \neq_Y f(x))$. If the apartness Y is tight, then holds $f(x) =_Y f(y)$. So, in this case when the apartness $\neq_X$ is tight, antiorders are compatible with the function f.

Lemma 0: *Let σ be a quasi-antiorder relation on an anti-ordered set $(X, =, \neq, \alpha)$. Then $q = \sigma \cup \sigma^{-1}$ is a coequality relation on X such that $(X/q, =_1, \neq_1)$ is an ordered set under the antiorder relation β defined by $(xq, yq) \in \beta \Longleftrightarrow (x, y) \in \sigma$.*
Proof: Let (uq, vq) be an arbitrary element of β , i.e. let $(u, v) \in \sigma$. Since $\sigma \subseteq q$, we have $uq \neq_1 vq$. Therefore, $\beta \subseteq \neq_1$ (in X/q). Let (xq, zq) and yqX/q, i.e. let (x, z) and yX. Since $(x, y)(y, z)$, we have (xq, yq) or (yq, zq). Let $(xq, yq) \in \beta$ and $aq, bq \in X/q$, i.e. $(x, y) \in \sigma$ and $a, b \in X$. Let $xq \neq_1 yq$, i.e. let $(x, y) \in q = \sigma \cup \sigma^{-1}$. Since $(x, y) \in \sigma$ or $(y, x) \in \sigma$, we have $(xq, yq) \in \beta$ or $(yq, xq) \in \beta$. So, the relation β is linear. Therefore, the relation β is an antiorder relation on X/q.
Now, suppose that $\sigma \cap \sigma^{-1} = \emptyset$. Then also $\beta \cap \beta^{-1} = \emptyset$. Indeed, let $(xq, yq) \in \beta$, i.e. let $(x, y) \in \sigma$. Then $\neg((y, x) \in \sigma)$, i.e. then $\neg((yq, xq) \in \beta)$. $\square$

Example VI: Let S, α and σ as in the example II. Then the relation $q = \sigma \cup \sigma^{-1} = \{(a, e),(b, e),(c, a),(c, b),(c, d),(c, e),(e, a),(e, b),(e, d),(e, a),(e, b),(a, c),(b, c),(d, c), (e, c),(a, e),(b, e),(d, e)\}$ is an anticongruence on S. Then $aq = \{c, e\}$, $bq = \{c, e\}$, $cq = \{a, b, d, e\}$, $dq = \{c, e\}$, $eq = \{a, b, c, d\}$ and $S/q = \{\{c, e\}, \{a, b, d, e\}, \{a, b, c, d\}\}$. So, the relation β is defined in the following way: $\beta =$

$\{(aq, eq), (bq, eq), (cq, aq), (cq, bq), (cq, dq), (cq, eq), (eq, aq), (eq, bq), (eq, dq)\}$.
♦

Corollary 0.1: *The mapping $\pi : X \longrightarrow X/q$ is a reverse isotone surjective function.*

Lemma 1: *If $\{\sigma_k\}_{k\in J}$ is a family of quasi-antiorders on a set $(X, =, \neq)$ relatively to a certain antiorder α, then $\bigcup_{k\in J} \sigma_k$ is a quasi-antiorder in X.*
Proof: Let (x, z) be an arbitrary elements of $X \times X$ such that $(x, z) \in \bigcup_{k\in J} \sigma_k$. Then there exists k in J such that $(x, z) \in \sigma_k$. Hence for every $y \in X$ we have $(x, y) \in \sigma_k \vee (y, z) \in \sigma_k$. So, $(x, y) \in \bigcup_{k\in J} \sigma_k \vee (y, z) \in \bigcup_{k\in J} \sigma_k$. At the other side, for every k in J holds $\sigma_k \subseteq \alpha$. From this we conclude $\bigcup_{k\in J} \sigma_k \subseteq \alpha$. □

3 The main results

First, we show a construction of maximal quasi-antiorder under a given relation:

Theorem 3: *Let R ($\subseteq \neq$) be a relation on a set $(X, =, \neq)$. Then for an inhabited family of quasi-antiorders under R there exists the biggest quasi-antiorder relation under R. That relation is exactly the relation $c(R)$.*
Proof: By Lemma 1, there exists the biggest quasi-antiorder relation on X under R. Let $\boldsymbol{A}_R$ be the inhabited family of all quasi-antiorder relation on X under R. With (R) we denote the biggest quasi-antiorder relation $\cup \boldsymbol{A}_R$ on X under R. The fulfillment $c(R) = \bigcap_{n\in N} {}^nR$ of the relation R is a cotransitive relation on set X under R. Therefore, $c(R) \subseteq (R)$ holds.
We need to show that $(R) \subseteq c(R)$. Let s be a quasi-antiorder relation in X under R. First, we have $s \subseteq R = {}^1R$. Let $(x, z) \in s$. Then from $(\forall y \in X)((x, y) \in s \vee (y, z) \in s)$ we conclude that for every y in X holds $(x, y) \in R \vee (y, z) \in R$, i.e. holds $(x, z) \in R * R = {}^2R$. So, $s \subseteq {}^2R$. Now, we will suppose that $s \subseteq {}^nR$ and let $(x, z) \in s$. Then from $(\forall y \in X)((x, y) \in s \vee (y, z) \in s)$ implies that $(x, y) \in R \vee (y, z) \in {}^nR$ holds for every $y \in X$. Therefore, $(x, z) \in {}^{n+1}R$. So, we have $s \subseteq {}^{n+1}R$. Thus, by induction, we have $s \subseteq \cap{}^nR$. Remember that s is an arbitrary quasi-antiorder on X under R. Hence, we proved that $(R) = \cup \boldsymbol{A}_R \subseteq c(R)$. □

Corollary 3.1: *Let $(X, =, \neq, \alpha)$ be an ordered set under an antiorder α . Then the family $\mathbf{A} = \{\tau : \tau$ is a quasi-antiorder on X under $\alpha\}$ is a complete lattice.*

Example VII ([18]): Let a and b be elements of semigroup S. Then ([18], Theorem 6) the set - $C_{(a)} = \{x \in S : x \bowtie SaS\}$ is a consistent subset of S such that :
- $a \bowtie C_{(a)}$;
- $C_{(a)} \neq \emptyset \Longrightarrow 1 \in C_{(a)}$;

- Let a be an invertible element of S. Then $C_{(a)} = \emptyset$;
- $(\forall x, y \in S)(C_{(a)} \subseteq C_{(xay)})$;
- $C_{(a)} \cup C_{(b)}C_{(ab)}$.

Let a be an arbitrary element of a semigroup S with apartness. The consistent subset $C_{(a)}$ is called a *principal* consistent subset of S generated by a. We introduce relation f, defined by $(a, b) \in f \iff b \in C_{(a)}$ and in the next assertion we will give some description of the relation f: The relation f has the following properties ([17], Theorem 7)

- f is a consistent relation ;
- $(a, b) \in f \Longrightarrow (\forall x, y \in S)((xay, b) \in f)$;
- $(a, b) \in f \Longrightarrow (\forall n \in N)((a^n, b) \in f)$;
- $(\forall x, y \in S)((a, xby) \in f \Longrightarrow (a, b) \in f)$;
- $(\forall x, y \in S)\neg((a, xay) \in f)$.

We can construct the cotransitive relation $c(f) = \bigcap_{n\in N} {}^n f$ as cotransitive fulfillment of the relation f ([7],[14],[18]). As corollary of theses assertions we have the following results: The relation $c(f)$ satisfies the following properties:

- $c(f)$ is a consistent relation on S ;
- $c(f)$ is a cotransitive relation ;
- $(\forall x, y \in S)((a, xay) \bowtie c(f))$;
- $(\forall n \in N)((a, an) \bowtie c(f))$;
- $(\forall x, y \in S)((a, b) \in c(f) \Longrightarrow (xay, b) \in c(f))$;
- $(\forall n \in N)((a, b) \in c(f) \Longrightarrow (a^n, b) \in c(f))$;
- $(\forall x, y \in S)((a, xby) \in c(f) \Longrightarrow (a, b) \in c(f))$.

For an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A_n(a) = \{x \in S : (a, x) \in {}^n f\},\ A(a) = \{x \in S : (a, x) \in c(f)\}$$

By the following results we will present some basic characteristics of these sets. Let a and b be elements of a semigroup S. Then:

- $A_1(a) = C_{(a)}$;
- $A_{n+1}(a) \subseteq A_n(a)$;
- $A_{n+1}(a) = \{x \in S : S = A_n(a) \cup B_1(x)\}$ where $B_1(x) = \{u \in S : (u, x) \in f\}$;
- $A(a) = \bigcap_{n\in N} A_n(a)$;
- $a \bowtie A(a)$;
- $A(a) \cup A(b) \subseteq A(ab)$;
- The set $A(a)$ is the maximal strongly extensional consistent subset of S such that $a \bowtie A(a)$. ♦

Example VIII: Let $S = \{a, b, c, d\}$ be a ordered semigroup with the Cayley table and antiorder shown below:

	a	b	c	d
a	b	b	c	c
b	b	b	c	c
c	c	c	c	c
d	c	c	c	c

$\alpha = \{(a,b),(a,c),(a,d),(b,a),(c,a),(c,b),(d,a),(d,b),(d,c)\}$. For relation α holds $\alpha \cap \alpha^{-1} \neq \emptyset$. If we put $\beta = \{(a,b),(a,c),(a,d),(c,b),(d,b),(d,c)\}$, then $\beta \subset \alpha$ and $\beta \cap \beta^{-1} = \emptyset$. ♦

Let $f : X \longrightarrow Y$ be a strongly extensional function. It is easy to verify that sets

$$Ker(f) = \{(a,b) \in X \times X : f(a) = f(b)\},$$

$$(q =)Coker(f) = \{(a,b) \in X \times X : f(a) \neq f(b)\}$$

are compatible equality and coequality relations on X and we can construct the factor-set X/q .

The following theorem is the main result in this paper:

Theorem 4: *Let $(X, =_X, \neq_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders, where the apartness $\neq_Y$ is tight. If $\varphi : X \longrightarrow Y$ is reverse isotone strongly extensional function, then there exists a strongly extensional and embedding reverse isotone bijection*

$$((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \longrightarrow (Im(\varphi), =_Y, \neq_Y, \beta)$$

where $c(R)$ is the biggest quasi-antiorder relation on X under $R = \alpha \cap Coker(\varphi)$, $q = c(R) \cup c(R)^{-1}$ and γ is the antiorder induced by the quasi-antiorder $c(R)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the there exists the isomorphism

$$(X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \cong (Im(\varphi), =_Y, \neq_Y, \beta).$$

Proof:
(1) Let $(X, =_X, \cong_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders α and β respectively, and $\varphi : X \longrightarrow Y$ a strongly extensional mapping. Then the relation $\varphi^{-1}(\beta) = \{(a,b) \in X \times X : (\varphi(a), \varphi(b)) \in \beta\}$ is a quasi-antiorder on X, the relation $Coker = \{(a,b) \in X \times X : \varphi(a) \neq_Y \varphi(b))\}$ is coequality relation on X compatible with equality relation $Ker\varphi = \varphi^{-1} \circ \varphi$, and $Coker\varphi \supseteq \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1}$ holds. Also, since the relation β is linear we have $Coker\varphi = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1}$. Indeed, $\alpha \subseteq \neq$ and $\varphi^{-1}(\beta) \subseteq \alpha$.
Since the relation β is linear, we have
$(a,b) \in Coker\varphi \Longleftrightarrow \varphi(a) \neq_Y \varphi(b)$
$\Longrightarrow (\varphi(a), \varphi(b)) \in \beta \vee (\varphi(b), \varphi(a)) \in \beta$
$\Longleftrightarrow (a,b) \in \varphi^{-1}(\beta) \vee (b,a) \in \varphi - 1(\beta)$.
At the other side, if $(a,b) \in \varphi - 1(\beta)$ or $(b,a) \in \varphi^{-1}(\beta)$, then $(\varphi(a), \varphi(b)) \in$

$\beta \subseteq \neq_Y$ or $(\varphi(b), \varphi(a)) \in \beta \subseteq \neq_Y$. Therefore, $Coker\varphi = \varphi^{-1}(\beta)(\varphi^{-1}(\beta))^{-1}$.

(2) It is easily to conclude that $Coker(\varphi) \subseteq \alpha \cup \alpha^{-1}$.

(3) The family $\boldsymbol{A}_R$ of quasi-antiorder relations on X under relation $R = \alpha \cap Coker(\varphi)$ is not empty, because $\varphi^{-1}(\beta) \subseteq R$. Then, by Theorem 3, there exists the biggest quasi-antiorder relation $c(R)$ under R. Put $q = c(R) \cup c(R)^{-1}$. We can construct, according Lemma 0, the ordered factor-set $((X, =_X, \neq_X, c(R))/q, =_1, \neq_1)$ under antiorder relation γ on X/q, defined by $(aq, bq) \in \gamma$ if and only if $(a, b) \in c(R)$.

(4) We wish to show that $Coker(\varphi) = q = c(R) \cup (c(R))^{-1}$. The first, by definition of $c(R)$, $c(R)$ is the biggest quasi-antiorder relation under R. So, we have $c(R) \subseteq Coker(\varphi)$ and $(c(R))^{-1} \subseteq (Coker(\varphi))^{-1} = Coker(\varphi)$ because the relation $Coker(\varphi)$ is symmetric. Therefore, holds $c(R) \cup (c(R))^{-1} \subseteq Coker(\varphi)$. The second, the relation $\varphi^{-1}(\beta)$ is a quasi-antiorder under $R = \alpha \cap Coker(\varphi)$. So, it must be $\varphi^{-1}(\beta) \subseteq c(R)$ because the relation $c(R)$ is the biggest under R. Thus, it must be $(\varphi^{-1}(\beta))^{-1} \subseteq (c(R))^{-1}$. Therefore, it must be $Coker(\varphi) = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} \subseteq c(R) \cup (c(R))^{-1}$.
If $\alpha \cap \alpha^{-1} = \emptyset$ holds, then easy to verify that $c(R) \cap (c(R))^{-1} = \emptyset$ holds too.

(5) By Lemma 0, the set $((X, =_X, \neq_X, \alpha)/q, =_1, \neq_1)$ is ordered set under the antiorder γ on X/q defined by

$$(aq, bq) \in \gamma \Longleftrightarrow (a, b) \in c(R).$$

If $\alpha \cap \alpha^{-1} = \emptyset$, then $c(R) \cap (c(R))^{-1} = \emptyset$, because $c(R) \cup (c(R))^{-1} \subseteq \alpha \cup \alpha^{-1}$. It remains to construct mapping $\phi : X/q \longrightarrow Im(\varphi)$ $(\subseteq Y)$. Define $\phi(aq) = \varphi(a)$ for any a in X.

(a) This mapping is well defined because if $aq =_1 bq$, i.e. if $(a, b) \bowtie q = c(R) \cup (c(R))^{-1} = Coker(\varphi)$, then $\neg(\varphi(a) \neq_Y \varphi(b))$ holds. Since the apartness $\neq_Y$ is tight, it implies that $\varphi(a) =_Y \varphi(b)$, i.e. $\phi(aq) =_Y \phi(bq)$.

(b) Suppose that $\phi(aq) \neq_Y \phi(bq)$, i.e. suppose that $\varphi(a) \neq_Y \varphi(b)$, i.e. suppose that $(a, b) \in Coker(\varphi)$. Then $aq \neq_1 bq$. Therefore, the mapping is strongly extensional function from set X/q into Y.

(c) If $y \in Im(\varphi)$, then for some $x \in X$, $\phi(xq) =_Y \varphi(x) =_Y y$. Thus, the mapping $\phi : X/q \longrightarrow Im(\varphi)$ is a strongly extensional and surjective function.

(d) If $\phi(aq) =_Y \phi(bq)$, then $\varphi(a) =_Y \varphi(b)$. Let (u, v) be an arbitrary element of $Coker(\varphi)$. Then from $\varphi(u) \neq_Y \varphi(v)$ follows

$$\varphi(u) \neq_Y \varphi(a) \vee \varphi(a) \neq_Y \varphi(b) \vee \varphi(b) \neq_Y \varphi(v).$$

Since $\varphi(a) \neq_Y \varphi(b)$ is impossible, we conclude that above disjunction follows

$$\varphi(u) \neq_Y \varphi(a) \varphi(b) \neq_Y \varphi(v)$$

and $u \neq_X a$ or $b \neq_X v$. So, $(u, v) \neq_{X \times Y} (a, b)$. This means $(a, b) \bowtie Coker(\varphi)$. Therefore $aq =_1 bq$. Hence, the mapping ϕ is an injective function.

(e) Now, let be $aq \neq_1 bq$. Then $(a, b) \in Coker(\varphi)$, i.e. then $\varphi(a) \neq_Y \varphi(b)$.

Therefore, in this case, we have $\phi(aq) \neq_Y \phi(bq)$. So, the function ϕ is an embedding.

(f) The first, we wish to prove that the function ϕ is reverse isotone bijection. If $(\phi(aq), \phi(bq)) \in \beta$, i.e. if $(\varphi(a), \varphi(b)) \in \beta$ $(\subseteq \neq_Y)$, then $(a, b) \in \varphi^{-1}(\beta) \subseteq c(R)$ by the second part of the point (3) of this proof. Therefore, $(aq, bq) \in \gamma$. So, the bijection is reverse isotone.

The second, we wish to prove that the function ϕ is isotone bijection. Let $(aq, bq) \in \gamma$ $(\subseteq \neq_1)$, i.e. let $(a, b) \in c(R)$ $(\subseteq \alpha$). Since the function ϕ is an embedding, then $\phi(aq) \neq_Y \phi(bq)$. So, must be $(\phi(aq), \phi(bq)) \in \beta$ or $(\phi(bq), \phi(aq)) \in \beta$. Suppose that $(\phi(bq), \phi(aq)) \in \beta$, i.e. suppose that $(\varphi(b), \varphi(a)) \in \beta$ holds. Thus we conclude that $(b, a) \in \alpha$ because the function φ is reverse isotone mapping. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the case $(\phi(bq), \phi(aq)) \in \beta$ is impossible. Now, we have to have $(\phi(aq), \phi(bq)) \in \beta$. So, in the case that the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, the mapping ϕ is isotone.

At end of this conclusion we have that there exists strongly extensional and embedding reverse isotone bijection from $((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1,)$ onto $(Im(\varphi), =_Y, \neq_Y, \gamma)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then there exists the isomorphism $((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \cong (Im(\varphi), =_Y, \neq_Y,)$. $\square$

Note. Let $(X, =, \neq, \alpha)$, $(Y, =, \neq, \beta)$ be ordered sets under antiorders α and β respective, and let $\varphi : X \longrightarrow Y$ be a strongly extensional mapping from X into Y. Then, by point (1) in the proof of the Theorem 4, the relation induced there $\varphi^{-1}(\beta)$ is quasi-antiorder relation on X. Then:
(i) φ is isotone if $\alpha \subseteq \varphi^{-1}(\beta)$;
(ii) φ is reverse isotone if and only if $\varphi^{-1}(\beta) \subseteq \alpha$.

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Address

Department of Mathematics and Informatics, Banja Luka University, 2 Mladen Stojanovic Street, 78000 Banja Luka, Bosnia and Herzegovina
E-mail: `bato49@hotmail.com`