NEW SUBCLASS OF GOODMAN-TYPE p-VALENT HARMONIC FUNCTIONS

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Abstract. In this paper, we have introduced a new subclass of p-valent harmonic functions that are orientation preserving in the open unit disk and are related to Goodman-type analytic uniformly starlike functions. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for the functions belonging to this class are obtained.

1. INTRODUCTION

A continuous complex-valued function \( f = u + iv \) defined in a simply connected complex domain \( D \) is said to be harmonic in \( D \), if both \( u \) and \( v \) are real harmonic in \( D \). There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions \( u \) and \( v \) there exist analytic functions \( U \) and \( V \) so that

\[
\begin{align*}
\Re(f) &= U(z) \\
\Im(f) &= V(z)
\end{align*}
\]

where \( U \) and \( V \) are respectively, the analytic functions \( (u + V/2) \) and \( (u - V/2) \). In this case, the Jacobian of \( f(z) = h(z) + g(z) \) is given by

\[
J_f(z) = |h'(z)|^2 - |g'(z)|^2.
\]

The mapping \( z \to f(z) \) is orientation preserving and locally one to one in \( D \), if and only if \( J_f(z) > 0 \) in \( D \). The necessity of this condition is a result of Lewy [6]. See also Clunie and Sheil-Small [2].
The function $f(z) = h(z) + \overline{g(z)}$ is said to be harmonic univalent in $D$, if the mapping $z \mapsto f(z)$ is orientation preserving, harmonic and one to one in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f(z) = h(z) + \overline{g(z)}$.

For fixed positive integer $p$, let $H(p)$ denote the family of functions $f(z) = h(z) + \overline{g(z)}$ that are harmonic, orientation preserving and $p$-valent in the open unit disk $U = \{z : |z| < 1\}$ with the normalization

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1}z^{n+p-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+p-1}z^{n+p-1}, \quad |b_p| < 1. \quad (1.1)$$

Motivated by recent work of Rosy et al [9], we define a new subclass as follows:

Let $G_H(p, \gamma)$ denote the subclass of $H(p)$ consisting of functions $f$ in $H(p)$ that satisfy the condition

$$\Re \left\{ (1 + e^{i\alpha}) \frac{zf''(z)}{zf'(z)} - pe^{i\alpha} \right\} \geq p\gamma, \quad (1.2)$$

where $z' = \frac{\partial}{\partial \theta}(z = re^{i\theta})$, $f'(z) = \frac{\partial}{\partial \theta}(f(z) = f(re^{i\theta}))$, $p \geq 1$, $0 \leq r < 1$ and $\alpha$, $\theta$ are real.

We further let $G_H(p, \gamma)$ denote the subclass of $G_H(p, \gamma)$, consisting of functions $f(z) = h(z) + \overline{g(z)}$ such that $h$ and $g$ are of the form

$$h(z) = z^p - \sum_{n=2}^{\infty} d_{n+p-1}z^{n+p-1}, \quad g(z) = \sum_{n=1}^{\infty} b_{n+p-1}z^{n+p-1}. \quad (1.3)$$

For $p=1$ and $g \equiv 0$ that is, if $f$ is analytic, the family $G_H(1, 0)$ is uniformly starlike in $U$ and was first studied by Goodman [3]. In [8], Ronning investigated the uniformly starlike functions of order $\gamma$, $0 \leq \gamma < 1$. Later, Jahangiri et al [5] constructed a class of harmonic close to convex functions and studied basic properties. Recently, Jahangiri [4], Silverman
Silverman and Silvia [11] studied the harmonic starlike functions. Ahuja and Jahangiri [1] proved that if, \( f(z) = h(z) + g(z) \) is given by (1.1) and if,

\[
\sum_{n=1}^{\infty} (n+m-1)(|a_{n+m-1}| + |b_{n+m-1}|) \leq 2m
\]  
(1.4)

then \( f \) is harmonic, \( p \)-valent and starlike of order \( \gamma \) in \( U \). This condition is proved to be also necessary if \( h \) and \( g \) are of the form (1.3). In the present paper we have obtained coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations for the class \( G_{p,h}(p,\gamma) \).

### 2. COEFFICIENT BOUNDS

We being with a sufficient coefficient bounds for the class \( G_{h}(p,\gamma) \). These conditions are shown to be necessary for the functions in \( G_{h}(p,\gamma) \).

**Theorem 1.** Let \( f = h + \overline{g} \) with \( h \) and \( g \) are given by (1.1). If

\[
\sum_{n=1}^{\infty} \left[ \frac{2n+p-2-\gamma}{p-\gamma} a_{n+p-1} + \frac{2n+3p-2+\gamma}{p-\gamma} b_{n+p-1} \right] \leq 2 ,
\]  
(2.1)

where \( |a_1| = 1, 0 \leq \gamma < 1 \). Then \( f \) is harmonic \( p \)-valent in \( U \) and \( f \in G_{h}(p,\gamma) \).

**Proof:** Suppose that (2.1) holds. Then we have

\[
\text{Re} \left\{ \frac{1 + e^{i\alpha} \left( zh'(z) - zg'(z) - pe^{i\alpha} (h(z) + \overline{g(z)}) \right)}{h(z) + \overline{g(z)}} \right\} = \text{Re} \frac{A(z)}{B(z)} \geq p\gamma,
\]  
(2.2)

as \( f(z) \in H(p) \), \( h(z) + \overline{g(z)} \neq 0 \).

where \( z = re^{i\theta}, \ 0 \leq r < 1, 0 \leq \gamma < 1, 0 \leq \theta < 2\pi \).

Here, we let
\[ A(z) = \left(1 + e^{ia}\right) \left(z h'(z) - z g'(z)\right) - p e^{ia} \left(h(z) + g(z)\right) \]

and

\[ B(z) = h(z) + g(z). \]

Using the fact that \( \text{Re} \omega \geq \rho \gamma \), if and only if \( |p - \gamma + \omega| \geq |p + \gamma - \omega| \), it suffices to show that

\[
\left| A(z) + (p - \gamma) B(z) \right| - \left| A(z) - (p + \gamma) B(z) \right| \geq 0. \tag{2.3}
\]

Substituting for \( A(z) \) and \( B(z) \) in (2.3), we obtain

\[
\left| \left(p - \gamma\right) h(z) + (1 + e^{ia}) z h'(z) - p e^{ia} h(z) + \left(p - \gamma\right) g(z) - (1 + e^{ia}) z g'(z) - p e^{ia} g(z) \right|
\]

\[
- \left| \left(p + \gamma\right) h(z) - (1 + e^{ia}) z h'(z) + p e^{ia} h(z) + \left(p + \gamma\right) g(z) + (1 + e^{ia}) z g'(z) + p e^{ia} g(z) \right|
\]

\[
= \left(2p - \gamma\right) z^n + \sum_{n=2}^{n} \left[ (n + 2p - 1 - \gamma) + e^{ia} (n - 1) \right] a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{n} \left[ (n - 1 + \gamma) + e^{ia} (n + 2p - 1) \right] b_{n+p-1} z^{n+p-1}
\]

\[
- \left( \gamma z^n - \sum_{n=2}^{n} \left[ (n - 1 - \gamma) + e^{ia} (n - 1) \right] a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{n} \left[ (n + 2p - 1 + \gamma) + e^{ia} (n + 2p - 1) \right] b_{n+p-1} z^{n+p-1} \right)
\]

\[
\geq 2 \left( p - \gamma \right) |z|^p - \sum_{n=2}^{n} \left[ (4n + 2p - 4 - 2\gamma) \right] a_{n+p-1} |z|^{n+p-1} - \sum_{n=1}^{n} \left[ (4n + 6p - 4 - 2\gamma) \right] b_{n+p-1} |z|^{n+p-1}
\]

\[
= 2 \left( p - \gamma \right) |z|^p \left\{ 1 - \sum_{n=2}^{n} \frac{2n + p - 2 - \gamma}{p - \gamma} a_{n+p-1} |z|^{n+p-1} + \sum_{n=1}^{n} \frac{2n + 3p - 2 + \gamma}{p - \gamma} b_{n+p-1} |z|^{n+p-1} \right\}
\]

\[
\geq 2 \left( p - \gamma \right) |z|^p \left\{ 1 - \left[ \sum_{n=2}^{n} \frac{2n + p - 2 - \gamma}{p - \gamma} a_{n+p-1} + \sum_{n=1}^{n} \frac{2n + 3p - 2 + \gamma}{p - \gamma} b_{n+p-1} \right] \right\} \geq 0, \text{ by (2.1).}
\]

The functions
where  

\[ \sum_{n=2}^{\infty} |x_{n+p-1}| + \sum_{n=2}^{\infty} |y_{n+p-1}| = 1, \]

show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.4) are in \( G_{\Pi} (p, \gamma) \) because

\[ \sum_{n=2}^{\infty} \left( \frac{2n + p - 2 - \gamma}{p - \gamma} |a_{n+p-1}| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} |b_{n+p-1}| \right) = 1 + \sum_{n=2}^{\infty} |x_{n+p-1}| + \sum_{n=2}^{\infty} |y_{n+p-1}| = 2. \]

We next show that the condition (2.1) is also necessary for the function in \( G_{\Pi} (p, \gamma) \).

**Theorem 2.** Let \( f = h + \overline{g} \) be so that \( h \) and \( g \) are given by (1.3). Then \( f(z) \in G_{\Pi} (p, \gamma) \), if and only if the inequality (2.1) holds for the coefficient of \( f = h + \overline{g} \).

**Proof.** In view of Theorem 1, we need only show that \( f(z) \not\in G_{\Pi} (p, \gamma) \) if the condition (2.1) does not holds. We note that a necessary condition for \( f = h + \overline{g} \) given by (1.3) to be in \( G_{\Pi} (p, \gamma) \) is that

\[ \Re \left\{ \left( 1 + e^{i\alpha} \right) \frac{z f'(z)}{z' f(z)} - pe^{i\alpha} \right\} \geq p \gamma. \]

This is equivalent to

\[ \Re \left\{ \left( 1 + e^{i\alpha} \right) \left( z h'(z) - z' g'(z) \right) - p e^{i\alpha} \left( h(z) + \overline{g(z)} \right) \right\} \]

\[ = \Re \left\{ \frac{2(p - \gamma) |\alpha| - \sum_{n=2}^{\infty} 2n + p - 2 - \gamma |a_{n+p-1}| \|z\|^{n+p-1} - \sum_{n=1}^{\infty} 2n + 3p - 2 + \gamma |b_{n+p-1}| \|z\|^{n+p-1}}{|\alpha| - \sum_{n=2}^{\infty} |a_{n+p-1}| \|z\|^{n+p-1} + \sum_{n=2}^{\infty} |b_{n+p-1}| \|z\|^{n+p-1}} \right\} \geq 0. \]
The above condition must hold for all values of $z$, $|z| = r < 1$.

Upon choosing the values of $z$ on the positive real axis, we must have

$$2(p - \gamma) - \sum_{n=2}^{\infty} 2n + p - 2 - \gamma |a_{n+p-1}| r^{n+p-2} - \sum_{n=1}^{\infty} 2n + 3p - 2 - \gamma |b_{n+p-1}| r^{n+p-2}$$

$$\geq 0. \quad (2.5)$$

If the condition (2.1) does not hold, then the numerator in (2.5) is negative for $r$ sufficiently close to 1. Thus there exists a $z_0 = r_0 > 1$, for which the quotient in (2.5) is negative. This contradicts the condition for $f(z) \in G_\Pi(p, \gamma)$ and so the proof is complete.

### 3. DISTORTION BOUNDS AND EXTREME POINTS

In this section, we shall obtain distortion bounds for functions in $G_\Pi(p, \gamma)$ and also we determine the extreme points of the closed convex hulls of denoted by $clco G_\Pi(p, \gamma)$.

**Theorem 3.** If $f(z) \in G_\Pi(p, \gamma)$, then

$$|f(z)| \leq \left(1 + |b_p|\right) r^p + \left(\frac{p - \gamma}{2 + p - \gamma} - \frac{3p + \gamma}{2 + p - \gamma} |b_p|\right) r^{p+1}, \quad |z| = r < 1$$

and

$$|f(z)| \geq \left(1 - |b_p|\right) r^p - \left(\frac{p - \gamma}{2 + p - \gamma} - \frac{3p + \gamma}{2 + p - \gamma} |b_p|\right) r^{p+1}, \quad |z| = r < 1.$$

**Proof.** We only prove the right hand inequality. The argument for left hand inequality is similar and will be omitted. Let $f(z) \in G_\Pi(p, \gamma)$. Taking the absolute value of $f$, we obtain
\[ |f(z)| \leq (1 + |b_p|)r^p + \sum_{n=2}^{\infty} \left( |a_{n+p-1}| + |b_{n+p-1}| \right) r^{n+p-1} \]

\[ \leq (1 + |b_p|)r^p + \sum_{n=2}^{\infty} \left( |a_{n+p-1}| + |b_{n+p-1}| \right) r^{n+p-1} \]

\[ = (1 + |b_p|)r^p + \frac{p - \gamma}{2 + p - \gamma} \sum_{n=2}^{\infty} \left[ \frac{2 + p - \gamma}{p - \gamma} |a_{n+p-1}| + \frac{3p + \gamma}{p - \gamma} |b_{n+p-1}| \right] r^{n+p-1} \]

\[ \leq (1 + |b_p|)r^p + \frac{p - \gamma}{2 + p - \gamma} \sum_{n=2}^{\infty} \left[ \frac{2 + p - 2 - \gamma}{p - \gamma} |a_{n+p-1}| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} |b_{n+p-1}| \right] r^{n+p-1} \]

\[ \leq (1 + |b_p|)r^p + \frac{p - \gamma}{2 + p - \gamma} \left( 1 - \frac{3p + \gamma}{p - \gamma} |b_{n+p-1}| \right) r^{p+1} \text{ by (2.1)} \]

\[ = (1 + |b_p|)r^p + \left( \frac{p - \gamma}{2 + p - \gamma} - \frac{3p + \gamma}{2 + p - \gamma} |b_p| \right) r^{p+1}. \]

**Theorem 4.** \( f \in clco G_\Pi(p, \gamma), \) if and only if \( f \) can be expressed as

\[ f(z) = \sum_{n=1}^{\infty} x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1} \quad (3.1) \]

where \( z \in U, \)

\[ h_{p-1}(z) = z^p, \quad h_{n+p-1}(z) = z^p - \frac{p - \gamma}{2n + p - 2 - \gamma} z^{n+p-1}. \]

\( (n = 2, 3, 4, \ldots), \quad g_{n+p-1}(z) = z^p + \frac{p - \gamma}{2n + 3p - 2 + \gamma} z^{n+p-1}. \)

\( (n = 1, 2, 3, 4, \ldots), \quad \sum_{n=1}^{\infty} (x_{n+p-1} + y_{n+p-1}) = 1, \quad x_{n+p-1} \geq 0 \text{ and } y_{n+p-1} \geq 0. \)

**Proof.** For the functions \( f \) given by (3.1), we may write
\[ f(z) = \sum_{n=1}^{\infty} \left( x_{n+p-1} z^{n+p-1} + y_{n+p-1} z^{n+p-1} \right) \]
\[ = x_{p-1} h_{p-1}(z) + y_{p-1} g_{p-1}(z) + \sum_{n=2}^{\infty} x_{n+p-1} \left( z^n + \frac{p-\gamma}{2n + p - 2 - \gamma} \right) z^{n+p-1} \]
\[ + \sum_{n=1}^{\infty} y_{n+p-1} \left( z^n - \frac{p-\gamma}{2n + 3p - 2 + \gamma} \right) z^{n+p-1} \]
\[ = \sum_{n=1}^{\infty} \left( x_{n+p-1} + y_{n+p-1} \right) z^n - \sum_{n=2}^{\infty} \frac{p-\gamma}{2n + p - 2 - \gamma} x_{n+p-1} z^{n+p-1} \]
\[ + \sum_{n=1}^{\infty} \frac{p-\gamma}{2n + 3p - 2 + \gamma} y_{n+p-1} z^{n+p-1}. \]

Then
\[ = \sum_{n=2}^{\infty} \frac{2n + p - 2 - \gamma}{p - \gamma} \left( \frac{p-\gamma}{2n + p - 2 - \gamma} x_{n+p-1} \right) + \sum_{n=1}^{\infty} \frac{2n + 3p - 2 + \gamma}{p - \gamma} \left( \frac{p-\gamma}{2n + 3p - 2 + \gamma} y_{n+p-1} \right) \]
\[ = \sum_{n=2}^{\infty} x_{n+p-1} + \sum_{n=2}^{\infty} y_{n+p-1} = 1 - x_i \leq 1, \]
and so \( f \in clco G_{\gamma}(p, \gamma). \)

Conversely, suppose that \( f \in clco G_{\gamma}(p, \gamma). \) Set
\[ x_{n+p-1} = \frac{2n + p - 2 - \gamma}{p - \gamma} \left| a_{n+p-1} \right| \quad (n = 2, 3, \ldots) \]
and
\[ y_{n+p-1} = \frac{2n + 3p - 2 + \gamma}{p - \gamma} \left| b_{n+p-1} \right| \quad (n = 1, 2, 3, \ldots). \]

Then note that by Theorem 2,
0 ≤ x_{p-1} ≤ 1 and y_{p-1} = 1 - x_{p-1} - \sum_{n=1}^{\infty} (x_{n+p-1} + y_{n+p-1}).

Consequently, we obtain \( f(z) = \sum_{n=1}^{\infty} \left( x_{n+p-1} h_{n+p-1} + y_{n+p-1} g_{n+p-1} \right) \). Using Theorem 2, it is easily seen that \( G_\Pi(p, \gamma) \) is convex and closed, so \( \text{clco} G_\Pi(p, \gamma) = G_\Pi(p, \gamma) \).

4. CONVOLUTION AND CONVEX LINEAR COMBINATION

In this section, we show that the class \( G_\Pi(p, \gamma) \) is invariant under convolution and convex combinations of its members.

For harmonic functions

\[
f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} b_{n+p-1} \bar{z}^{n+p-1} \quad \text{and} \quad F(z) = z^p - \sum_{n=1}^{\infty} A_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} B_{n+p-1} \bar{z}^{n+p-1}
\]

we define the convolution of \( f \) and \( F \) as

\[
(f * F)(z) = z^p - \sum_{n=1}^{\infty} a_{n+p-1} A_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} b_{n+p-1} B_{n+p-1} \bar{z}^{n+p-1}.
\]

Using this definition, we show that the class \( G_\Pi(p, \gamma) \) is closed under convolution.

**Theorem 5.** For \( 0 ≤ \beta ≤ \gamma < 1 \), let \( f(z) \in G_\Pi(p, \gamma) \) and \( F(z) \in G_\Pi(p, \beta) \). Then

\[
f * F \in G_\Pi(p, \gamma) \subset G_\Pi(p, \beta).
\]

**Proof.** Let

\[
f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} b_{n+p-1} \bar{z}^{n+p-1} \quad \text{be in} \quad G_\Pi(p, \gamma)
\]

and

\[
F(z) = z^p - \sum_{n=1}^{\infty} A_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} B_{n+p-1} \bar{z}^{n+p-1} \quad \text{be in} \quad G_\Pi(p, \beta).
\]
Note that $A_{n+p-1} \leq 1$ and $B_{n+p-1} \leq 1$. Obviously, the coefficients of $f$ and $F$ must satisfy conditions similar to the inequality (2.1). So for the coefficients of $f \ast F$ we can write

$$\sum_{n=1}^{\infty} \left[ \frac{2n + p - 2 - \gamma}{p - \gamma} |a_{n+p-1}| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} |b_{n+p-1}| \right]$$

$$\leq \sum_{n=1}^{\infty} \left[ \frac{2n + p - 2 - \gamma}{p - \gamma} |a_{n+p-1}| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} |b_{n+p-1}| \right].$$

This right hand side of the above inequality is bounded by 2 because $f(z) \in G_{p,\gamma}$. By the same token, we then conclude that $f \ast F \in G_{p,\gamma} \subset G_{p,\beta}$.

Finally, we show that $G_{p,\gamma}$ is closed under convex combination of its members.

**Theorem 6.** The family $G_{p,\gamma}$ is closed under convex combination.

**Proof.** For $i = 1, 2, 3, ..., let f_i \in G_{p,\gamma}$ where $f_i$ is given by

$$f_i(z) = z^p - \sum_{n=2}^{\infty} |a_{i,n+p-1}| z^{n+p-1} + \sum_{n=1}^{\infty} |b_{i,n+p-1}| z^{n+p-1}.$$ 

Then, by (2.1),

$$\sum_{n=1}^{\infty} \frac{2n + p - 2 - \gamma}{p - \gamma} |a_{i,n+p-1}| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} |b_{i,n+p-1}| \leq 2, \quad (4.2)$$

for $\sum_{i=1}^{\infty} t_i = 1, \ 0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{n=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_{i,n+p-1}| \right) z^{n+p-1} + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_{i,n+p-1}| \right) z^{n+p-1}.$$ 

Then, by (4.2),

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\[
\sum_{n=1}^{\infty} \left[ \frac{2n + p - 2 - \gamma}{p - \gamma} \left| \sum_{i=1}^{\infty} t_i \left| a_{i,n+p-1} \right| \right| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} \left| \sum_{n=1}^{\infty} t_i \left| b_{i,n+p-1} \right| \right| \right] \\
= \sum_{i=1}^{\infty} t_i \left[ \sum_{n=1}^{\infty} \frac{2n + p - 2 - \gamma}{p - \gamma} \left| a_{i,n+p-1} \right| + \frac{2n + 3p - 2 + \gamma}{p - \gamma} \left| b_{i,n+p-1} \right| \right] \\
\leq 2 \sum_{n=1}^{\infty} t_i = 2.
\]

This is the condition required by (2.1) and so \( \sum_{i=1}^{\infty} t_i f_i \in G_{p}(p, \gamma) \).

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