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π-NORMAL TOPOLOGICAL SPACES

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Abstract

A topological space X is called π-*normal* if for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. We will present some characterizations of π-normality and some examples to show relations between π-normality and other weaker version of normality such as mild normality, almost normality, and quasi-normality.

We investigate in this paper a weaker version of normality called π-normality. We will prove that π-normality is a property which lies between almost normality and normality. We will present some characterizations of π-normality and some examples to show relations between π-normality and other weaker versions of normality such as mild normality, almost normality, and quasi-normality.

We will denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space and a Tychonoff space is a T_1 completely regular space. The interior of a set A will be denoted by $\mathrm{int}A$, and the closure of a set A will be denoted by $\overline{A}$.

1 Definition:

A subset A of a topological space X is called *regularly closed* (called also, *closed domain*) if $A = \overline{\mathrm{int}\, A}$. A subset A is called *regularly open* (called also, *open domain*) if $A = \mathrm{int}(\,\overline{A}\,)$. A finite union of regular open sets is called π-*open* set and a finite intersection of regular closed sets is called π-*closed* set. Two

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subsets A and B in a topological space X are said to be *separated* if there exist two disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.

Observe that an intersection of two regularly closed sets need not be regularly closed, for example, in $\mathbb{R}$ with the usual metric topology, any closed interval is a regularly closed set but $[0,1] \cap [1,2] = \{1\}$ and the singleton $\{1\}$ is not regularly closed. A dual argument holds for regularly open sets. So, we always have

$$\text{regularly closed} \implies \pi\text{-closed} \implies \text{closed}.$$

Observe that the complement of a π-open set is π-closed and the complement of a π-closed set is π-open, the finite union (intersection) of π-closed sets is π-closed, but the infinite union (intersection) of π-closed sets need not be π-closed. An example for the union is the rationals $\mathbb{Q}$ in $\mathbb{R}$ with its usual topology as $\{q\}$ is π-closed for each $q \in \mathbb{Q}$, but $\mathbb{Q}$ is not closed hence not π-closed. An example for the intersection is the singleton $\{\omega_1\}$ in the ordinal space $\omega_1 + 1$. Observe that for each $\alpha < \omega_1$ we have $(\alpha, \omega_1]$ is a clopen subset of $\omega_1 + 1$, so $(\alpha, \omega_1]$ is regularly closed in $\omega_1 + 1$ for each $\alpha < \omega_1$. Now $\bigcap_{\alpha<\omega_1} (\alpha, \omega_1] = \{\omega_1\}$, which is not π-closed. To see this, suppose, without loss of generality, $\{\omega_1\} = F \cap E$, where $F = \overline{\text{int}\, F}$ and $E = \overline{\text{int}\, E}$ are regularly closed in ω_1+1. Let $A = F \setminus \{\omega_1\}$ and $B = E \setminus \{\omega_1\}$, then A and B are closed sets in ω_1, because if $\alpha \in \omega_1 \setminus A$, then $\alpha \in \omega_1 + 1 \setminus F$. Since F is closed in ω_1+1, so there exists an $\alpha_1 < \alpha$ such that $(\alpha_1, \alpha] \cap F = \emptyset$, then $(\alpha_1, \alpha] \cap A = \emptyset$. Thus $(\alpha_1, \alpha] \subseteq \omega_1 \setminus A$. Hence $\omega_1 \setminus A$ is an open set in ω_1 and so A is closed in ω_1. Similarly B is closed in ω_1. Now A and B are also unbounded in ω_1, because if $\alpha < \omega_1$ then $(\alpha, \omega_1] \cap \text{int}\, F \neq \emptyset$,so there exists a $\beta \in \text{int}\, F$ such that $\alpha < \beta$. Thus $\beta \in F$ such that $\alpha < \beta$, hence $\beta \in A$ such that $\alpha < \beta$. Thus A is unbounded, similarly B is unbounded. We have now A and B are both closed and unbounded in ω_1, then A and B are clubs in ω_1, so $A \cap B$ is club in ω_1. Thus there is a $\gamma \in A \cap B$ such that $\gamma < \omega_1$, this implies that $\gamma \neq \omega_1$ and $\gamma \in F \cap E$ a contradiction. Therefore $\{\omega_1\}$ is a closed set in $\omega_1 + 1$ which is not π-closed.

2 Definition: (Singal and Singal)
A topological space X is called *mildly normal* if for any two disjoint regularly closed subsets A and B of X, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. i.e., any two disjoint regularly closed subsets are separated.

Shchepin introduced the same notion in the class of regular spaces, see [6], he called it κ-normality. It is clear from the definitions that any normal space is mildly normal. The converse is not always true. The space $\omega_1 \times \omega_1 + 1$ is mildly normal, see [2] and [3], but not normal.

3 Definition: (Singal and Arya)
A topological space X is called *almost normal* if for any two disjoint closed subsets A and B of X, one of which is regularly closed, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

It is clear from the definitions that any normal space is almost normal and any almost normal space is mildly normal. The converse is not always true. The space $\omega_1 \times \omega_1 + 1$ is mildly normal, but not almost normal because the closed subset $A = \omega_1 \times \{\omega_1\}$ is disjoint from the regularly closed subset $B = \{\langle \alpha, \alpha \rangle : \alpha < \omega_1\}$ and they cannot be separated by two disjoint open subsets, see [1]. We do not assume regularity in the definition of almost normality, so any finite complement topology on an infinite set is an example of an almost normal space which is not normal. There is a Hausdorff space which is almost normal but not regular, see [4]. And there is a Tychonoff space which is almost normal but not normal, see [4].

4 Definition:
A topological space X is called *quasi-normal* if any two disjoint π-closed subsets A and B of X there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Zaitsev introduced the above notion in the class of regular spaces, see [10]. It is clear that any normal space is quasi-normal and any quasi-normal space is mildly normal. The converse is not always true. Shchepin, see [6], illustrated an example of a mildly normal space which is not quasi-normal and also he proved that the Niemytzki plane is quasi-normal.

5 Definition:
A topological space X is called *π-normal* if for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

We do not assume regularity or any separation axiom in the definition of π-normality. The following implications are clear:

$$\text{normal} \Longrightarrow \pi\text{-normal} \Longrightarrow \text{almost normal} \Longrightarrow \text{mildly normal.}$$

$$\text{normal} \Longrightarrow \pi\text{-normal} \Longrightarrow \text{quasi-normal} \Longrightarrow \text{mildly normal.}$$

We will study some of the converse of the above implications. First, consider the finite complement topology on any infinite set X. This space is T_1 but not T_2 and any regularly closed set is either the empty set or the whole space X, as any infinite subset is dense in this space. Thus it is π-normal but not normal. Now, we will present a quasi-normal space which is not almost normal, hence not π-normal.

6 Example:
Let $P = \{\langle x, y\rangle : x, y \in \mathbb{R}, y > 0\}$ be the open upper half-plane and let $L = \{\langle x, 0\rangle : x \in \mathbb{R}\}$ be the x-axis. Let $X = P \cup L$. Let τ be the topology on X generated by the following neighborhood system: For any $z \in P$, a basic open neighborhood of z is any basic open disc around z contained in P with its usual Euclidean topology. For $z \in L$, a basic open neighborhood of z is of the form $\{z\} \cup (P \cap D)$, where D is any open disc around z in the plane $\mathbb{R}^2$ with its usual Euclidean topology. This topological space (X, τ) is called the *Half-Disc* topological space. Observe that P as a subspace of X is the same as the usual Euclidean space. This space is not regular. For more information about this space, see [9].

It is clear that $\mathbb{Q}^2 \cap P$ is countable and dense in X. Thus X is separable. Also, L is a closed uncountable discrete subset of X. Thus, by Jones' Lemma, X is not normal. Now, let $p : X \longrightarrow L$ be the usual projection function from $X \subset \mathbb{R}^2$ onto the x-axis L.

Claim 1: Let E and F be any two closed disjoint subsets of X. Then E and F are unseparated if and only if either (there exists $\langle x, 0\rangle \in E$ such that $x \in \overline{p(F \cap L)}$) or (there exists $\langle x, 0\rangle \in F$ such that $x \in \overline{p(E \cap L)}$), where the closure is taken in the x-axis L with its usual Euclidean topology.

Proof of Claim 1: If the condition holds, the it is clear that E and F are unseparated. For the converse, suppose that E and F are unseparated but the condition does not hold. Then we would have that (for all $\langle x, 0\rangle \in E$, we have $x \notin \overline{p(F \cap L)}$) and (for all $\langle x, 0\rangle \in F$, we have $x \notin \overline{p(E \cap L)}$). Since $E \cap P$ and $F \cap P$ are closed disjoint subsets of P with its usual Euclidean topology, then there are two open sets U_1 and V_1, open in P and hence open in X, such that $(E \cap P) \subseteq U_1$, $(F \cap P) \subseteq V_1$ and $\overline{U_1}^X \cap \overline{V_1}^X = \emptyset$. Now, for all $\langle x, 0\rangle \in E$ there exists an $\epsilon_x > 0$ such that the open disc D_{ϵ_x} around $\langle x, 0\rangle$ of radius ϵ_x is disjoint from $\overline{V_1}$ and there exists a $\delta_x > 0$ such that the open interval $(x - \delta_x, x + \delta_x)$ is disjoint from $p(F \cap L)$. Let $r_x = \min\{\frac{\epsilon_x}{2}, \frac{\delta_x}{2}\}$ and let D_x be the open disc around $\langle x,0\rangle$ of radius r_x and let $U_x = \{\langle x,0\rangle\} \cup (P \cap D_x)$. Similarly, for all $\langle x,0\rangle \in F$ there exists an $\epsilon_x > 0$ such that the open disc D_{ϵ_x} around $\langle x,0\rangle$ of radius ϵ_x is disjoint from $\overline{U_1}$ and there exists a $\delta_x > 0$ such that the open interval $(x-\delta_x, x+\delta_x)$ is disjoint from $p(E \cap L)$. Let $r_x = \min\{\frac{\epsilon_x}{2}, \frac{\delta_x}{2}\}$ and let D_x be the open disc around $\langle x,0\rangle$ of radius r_x and let $V_x = \{\langle x,0\rangle\} \cup (P \cap D_x)$. Define

$$U = U_1 \cup (\cup_{\langle x,0\rangle \in E} U_x) \text{ and } V = V_1 \cup (\cup_{\langle x,0\rangle \in F} V_x).$$

Then $E \subseteq U$ and $F \subseteq V$ where U and V are open in X and disjoint. Thus E and F are separated which is a contradiction. Thus Claim 1 is proved.

Now, suppose that X is not quasi-normal. Then there would be two disjoint unseparated π-closed sets $E = \cap_{i=1}^{n} \overline{\text{int}E_i}$ and $F = \cap_{i=1}^{m} \overline{\text{int}F_i}$. By Claim 1, we may assume, without loss of generality, that there exists an element $\langle x,0\rangle \in E$ such that $x \in \overline{p(F \cap L)}$. For an $\epsilon > 0$ and $z \in \mathbb{R}^2$, let us denote the open disc centered at z of radius ϵ by $B(z;\epsilon)$. Observe that for each $k \in \mathbb{N}$ there exists $y_k \in (p(F \cap L)) \cap (x - \frac{1}{k}, x + \frac{1}{k})$. Thus for each $k \in \mathbb{N}$ and for each $i \in \{1, ..., m\}$ we have that $\langle y_k, 0\rangle \in \overline{\text{int}F_i}$ and in L, the x-axis, we have $y_k \longrightarrow x$. Thus, for each $K \in \mathbb{N}$ and for each $i \in \{1, ..., m\}$ pick

$$z_k^i \in ((B(\langle y_k,0\rangle; \frac{1}{k}) \cap P) \cup \{\langle y_k,0\rangle\}) \bigcap \text{int}F_i.$$

Now, let $\epsilon > 0$ be arbitrary. Pick $j \in \mathbb{N}$ so that $y_k \in (x-\epsilon, x+\epsilon)$ for each $k \geq j$ and $B(\langle y_k,0\rangle; \frac{1}{k}) \subseteq B(\langle x,0\rangle;\epsilon)$ for each $k \geq j$. Then

$$z_k^i \in ((B(\langle y_k,0\rangle; \frac{1}{k}) \cap P) \cup \{\langle y_k,0\rangle\}) \cap \text{int}F_i \subseteq (B(\langle x,0\rangle;\epsilon) \cap P).$$

That is

$$((B(\langle x,0\rangle;\epsilon) \cap P) \cup \{\langle x,0\rangle\}) \cap \text{int}F_i \neq \emptyset \text{ for each } i \in \{1, ..., m\}.$$

Since ϵ was arbitrary, we conclude that $\langle x,0\rangle \in \overline{\text{int}F_i}$ for each $i \in \{1, ..., m\}$. Thus $\langle x,0\rangle \in \cap_{i=1}^{m} \overline{\text{int}F_i} = F$. Thus $E \cap F \neq \emptyset$ which is a contradiction. Therefore, X is quasi-normal.

Now, let $U = \{\langle x,y\rangle : x^2 + (y-1)^2 < 1\}$ be points of the interior of the circle of radius 1 centered at $\langle 0,1\rangle$. Let $E = \overline{U} = \{\langle x,u\rangle : x^2 + (y-1)^2 \leq 1\}$, then E is a regularly closed set being a closure of an open set. Observe that $\langle 0,0\rangle \in \overline{U} = E$. Let $F = \{\langle \frac{1}{n}, 0\rangle : n \in \mathbb{N}\}$. Then F is closed. By Claim

1, E and F are unseparated. Thus X is not almost normal. Hence X is not π-normal. ■

Let us now give some characterization of π-normality.

7 Theorem:
For a space X, the following are equivalent:

1. X is π-normal
2. For every π-closed set A and every open set B with $A \subseteq B$, there exists an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B$.
3. For every closed set A and every π-open set B with $A \subseteq B$, there exists an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq B$.
4. For every pair consisting of disjoint sets A and B, one of which is π-closed and the other is closed, there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Proof:
(1) $\Longrightarrow$ (2) Assume (1). Let A be any π-closed set and B be any open set such that $A \subseteq B$. Then $A \cap (X \setminus B) = \emptyset$, where $X \setminus B$ is closed. Then there exist open disjoint sets U and V such that $A \subset U$ and $(X \setminus B) \subseteq V$. Since $U \cap V = \emptyset$, then $\overline{U} \cap V = \emptyset$. Thus $\overline{U} \subseteq X \setminus V \subseteq X \setminus (X \setminus B) = B$. Therefore, $A \subseteq U \subseteq \overline{U} \subseteq B$.

(2) $\Longrightarrow$ (3) Assume (2). Let A be any closed set and B be any π-open set such that $A \subseteq B$. Then $X \setminus B \subseteq X \setminus A$ where $X \setminus B$ is π-closed and $X \setminus A$ is open. Thus, by (2), there exists open W such that $X \setminus B \subseteq W \subseteq \overline{W} \subseteq X \setminus A$. Thus $A \subseteq X \setminus \overline{W} \subseteq X \setminus W \subseteq B$. So, we let $U = X \setminus \overline{W}$ which is open and since $W \subseteq \overline{W}$, then $X \setminus \overline{W} \subseteq X \setminus W$. Thus $U \subseteq X \setminus W$, hence $\overline{U} \subseteq \overline{X \setminus W} = X \setminus W \subseteq B$.

(3) $\Longrightarrow$ (4) Assume (3). Let A be any closed set and B be any π-closed set with $A \cap B = \emptyset$. Then $A \subseteq X \setminus B$ where $X \setminus B$ is π-open. By (3), there exists an open U such that $A \subseteq U \subseteq \overline{U} \subseteq X \setminus B$. Now, $\overline{U}$ is closed. Applying (3) again we get an open W such that $A \subseteq U \subseteq \overline{U} \subseteq W \subseteq \overline{W} \subseteq X \setminus B$. Let $V = X \setminus \overline{W}$, then V is open and $B \subseteq V$. We have $X \setminus \overline{W} \subseteq X \setminus W$, hence $V \subseteq X \setminus W$, thus $\overline{V} \subseteq \overline{X \setminus W} = X \setminus W$. So, we have $\overline{U} \subseteq W$ and $\overline{V} \subseteq X \setminus W$. Therfore $\overline{U} \cap \overline{V} = \emptyset$.

(4) $\Longrightarrow$ (1) is clear. ■

Using Theorem 7, it is easy to show the following theorem which is a Urysohn's Lemma version for π-normality. A proof can be established by a similar way of the normal case.

8 Theorem:
A space X is π-normal if and only if for every pair of disjoint closed sets A and B, one of which is π-closed, there exists a continuous function f on X into $[0, 1]$, with its usual topology, such that $f(A) = \{0\}$ and $f(B) = \{1\}$. ■

It is easy to see that the inverse image of a regularly closed set under an open continuous function is regularly closed and the inverse image of a π-closed set under an open continuous function is π-closed. We will use that in the next theorem.

9 Theorem:
Let X be a π-normal space and $f : X \longrightarrow Y$ be an open continuous injective function. Then $f(X)$ is a π-normal space.

Proof: Let A be any π-closed subset in $f(X)$ and let B be any closed subset in $f(X)$ such that $A \cap B = \emptyset$. Then $f^{-1}(A)$ is a π-closed set in X which is disjoint from the closed set $f^{-1}(B)$. Since X is π-normal, there are two disjoint open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is 1-1 and open, result follows. ■

10 Corollary:
π-normality is a topological property. ■

π-normality is not productive, for example $\omega_1 \times \omega_1 + 1$. It is also not inherited, for example $(\omega_1 + 1)^2$, or any compactification of a Tychonoff non-π-normal space. But it is clearly additive.

Recall that a space X is called *extremally disconnected space* if the closure of any open set in X is open. It is easy to see that any π-open subset of an extremally disconnected space is an open domain. Thus any π-closed subset of an extremally disconnected space is a closed domain. The proof of the following theorem is straight.

11 Theorem:
Any extremally disconnected space is π-normal space.

Any Tychonoff extremally disconnected non-normal space is an example of a Tychonoff π-normal non-normal space. There are finite spaces which are almost normal but not π-normal. We still do not know if there exists an infinite Tychonoff space which is almost normal but not π-normal, even a consistent example.

Shchepin, see [6], used the notion of κ-metrizablity to show that the Sorgenfrey line square is κ-normal. We still do not know the following:

Open Problem:

1. Is the Sorgenfrey line square almost normal? π-normal?
2. Is the Niemytzki plane almost normal? π-normal?

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