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ON Δ-LACUNARY STATISTICAL ASYMPTOTICALLY EQUIVALENT SEQUENCES

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Abstract

This paper presents new definitions which are a natural combination of the definition for asymptotically equivalence $\Delta-$lacunary statistically convergence. Using this definitions we have proved the $S_{\theta}^{L}(\Delta)$-asymptotically equivalence analogues theorems of [5] and [6].

1 Introduction

Let w be the set of all sequences of real or complex numbers and l_{∞}, c and c_0 be, respectively, the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\| = \sup\limits_{k} |x_k|$.

The idea of difference sequences was introduced by Kızmaz [1]. In 1981, Kızmaz defined the following sequence spaces

$$\begin{aligned} l_{\infty}(\Delta) &= \{x=(x_k) : \Delta x \in l_{\infty}\}, c(\Delta) = \{x=(x_k) : \Delta x \in c\} \\ c_0(\Delta) &= \{x=(x_k) : \Delta x \in c_0\} \end{aligned}$$

where $\Delta x = (\Delta x) = (x_k - x_{k+1})$ and showed that these are Banach spaces with norm $\|x\|_{\Delta} = |x_1| + \|\Delta x\|_{\infty}$.

We call these sequence spaces $\Delta-$bounded, $\Delta-$convergent and $\Delta-$null sequences.

Subsequently difference sequence spaces has been discussed in Çolak [2], Et and Başarır [3].

The idea of statistical convergence was introduced by Fast [8] and studied by various authors (see [9], [10], [11]). A sequence $x=(x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \mu(\{k \le n : |x_k - L| \ge \varepsilon\}) = 0$$

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where $\mu(\{k \le n : |x_k - L| \ge \varepsilon\})$ denotes the number of element belonging to $\{k \le n : |x_k - L| \ge \varepsilon\}$. In this case, we write $S - \lim x = L$ $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

In 1993, Marouf [12] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [13] extend these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Furthermore, asymptotically equivalent sequences has been studied in [14], [15], [16], [17], [18] and [19].

This paper presents new definitions which are a natural combination of the definition for asymptotically equivalence and $\Delta-$lacunary statistically convergence. In addition to these definitions, some connections between $\Delta-$lacunary statistical asymptotically equivalence and $\Delta-$lacunary asymptotically equivalence have also been presented.

2 Definitions and notations

Definition 1 *Two nonnegative sequences x, y are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$),[12].

Definition 2 *Two nonnegative sequences x, y are said to be statistical asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} \mu\left(\left\{k \le n : \left|\frac{x_k}{y_k} - L\right| \ge \varepsilon\right\}\right) = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply statistical asymptotically equivalent, if L=1, [13].

Definition 3 *A sequence $x = (x_n)$ is said to be Δ-statistically convergent to L if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} \mu\left(\{k \le n : |\Delta x_k - L| \ge \varepsilon\}\right) = 0.$$

We denote the set of these sequences by $S(\Delta)$,[4].

Definition 4 *Let θ be a lacunary sequence. A sequence $x = (x_n)$ is said to be $\Delta-$lacunary statistically convergent to L if for every $\varepsilon > 0$*

$$\lim_{r\to\infty}\frac{1}{h_r}\mu\left(\{k\in I_r : |\Delta x_k - L| \geq \varepsilon\}\right) = 0.$$

We denote the set of these sequences by $S_\theta(\Delta)$,[5].

Definition 5 *A sequence $x = (x_n)$ is said to be $\Delta-$Cesaro summable to L if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(\Delta x_k - L) = 0.$$

We denote the set of these sequences by $(\sigma_1)(\Delta)$, [5].

Definition 6 *A sequence $x = (x_n)$ is said to be strongly $\Delta-$Cesaro summable to L if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\Delta x_k - L| = 0.$$

We denote the set of these sequences by $|\sigma_1|(\Delta)$, [5].

Definition 7 *Let θ be a lacunary sequence. A sequence $x = (x_n)$ is said to be strongly $\Delta-$lacunary strongly convergent to L if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|\Delta x_k - L| = 0.$$

We denote the set of these sequences by $N_\theta(\Delta)$, [5].

Definition 8 *A sequence $x = (x_n)$ is said to be strongly $\Delta-$almost convergent to L if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\Delta x_{k+m} - L| = 0$$

uniformly in m. We denote the set of these sequences by $|AC|(\Delta)$, [5].

Following this definitions which are given above, we shall now introduce following new notions $\Delta-$asymptotically equivalence, $\Delta-$statistical asymptotically equivalent of multiple L, $\Delta-$lacunary statistical asymptotically equivalent of multiple L and $\Delta-$lacunary asymptotically equivalent of multiple L, $\Delta-$Cesaro asymptotically equivalent of multiple L, strongly $\Delta-$Cesaro asymptotically equivalent of multiple L, strongly $\Delta-$almost asymptotically equivalent of multiple L.

Definition 9 *Two nonnegative sequences x, y are said to be* $\Delta-$*asymptotically equivalent if*

$$\lim_k \frac{\Delta x_k}{\Delta y_k} = 1$$

(denoted by $x \overset{\Delta}{\sim} y$).

Definition 10 *Two nonnegative sequences x, y are* $\Delta-$*statistical asymptotically equivalent of multiple L provided that for every* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}\mu\left(\left\{k \leq n : \left|\frac{\Delta x_k}{\Delta y_k} - L\right| \geq \varepsilon\right\}\right) = 0$$

(denoted by $x \overset{S^L(\Delta)}{\sim} y$*) and simply* $\Delta-$*statistical asymptotically equivalent, if L=1.*

Definition 11 *Let* θ *be a lacunary sequence. Two nonnegative sequences x, y are* $\Delta-$*lacunary statistical asymptotically equivalent of multiple L provided that for every* $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r}\mu\left(\left\{k \in I_r : \left|\frac{\Delta x_k}{\Delta y_k} - L\right| \geq \varepsilon\right\}\right) = 0$$

(denoted by $x \overset{S_\theta^L(\Delta)}{\sim} y$*) and simply* $\Delta-$*lacunary statistical asymptotically equivalent, if L=1.*

Definition 12 *Let* θ *be a lacunary sequence. Two nonnegative sequences x, y are* $\Delta-$*lacunary strongly asymptotically equivalent of multiple L provided that*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left|\frac{\Delta x_k}{\Delta y_k} - L\right| = 0$$

(denoted by $x \overset{N_\theta^L(\Delta)}{\sim} y$*) and simply* $\Delta-$*lacunary strongly asymptotically equivalent, if L=1.*

Definition 13 *Two nonnegative sequences x, y are* $\Delta-$*Cesaro asymptotically equivalent of multiple L provided that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(\frac{\Delta x_k}{\Delta y_k} - L\right) = 0$$

(denoted by $x \overset{(\sigma_1)^L(\Delta)}{\sim} y$*) and simply* $\Delta-$*Cesaro asymptotically equivalent, if L=1.*

Definition 14 *Two nonnegative sequences x, y are $\Delta-$strongly Cesaro asymptotically equivalent of multiple L provided that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{\Delta x_k}{\Delta y_k}-L\right|=0$$

(denoted by $x \overset{|\sigma_1|^L(\Delta)}{\sim} y$) and simply $\Delta-$strongly Cesaro asymptotically equivalent, if L=1.

Definition 15 *Two nonnegative sequences x, y are $\Delta-$strongly almost asymptotically equivalent of multiple L provided that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{\Delta x_{k+m}}{\Delta y_{k+m}}-L\right|=0$$

uniformly in m (denoted by $x \overset{|AC|^L(\Delta)}{\sim} y$) and simply $\Delta-$strongly asymptotically equivalent, if L=1.

3 Main results

Theorem 1 *If x and y $\Delta-$bounded sequences are $\Delta-$statistical asymptotically equivalent of multiple L then they are Δ-Cesaro asymptotically equivalent of multiple L.*

Proof. Suppose x, y are in $l_\infty(\Delta)$ and $x \overset{S^L(\Delta)}{\sim} y$. Then we can assume that

$$\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\leq M$$

for almost all k. Given $\varepsilon>0$

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\Delta x_k}{\Delta y_k}-L\right)\right| &\leq \frac{1}{n}\sum_{k=1}^{n}\left|\frac{\Delta x_k}{\Delta y_k}-L\right| \\
&= \frac{1}{n}\sum_{\substack{k=1\\ \frac{\Delta x_k}{\Delta y_k}-L\ \geq\varepsilon}}^{n}\left|\frac{\Delta x_k}{\Delta y_k}-L\right|+\frac{1}{n}\sum_{\substack{k=1\\ \frac{\Delta x_k}{\Delta y_k}-L\ <\varepsilon}}^{n}\left|\frac{\Delta x_k}{\Delta y_k}-L\right| \\
&< \frac{1}{n}M\mu\left(\left\{k\leq n:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\geq\varepsilon\right\}\right)+\frac{1}{n}n\varepsilon.
\end{aligned}$$

Thus $x \overset{(\sigma_1)^L(\Delta)}{\sim} y$. ■

Theorem 2 *Let $\theta = \{k_r\}$ be a lacunary sequence with* $\liminf q_r > 1$ *then*

$$x \overset{S^L(\Delta)}{\sim} y \text{ implies } x \overset{S_\theta^L(\Delta)}{\sim} y.$$

Proof. Suppose first that $\liminf q_r > 1$ then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $x \overset{S^L(\Delta)}{\sim} y$ then for every $\varepsilon > 0$ and for sufficiently large r we have

$$\frac{1}{k_r}\mu\left(\left\{k \leq k_r : \left|\frac{\Delta x_k}{\Delta y_k} - L\right| \geq \varepsilon\right\}\right) \geq \frac{1}{k_r}\mu\left(\left\{k \in I_r : \left|\frac{\Delta x_k}{\Delta y_k} - L\right| \geq \varepsilon\right\}\right)$$
$$\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\mu\left(\left\{k \in I_r : \left|\frac{\Delta x_k}{\Delta y_k} - L\right| \geq \varepsilon\right\}\right).$$

This completes the proof. ■

Theorem 3 *Let $\theta = \{k_r\}$ be a lacunary sequence with* $\limsup q_r < \infty$ *then*

$$x \overset{S_\theta^L(\Delta)}{\sim} y \text{ implies } x \overset{S^L(\Delta)}{\sim} y.$$

Proof. If $\limsup q_r < \infty$ then there exists B>0 such that $q_r < B$ for all r. Let $x \overset{S_\theta^L(\Delta)}{\sim} y$ and $\varepsilon_1 > 0$. There exist R>0 and $\varepsilon > 0$ such that for every $j \geq R$

$$A_j = \frac{1}{h_j}\mu\left(\left\{k \in I_j : \left|\frac{\Delta x_k}{\Delta y_k} - L\right| \geq \varepsilon\right\}\right) < \varepsilon_1.$$

We can also find $K > 0$ such that $A_j < K$ for all j=1, 2, ... Now let n be any integer with $k_{r-1} < n < k_r$, where $r > R$. Then

$$\begin{aligned}
&\frac{1}{n}\mu\left(\left\{k\le n:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\le\frac{1}{k_{r-1}}\mu\left(\left\{k\le k_r:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
=\;&\frac{1}{k_{r-1}}\mu\left(\left\{k\in I_1:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
&+\frac{1}{k_{r-1}}\mu\left(\left\{k\in I_2:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
&+\ldots+\frac{1}{k_{r-1}}\mu\left(\left\{k\in I_r:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
=\;&\frac{k_1}{k_{r-1}k_1}\mu\left(\left\{k\in I_1:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
&+\frac{k_2-k_1}{k_{r-1}\left(k_2-k_1\right)}\mu\left(\left\{k\in I_2:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
&+\ldots+\frac{k_R-k_{R-1}}{k_{r-1}\left(k_R-k_{R-1}\right)}\mu\left(\left\{k\in I_R:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
&+\ldots+\frac{k_r-k_{r-1}}{k_{r-1}\left(k_r-k_{r-1}\right)}\mu\left(\left\{k\in I_r:\left|\frac{\Delta x_k}{\Delta y_k}-L\right|\ge\varepsilon\right\}\right)\\
=\;&\frac{k_1}{k_{r-1}}A_1+\frac{k_2-k_1}{k_{r-1}}A_2\\
&+\ldots+\frac{k_R-k_{R-1}}{k_{r-1}}A_R+\frac{k_{R+1}-k_R}{k_{r-1}}A_{R+1}\\
&+\ldots+\frac{k_r-k_{r-1}}{k_{r-1}}A_r\\
\le\;&\left\{\sup_{j\ge1}A_j\right\}\frac{k_R}{k_{r-1}}+\left\{\sup_{j\ge R}A_j\right\}\frac{k_r-k_R}{k_{r-1}}\\
\le\;&K\frac{k_R}{k_{r-1}}+\varepsilon_1 B.
\end{aligned}$$

Since $k_r\to\infty$ we have $\frac{k_R}{k_{r-1}}\to 0$. This conclude the proof. ■

Theorem 4 *Let $\theta=\{k_r\}$ be a lacunary sequence with $1<\liminf q_r\le\limsup q_r<\infty$ then*

$$x\overset{S_\theta^L(\Delta)}{\sim}y\iff x\overset{S^L(\Delta)}{\sim}y.$$

Proof. The result clearly follows from Theorem 2 and Theorem 3. ■

Theorem 5 *Let $\theta=\{k_r\}$ be a lacunary sequence then* *(i) If* $x \overset{N_\theta^L(\Delta)}{\sim} y$ *then* $x \overset{S_\theta^L(\Delta)}{\sim} y$

(ii) If x,y are $\Delta-$*bounded and* $x \overset{S_\theta^L(\Delta)}{\sim} y$ *then* $x \overset{N_\theta^L(\Delta)}{\sim} y$

(iii) Under the condition that x, y are $\Delta-$*bounded, we have the equivalence* $x \overset{S_\theta^L(\Delta)}{\sim} y \cap l_\infty(\Delta) = x \overset{N_\theta^L(\Delta)}{\sim} y \cap l_\infty(\Delta)$.

Proof. (i) If $\varepsilon > 0$ and $x \overset{N_\theta^L(\Delta)}{\sim} y$ then

$$\sum_{k\in I_r} \left| \frac{\Delta x_k}{\Delta y_k} - L \right| \geq \sum_{\substack{k\in I_r \\ \frac{\Delta x_k}{\Delta y_k} - L \geq \varepsilon}} \left| \frac{\Delta x_k}{\Delta y_k} - L \right| \geq \varepsilon \mu \left(\left\{ k \in I_r : \left| \frac{\Delta x_k}{\Delta y_k} - L \right| \geq \varepsilon \right\} \right).$$

Therefore $x \overset{S_\theta^L(\Delta)}{\sim} y$.

(ii) Suppose x,y are in $l_\infty(\Delta)$ and $x \overset{S_\theta^L(\Delta)}{\sim} y$.Then we can assume that

$$\left| \frac{\Delta x_k}{\Delta y_k} - L \right| \leq M$$

for almost all k. Given $\varepsilon > 0$

$$\begin{aligned} \frac{1}{h_r} \sum_{k\in I_r} \left| \frac{\Delta x_k}{\Delta y_k} - L \right| &= \frac{1}{h_r} \sum_{\substack{k\in I_r \\ \frac{\Delta x_k}{\Delta y_k} - L \geq \varepsilon}} \left| \frac{\Delta x_k}{\Delta y_k} - L \right| + \frac{1}{h_r} \sum_{\substack{k\in I_r \\ \frac{\Delta x_k}{\Delta y_k} - L < \varepsilon}} \left| \frac{\Delta x_k}{\Delta y_k} - L \right| \\ &\leq \frac{M}{h_r} \mu \left(\left\{ k \in I_r : \left| \frac{\Delta x_k}{\Delta y_k} - L \right| \geq \varepsilon \right\} \right) + \varepsilon. \end{aligned}$$

Therefore $x \overset{N_\theta^L(\Delta)}{\sim} y$.

(iii) This immediately follows from (i) and (ii).

In order to show that the converse of Theorem 5 (i) is not generally true, we now give the following example. ■

Example 1 *Take L=1, let* $\theta = \{k_r\}$ *be given and define* Δx_k *to be*

$$1, 2, \ldots, \left[\sqrt{h_r}\right];$$

for $k = k_{r-1}+1, k_{r-1}+2, ..., k_{r-1} + \left[\sqrt{h_r}\right]$; *and* $\Delta x_k = 1$ *otherwise (where [ ] denotes the greatest integer function) and* $\Delta y_k = 1$ *for all k. Note that x is not* $\Delta-$*bounded.*

Further, for $\varepsilon > 0$, *we have*

$$\frac{1}{h_r} \mu \left(\left\{ k \in I_r : \left| \frac{\Delta x_k}{\Delta y_k} - 1 \right| \geq \varepsilon \right\} \right) = \frac{\left[\sqrt{h_r}\right]}{h_r} \to 0 \text{ as } r \to \infty,$$

i.e., $x \overset{S_\theta(\Delta)}{\sim} y$. *On the other hand, since* I_r *is the union of the intervals* $[k_{r-1}+i, k_{r-1}+i+1]$ *for* $i = 0, 1, \ldots, \left[\sqrt{h_r}\right] - 1$ *we have*

$$\sum_{k \in I_r} \left| \frac{\Delta x_k}{\Delta y_k} - 1 \right| = 0 + 1 + 2 + \ldots + \left(\left[\sqrt{h_r} \right] - 1 \right) = \frac{\left[\sqrt{h_r} \right] \left(\left[\sqrt{h_r} \right] - 1 \right)}{2}.$$

Hence $x \overset{N_\theta(\Delta)}{\nsim} y$ *(x,y are not simply* $N_\theta(\Delta)-$*asymptotically equivalent).*

Note that any $\Delta-$bounded, $\Delta-$lacunary statistical asymptotically equivalent of multiple L sequences are $\Delta-$Cesaro lcunary asymptotically equivalent of multiple L.

Theorem 6 *For every lacunary* $\theta = \{k_r\}$

$$x \overset{|AC|^L(\Delta)}{\sim} y \text{ implies } x \overset{N_\theta^L(\Delta)}{\sim} y.$$

Proof. If $x \overset{|AC|^L(\Delta)}{\sim} y$ and $\varepsilon > 0$ there exist $N > 0$ and L such that

$$\frac{1}{n} \sum_{i=m+1}^{m+n} \left| \frac{\Delta x_i}{\Delta y_i} - L \right| < \varepsilon$$

for $n > N$, $m = 0, 1, 2, \ldots$.

Since θ is lacunary, we can choose $R > 0$ such that $r \geq R$ implies $h_r > N$ and consequently $\tau_r = \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{\Delta x_k}{\Delta y_k} - L \right| < \varepsilon$. Thus $x \overset{N_\theta^L(\Delta)}{\sim} y$. ∎

To show the converse of Theorem 6 is not generally true, we have to obtain x and y sequences that $x \overset{N_\theta^L(\Delta)}{\sim} y$ and $x \overset{|AC|^L(\Delta)}{\nsim} y$. Take $L = 1$ and define $\Delta x = (\Delta x_i)$ by

$$\Delta x_i = \begin{matrix} 2, & k_{r-1} < i \leq k_{r-1} + \left[\sqrt{h_r} \right] \\ 1, & \text{otherwise} \end{matrix}$$

and $\Delta y_i = 1$ for $i = 1, 2, \ldots$.

Then $x \overset{N_\theta(\Delta)}{\sim} y$ since

$$\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{\Delta x_i}{\Delta y_i} - 1 \right| = \frac{1}{h_r} \left[\sqrt{h_r} \right]$$

(where [ ] denotes the greatest integer function), which converges to 0 as $r \to \infty$.

Theorem 7 *Let* $\theta = \{k_r\}$ *be a lacunary sequence*

(i) $\liminf q_r > 1$ *then* $x \overset{|\sigma_1|^L(\Delta)}{\sim} y$ *implies* $x \overset{N_\theta^L(\Delta)}{\sim} y$

(ii) $\limsup q_r < \infty$ *then* $x \overset{N_\theta^L(\Delta)}{\sim} y$ *implies* $x \overset{|\sigma_1|^L(\Delta)}{\sim} y$

(iii) $1 < \liminf q_r \leq \limsup q_r < \infty$ *then* $x \overset{N_\theta^L(\Delta)}{\sim} y \Leftrightarrow x \overset{|\sigma_1|^L(\Delta)}{\sim} y$.

Proof. (i) Suppose $\liminf q_r > 1$. There exists $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r. We have

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

Now write

$$\begin{aligned}
\frac{1}{k_r}\sum_{k=1}^{k_r}\left|\frac{\Delta x_k}{\Delta y_k}-L\right| &\geq \frac{1}{k_r}\sum_{k\in I_r}\left|\frac{\Delta x_k}{\Delta y_k}-L\right| = \frac{h_r}{k_r}\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{\Delta x_k}{\Delta y_k}-L\right| \\
&\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{\Delta x_k}{\Delta y_k}-L\right|
\end{aligned}$$

from which we deduce that $x \overset{|\sigma_1|^L(\Delta)}{\sim} y$ implies $x \overset{N_\theta^L(\Delta)}{\sim} y$.
This completes the proof.

(ii) If $\limsup q_r < \infty$ then there exists $M > 0$ such that $q_r < M$ for all r. Let $\varepsilon > 0$, $x \overset{N_\theta^L(\Delta)}{\sim} y$ and $\tau_i = \frac{1}{h_i}\sum\limits_{k\in I_i}\left|\frac{\Delta x_k}{\Delta y_k}-L\right|$. We can then find $R > 0$ and $K > 0$ such that $\sup\limits_{i\geq R}\tau_i < \varepsilon$ and $\tau_i < K$ for all $i = 1, 2, \ldots$. Then if t is any integer with $k_{r-1} < t \leq k_r$, where $r > R$, we can write

$$\begin{aligned}
&\frac{1}{t}\sum_{i=1}^{t}\left|\frac{\Delta x_i}{\Delta y_i}-L\right| \leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r}\left|\frac{\Delta x_i}{\Delta y_i}-L\right| \\
= \;& \frac{1}{k_{r-1}}\left(\sum_{i\in I_1}\left|\frac{\Delta x_i}{\Delta y_i}-L\right| + \sum_{i\in I_2}\left|\frac{\Delta x_i}{\Delta y_i}-L\right| + \ldots + \sum_{i\in I_r}\left|\frac{\Delta x_i}{\Delta y_i}-L\right|\right) \\
= \;& \frac{k_1}{k_{r-1}}\tau_1 + \frac{k_2-k_1}{k_{r-1}}\tau_2 \\
& + \ldots + \frac{k_R-k_{R-1}}{k_{r-1}}\tau_R + \frac{k_{R+1}-k_R}{k_{r-1}}\tau_{R+1} \\
& + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}}\tau_r \\
\leq \;& \left\{\sup_{i\geq 1}\tau_i\right\}\frac{k_R}{k_{r-1}} + \left\{\sup_{i\geq R}\tau_i\right\}\frac{k_r-k_R}{k_{r-1}} \\
< \;& K\frac{k_R}{k_{r-1}} + \varepsilon M.
\end{aligned}$$

Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that $\frac{1}{t}\sum\limits_{i=1}^{t}\left|\frac{\Delta x_i}{\Delta y_i}-L\right| \to 0$ and consequently $x \overset{|\sigma_1|^L(\Delta)}{\sim} y$.

(iii) The result clearly follows from (i) and (ii). ■

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