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ON MIXED AND COMPONENTWISE CONDITION NUMBERS FOR HYPERBOLIC QR FACTORIZATION

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Abstract

We present normwise and componentwise perturbation bounds for the hyperbolic QR factorization by using a new approach. The explicit expressions of mixed and componentwise condition numbers for the hyperbolic QR factorization are derived.

1 Introduction

The indefinite least squares problem (ILS) has the form

$$\text{ILS}: \quad \min_x (b - Ax)^T J(b - Ax), \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ are given and J is the signature matrix

$$J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p + q = m. \tag{1.2}$$

This problem was introduced by Chandrasekaran, Gu and Sayed [3] and further studied by Bojanczyk, Higham and Patel [1]. The theory and algorithms for the equality constrained indefinite least squares problem are presented in [2].

A matrix $Q \in \mathbb{R}^{m\times m}$ is J-orthogonal if

$$Q^T JQ = J. \tag{1.3}$$

Clearly, Q is nonsingular and $QJQ^T = J$. For properties of J-orthogonal matrices see [8].

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Consider the downdating problem of computing the Cholesky factorization of a positive definite matrix $C = A^T JA = A_1^T A_1 - A_2^T A_2$, where $A_1 \in \mathbb{R}^{p\times n}$ ($p \geq n$) and $A_2 \in \mathbb{R}^{q\times n}$. If there exists a J-orthogonal matrix Q such that

$$Q^T \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{1.4}$$

with $R \in \mathbb{R}^{n\times n}$ upper triangular, then

$$C = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T J \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^T QJQ^T \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = R^T R,$$

so R is the desired Cholesky factor. The factorization (1.4) is a hyperbolic QR factorization; for details of how to compute it see, for example, [1].

Note that $Q^{-1} = JQ^T J$, let $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$. Then $Q^{-1} = \begin{bmatrix} Q_{11}^T & -Q_{21}^T \\ -Q_{12}^T & Q_{22}^T \end{bmatrix}$. From (1.4), the hyperbolic QR factorization can be rewritten as

$$A = Q_1 R = \begin{matrix} \\ p \\ q \end{matrix} \begin{matrix} n \\ \begin{bmatrix} Q_{1*} \\ -Q_{2*} \end{bmatrix} \end{matrix} R, \qquad R \in \mathbb{R}^{n\times n}. \tag{1.5}$$

This factorization yields

$$A^T JA = R^T \begin{bmatrix} Q_{1*} \\ -Q_{2*} \end{bmatrix}^T J \begin{bmatrix} Q_{1*} \\ -Q_{2*} \end{bmatrix} R = R^T (Q_{1*}^T Q_{1*} - Q_{2*}^T Q_{2*}) R = R^T R.$$

Let $\tilde{A} = A + \Delta A$ be a perturbation of A. We assume that $\tilde{A}$ satisfies the uniqueness condition $\tilde{A}^T J\tilde{A}$ is positive definite, which will always be the case for ΔA sufficiently small in norm. Then $\tilde{A}$ also has the unique hyperbolic QR factorization:

$$A + \Delta A = (Q_1 + \Delta Q_1)^T (R + \Delta R), \tag{1.6}$$

where $Q_1 + \Delta Q_1$ is the first n columns of J-orthogonal matrix $Q + \Delta Q$.

In this paper, using a new approach (i.e., the columns of a new matrix is given by choosing appropriate columns from two Kronecker product matrices), we derive the explicit perturbation expressions. Secondly, using the mixed and componentwise condition numbers defined in [5], the mixed and componentwise perturbation bounds for the hyperbolic QR factorization are given.

Throughout this paper, we use $\mathbb{R}^{m\times n}$ to denote the set of real $m\times n$ matrices, A^T denotes the transpose of the matrix A, I stands for the identity matrix, and 0 the null matrix. The symbol $\|\cdot\|_F$ stands for the Frobenius norm, and $\|\cdot\|_2$ the spectral norm and the Euclidean vector norm. For $A = [a_1, a_2, \ldots, a_n] = (a_{ij}) \in \mathbb{R}^{m\times n}$ and a matrix B, $A \otimes B = (a_{ij}B)$ is a Kronecker product, and $\mathrm{vec}(A)$ is a vector defined by $\mathrm{vec}(A) = [a_1^T, a_2^T, \ldots, a_n^T]^T$ (see [6, 10] for properties of the Kronecker product and vec operation).

2 Preliminaries

To define mixed and componentwise condition numbers, the following form of "distance" function will be useful. For any points $a, b \in \mathbb{R}^n$, we define $\frac{a}{b} = (c_1, c_2, \ldots, c_n)^T$ with

$$c_i = \begin{cases} a_i/b_i, & \text{if } b_i \neq 0, \\ 0, & \text{if } a_i = b_i = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then we define the componentwise relative "distance" between a and b by

$$d(a,b) = \left\| \frac{a-b}{b} \right\|_\infty = \max_{i=1,2,\ldots,n} \left\{ \frac{|a_i - b_i|}{|b_i|} \right\}.$$

Note that if $d(a,b) < \infty$,

$$d(a,b) = \min\{v \geq 0 \mid |a_i - b_i| \leq v|b_i|, \quad \text{for } i = 1,2,\ldots,n\}.$$

The distance of two matrices is defined as

$$d(A,B) = d(\text{vec}(A), \text{vec}(B)).$$

It is easy to know that $\|\text{vec}(A)\|_\infty = \|A\|_{\max}$, where $\|\cdot\|_{\max}$ is the max norm given by

$$\|A\|_{\max} = \max_{i,j} |a_{ij}|.$$

We need the definition 2.1 below given in [5].

For $\varepsilon > 0$ we denote $B^0(a,\varepsilon) = \{x \mid d(x,a) \leq \varepsilon\}$. For a partial function $F: \mathbb{R}^p \to \mathbb{R}^q$, we denote by $\text{Dom}(F)$ the domain of definition of F.

Definition 2.1 *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set* $\text{Dom}(F) \subset \mathbb{R}^p$ *such that* $0 \notin \text{Dom}(F)$. *Let* $a \in \text{Dom}(F)$ *such that* $F(a) \neq 0$.

(i) The mixed condition number of F at a is defined by

$$m(F,a) = \lim_{\varepsilon \to 0} \sup_{\substack{x \in B^0(a,\varepsilon) \\ x \neq a}} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} \frac{1}{d(x,a)}.$$

(ii) Suppose that $F(a) = (f_1(a), f_2(a), \cdots, f_q(a))$ is such that $f_j(a) \neq 0$ for $j = 1,2,\ldots,q$. Then the componentwise condition number of F at a is

$$c(F,a) = \lim_{\varepsilon \to 0} \sup_{\substack{x \in B^0(a,\varepsilon) \\ x \neq a}} \frac{d(F(x), F(a))}{d(x,a)}.$$

The explicit expressions of the mixed and componentwise condition numbers of F at a are given by the following lemma.

Lemma 2.2 [5] *Suppose F is Fréchet differentiable at a. Then,*

(a) If $F(a) \neq 0$, then

$$m(F,a) = \frac{\|F'(a)\mathrm{Dg}(a)\|_\infty}{\|F(a)\|_\infty} = \frac{\||F'(a)|\ |a|\|_\infty}{\|F(a)\|_\infty}.$$

(b) If $(F(a))_i \neq 0$ for $i = 1, 2, \ldots, q$, then

$$c(F,a) = \|\mathrm{Dg}^{-1}(F(a))F'(a)\mathrm{Dg}(a)\|_\infty = \left\| \frac{|F'(a)|\ |a|}{|F(a)|} \right\|_\infty,$$

where $\mathrm{Dg}(a)$ is the $p\times p$ diagonal matrix with $a_1, a_2, \cdots, a_p$ in the diagonal.

Remark 2.3 *In the rest of this paper we assume when we deal with componentwise condition numbers, the computed solution has no zero components.*

3 Condition numbers for hyperbolic QR factorization

The mappings are defined as follows

$$\varphi_R : \ \mathrm{vec}(A) \to \mathrm{vec}(R),$$
$$\varphi_{Q_1} : \ \mathrm{vec}(A) \to \mathrm{vec}(Q_1),$$

where Q_1 and R are the hyperbolic QR factors of A.

3.1 The factor R

From (1.6), we have

$$(A + \Delta A)^T J(A + \Delta A) = (R + \Delta R)^T (R + \Delta R), \tag{3.1}$$

omitting the second-order term, which turns to

$$R^T(\Delta R) + (\Delta R)^T R \approx A^T J(\Delta A) + (\Delta A)^T JA. \tag{3.2}$$

Using the vec function, we have

$$(I \otimes R^T)\mathrm{vec}(\Delta R) + (R^T \otimes I)\mathrm{vec}((\Delta R)^T) \tag{3.3}$$
$$\approx (I \otimes (A^T J))\mathrm{vec}(\Delta A) + ((A^T J) \otimes I)\mathrm{vec}((\Delta A)^T). \tag{3.4}$$

Let $A \in \mathbb{R}^{m\times n}$. Then we have (see [9])

$$\mathrm{vec}((\Delta A)^T) = \Pi\ \mathrm{vec}(\Delta A),$$

where the ver-permutation matrix Π is expressed by

$$\Pi = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T,$$

where each $E_{ij} \in \mathbb{R}^{m\times n}$ has entry "1" in position (i,j) and all other entries are zero.

From (3.3), we have

$$(I\otimes R^T)\mathrm{vec}(\Delta R)+(R^T\otimes I)\mathrm{vec}((\Delta R)^T) \approx [(I\otimes(A^TJ))+((A^TJ)\otimes I)\Pi]\mathrm{vec}(\Delta A). \tag{3.5}$$

Choose

$$D_1 = \mathrm{diag}(\underbrace{\frac{1}{2},0,\dots,0}_{n},\underbrace{1,\frac{1}{2},0,\dots,0}_{n},\dots,\underbrace{1,1,\dots,\frac{1}{2}}_{n}),$$

where each element "1" of D_1 corresponds to the nonzero element of $\mathrm{vec}(\bar{R})$ (i.e., the strictly upper triangular part of R). Similarly, we choose

$$D_2 = \mathrm{diag}(\underbrace{\frac{1}{2},1,\dots,1}_{n},\underbrace{0,\frac{1}{2},1,\dots,1}_{n},\dots,\underbrace{0,0,\dots,\frac{1}{2}}_{n}),$$

where each element "1" of D_2 corresponds to the nonzero element of $\mathrm{vec}(\bar{R}^T)$ (i.e., the strictly lower triangular part of R^T). "$\frac{1}{2}$" corresponds to the each diagonal element of R.

For any matrices S and T, $SD_1 + TD_2$ is consisting of columns of S and T corresponding to the nonzero elements of D_1 and D_2. Let $n^2 \times n^2$ matrices

$$S = [s_{11},\cdots,s_{n1},\ s_{12},\cdots,s_{n2},\cdots\ s_{n1},\cdots,s_{nn}]$$

and

$$T = [t_{11},\cdots,t_{n1},\ t_{12},\cdots,t_{n2},\cdots,t_{n1},\cdots,t_{nn}],$$

where s_{ij} and t_{ij} are the $((j-1)n+i)$-th column of S and T, respectively. We have

$$S\cdot\mathrm{vec}(\Delta R)+T\cdot\mathrm{vec}(\Delta R^T) = \sum_{i,j} s_{ij}(\delta r_{ij})+\sum_{i,j} t_{ij}(\delta r_{ji}) = \sum_{i,j}(s_{ij}(\delta r_{ij})+t_{ij}(\delta r_{ji})),$$

where δr_{ij} is the element of $\mathrm{vec}(\Delta R)$. Note that ΔR is a upper triangular matrix, i.e., $\delta r_{ij} = 0$, for $i > j$. Thus we obtain

$$s_{ij}(\delta r_{ij}) + t_{ij}(\delta r_{ji}) = \begin{cases} t_{ij}(\delta r_{ij} + \delta r_{ji}), & i > j, \\ s_{ij}(\delta r_{ij} + \delta r_{ji}), & i < j, \\ \frac{1}{2}(s_{ii} + t_{ii})(\delta r_{ii} + \delta r_{ii}), & i = j. \end{cases} \tag{3.6}$$

From (3.5), we can get

$$[(I\otimes R^T)D_1+(R^T\otimes I)D_2][\mathrm{vec}(\Delta R)+\mathrm{vec}((\Delta R)^T)]\approx \mathrm{vec}(\delta A), \tag{3.7}$$

where $\mathrm{vec}(\delta A)=[(I\otimes(A^TJ))+((A^TJ)\otimes I)\Pi]\mathrm{vec}(\Delta A)$. It is easy to observe that $(I\otimes R^T)D_1+(R^T\otimes I)D_2$ is lower triangular with diagonal elements

$$\underbrace{r_{11},r_{11},\cdots,r_{11}}_{n},\ \underbrace{r_{22},r_{22},\cdots,r_{22}}_{n},\ \cdots,\ \underbrace{r_{n,n},r_{n,n},\cdots,r_{n,n}}_{n}.$$

Note that $(I\otimes R^T)D_1+(R^T\otimes I)D_2$ is a nonsingular lower triangular matrix, and from (3.7), we have

$$\mathrm{vec}(\Delta R)+\mathrm{vec}((\Delta R)^T)\approx[(I\otimes R^T)D_1+(R^T\otimes I)D_2]^{-1}\mathrm{vec}(\delta A). \tag{3.8}$$

The solution of triangular systems are usually computed with high accuracy even if they are ill-conditioned [7]. Note that the structure of the triangular matrix, the triangular systems (3.8) can be easily solved.

Note that $\mathrm{vec}(\Delta R)$ corresponds to upper triangular matrix. We have

$$\begin{array}{lcl}\mathrm{vec}(\Delta R) & \approx & D_1[(I\otimes R^T)D_1+(R^T\otimes I)D_2]^{-1}\mathrm{vec}(\delta A),\\ \mathrm{vec}((\Delta R)^T) & \approx & D_2[(I\otimes R^T)D_1+(R^T\otimes I)D_2]^{-1}\mathrm{vec}(\delta A).\end{array} \tag{3.9}$$

The normwise and componentwise perturbation bounds can be derived as follows:

$$\|\Delta R\|_F\lesssim\|D_1[(I\otimes R^T)D_1+(R^T\otimes I)D_2]^{-1}\|_2\|\delta A\|_F, \tag{3.10}$$

and

$$\mathrm{vec}(|\Delta R|)\lesssim|D_1[(I\otimes R^T)D_1+(R^T\otimes I)D_2]^{-1}|\mathrm{vec}(|\delta A|). \tag{3.11}$$

Using the hyperbolic QR factorization of $\tilde{A}$ in the δA, the rounding-error of perturbation bounds will be smaller.

The mixed and componentwise condition numbers for the factor R are defined as follows:

$$m_R(A)=\lim_{\varepsilon\to 0}\sup_{\|\Delta A/A\|_{\max}\le\varepsilon}\frac{\|\Delta R\|_{\max}}{\|R\|_{\max}}\frac{1}{\|\Delta A/A\|_{\max}},$$
$$c_R(A)=\lim_{\varepsilon\to 0}\sup_{\|\Delta A/A\|_{\max}\le\varepsilon}\frac{1}{\|\Delta A/A\|_{\max}}\left\|\frac{\Delta R}{R}\right\|_{\max}.$$

Here $\frac{B}{A}$ is an entrywise division defined by $\frac{B}{A}:=\mathrm{vec}^{-1}(\mathrm{vec}(B)/\mathrm{vec}(A))$.

The main result in this subsection is the following theorem. It presents explicit expressions for the condition numbers we defined for the factor R.

Theorem 3.1 *Let $A\in\mathbb{R}^{m\times n}$ with A^TJA is positive definite and $A=Q_1R$ be the hyperbolic QR factorization. Then*

(a)

$$m_R(A) = \frac{\| |D_1 N_R| \mathrm{vec}(|A|) \|_\infty}{\|\mathrm{vec}(R)\|_\infty}, \tag{3.12}$$

(b)

$$c_R(A) = \left\| \frac{|D_1 N_R| \mathrm{vec}(|A|)}{\mathrm{vec}(R)} \right\|_\infty, \tag{3.13}$$

where $N_R = [(I \otimes R^T)D_1 + (R^T \otimes I)D_2]^{-1}[(I \otimes (A^T J)) + ((A^T J) \otimes I)\Pi]$.

Proof. It follows from (3.9) that

$$\varphi_R'(A) = D_1[(I \otimes R^T)D_1 + (R^T \otimes I)D_2]^{-1}[(I \otimes (A^T J)) + ((A^T J) \otimes I)\Pi].$$

From Definition 2.1 and (a) of Lemma 2.2, we obtain

$$m_R(A) = m(\varphi_R; a) = \frac{\| |\varphi_R'(a)| \ |a| \|_\infty}{\|\varphi_R(a)\|_\infty} = \frac{\| |D_1 N_R| \mathrm{vec}(|A|) \|_\infty}{\|\mathrm{vec}(R)\|_\infty},$$

and

$$c_R(A) = c(\varphi_R; a) = \left\| \frac{|D_1 N_R| \ |a|}{|\varphi_R(a)|} \right\|_\infty = \left\| \frac{|D_1 N_R| \mathrm{vec}(|A|)}{\mathrm{vec}(R)} \right\|_\infty,$$

where a denotes $\mathrm{vec}(A)$. □

Theorem 3.1 gives explicit expressions for the condition numbers $m_R(A)$ and $c_R(A)$. While these expressions are sharp they may not be easily computed by their dependance on the vec-permutation matrix Π and Kronecker products. We need a lemma in [4].

Lemma 3.2 [4] *For any matrices* $M, N, P, Q, R,$ *and* S *with dimensions making the following well defined*

$$[M \otimes N + (P \otimes Q)\Pi]\mathrm{vec}(R), \ \ \frac{[M \otimes N + (P \otimes Q)\Pi]\mathrm{vec}(R)}{S}, \ \ NRM^T \text{ and } QR^T P^T,$$

we have

$$\| |[M \otimes N + (P \otimes Q)\Pi]| \mathrm{vec}(|R|) \|_\infty \le \|\mathrm{vec}(|N| \ |R| \ |M|^T + |Q| \ |R|^T |P|^T)\|_\infty,$$

and

$$\left\| \frac{|[M \otimes N + (P \otimes Q)\Pi]| \mathrm{vec}(|R|)}{|S|} \right\|_\infty \le \left\| \frac{\mathrm{vec}(|N| \ |R| \ |M|^T + |Q| \ |R|^T |P|^T)}{|S|} \right\|_\infty.$$

The following corollary gives computable upper bounds for these condition numbers.

Corollary 3.3 *In the hypothesis of Theorem 3.1, assume that the upper triangular part of* R *has no zero components. We have*

(a)

$$m_R(A) \le \frac{\|D_1 S\|_\infty \|2|A|^T |A|\|_{\max}}{\|R\|_{\max}}, \tag{3.14}$$

(b)

$$c_R(A) \le \|\mathrm{Dg}^{\dagger}(\mathrm{vec}(R))D_1 S\|_\infty \left\|2|A|^T|A|\right\|_{\max}, \tag{3.15}$$

where $S = [(I \otimes R^T)D_1 + (R^T \otimes I)D_2]^{-1}$ *and* $\mathrm{Dg}^{\dagger}(a)$ *is the Moore-Penrose inverse of the diagonal matrix* $\mathrm{diag}(a)$*.*

3.2 The factor Q

From (1.6), omitting the second-order term, which changes to

$$Q_1(\Delta R) + (\Delta Q_1)R \approx \Delta A. \tag{3.16}$$

Note that R is nonsingular in (3.16), right-multiplying by R^{-1} leads to

$$\Delta Q_1 \approx (\Delta A)R^{-1} - Q_1(\Delta R)R^{-1}. \tag{3.17}$$

Using the vec function, we have

$$\mathrm{vec}(\Delta Q_1) \approx (R^{-T} \otimes I)\mathrm{vec}(\Delta A) - (R^{-T} \otimes Q_1)\mathrm{vec}(\Delta R). \tag{3.18}$$

Substituting (3.9) into (3.18), we get

$$\mathrm{vec}(\Delta Q_1) \approx \{(R^{-T} \otimes I) - (R^{-T} \otimes Q_1)D_1 N_R\}\mathrm{vec}(\Delta A). \tag{3.19}$$

The normwise and componentwise perturbation bounds can be derived as follows:

$$\|\Delta Q_1\|_F \lesssim \|(R^{-T} \otimes I) - (R^{-T} \otimes Q_1)D_1 N_R\|_2 \|\Delta A\|_F, \tag{3.20}$$

and

$$\mathrm{vec}(|\Delta Q_1|) \lesssim |(R^{-T} \otimes I) - (R^{-T} \otimes Q_1)D_1 N_R|\mathrm{vec}(|\Delta A|). \tag{3.21}$$

The mixed and componentwise condition numbers for the factor Q_1 are defined as follows:

$$m_{Q_1}(A) = \lim_{\varepsilon \to 0} \sup_{\|\Delta A/A\|_{\max} \le \varepsilon} \frac{\|\Delta Q_1\|_{\max}}{\|Q_1\|_{\max}} \frac{1}{\|\Delta A/A\|_{\max}},$$
$$c_{Q_1}(A) = \lim_{\varepsilon \to 0} \sup_{\|\Delta A/A\|_{\max} \le \varepsilon} \frac{1}{\|\Delta A/A\|_{\max}} \left\|\frac{\Delta Q_1}{Q_1}\right\|_{\max}.$$

Here $\frac{B}{A}$ is an entrywise division defined by $\frac{B}{A} := \mathrm{vec}^{-1}(\mathrm{vec}(B)/\mathrm{vec}(A))$.

The main result in this subsection is the following theorem. It presents explicit expressions for the condition numbers we defined for the factor Q_1.

Theorem 3.4 *Let* $A \in \mathbb{R}^{m \times n}$ *with* $A^T JA$ *is positive definite and* $A = Q_1 R$ *be the hyperbolic QR factorization. Then*

(a)

$$m_{Q_1}(A) = \frac{\||(R^{-T} \otimes I) - (R^{-T} \otimes Q_1)D_1 N_R|\mathrm{vec}(|A|)\|_\infty}{\|\mathrm{vec}(Q_1)\|_\infty}, \tag{3.22}$$

(b)

$$c_{Q_1}(A) = \left\| \frac{|(R^{-T} \otimes I) - (R^{-T} \otimes Q_1) D_1 N_R| \mathrm{vec}(|A|)}{\mathrm{vec}(Q_1)} \right\|_\infty , \tag{3.23}$$

where $N_R = [(I \otimes R^T) D_1 + (R^T \otimes I) D_2]^{-1} [(I \otimes (A^T J)) + ((A^T J) \otimes I)\Pi]$.

Proof. The proof is similar to Theorem 3.1. □

The following corollary gives computable upper bounds for these condition numbers.

Corollary 3.5 *In the hypothesis of Theorem 3.4, we have*

(a)

$$m_{Q_1}(A) \leq \frac{\| |A| \ |R^{-1}| \|_{\max} + \|(R^{-T} \otimes Q_1) D_1 S\|_\infty \|2|A|^T |A|\|_{\max}}{\|Q_1\|_{\max}}, \tag{3.24}$$

(b)

$$c_{Q_1}(A) \leq \left\| \frac{|A| \ |R^{-1}|}{Q_1} \right\|_{\max} + \|\mathrm{Dg}^{-1}(\mathrm{vec}(Q_1))(R^{-T} \otimes Q_1) D_1 S\|_\infty \left\| 2|A|^T |A| \right\|_{\max}, \tag{3.25}$$

where $S = [(I \otimes R^T) D_1 + (R^T \otimes I) D_2]^{-1}$.

We give a simple example as the following. All computations are performed in MATLAB 6.5, with precision 2.22×10^{-16}.

Example 3.6 *Let*

$$A = \begin{bmatrix} 7 & 8 \\ 2 & 1 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} I_3 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A^T JA \text{ is positive definite.}$$

The mixed and componentwise condition numbers of the hyperbolic QR factorization are shown in Table 1.

Table 1. Mixed and componentwise condition numbers

$m_R(A)$	$m_R^{upper}(A)$	$c_R(A)$	$c_R^{upper}(A)$
1.4239	13.7987	4.5463	44.0578
$m_{Q_1}(A)$	$m_{Q_1}^{upper}(A)$	$c_{Q_1}(A)$	$c_{Q_1}^{upper}(A)$
2.1023	51.9557	245.9453	387.1674

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