Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **22:1** (2008), 13–21

SOME FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MULTIVALUED MAPPINGS SATISFYING AN IMPLICIT RELATION

Ishak Altun and Duran Turkoglu

Abstract

In this paper, we prove a common fixed point theorem for multivalued mappings under the condition of weak compatibility. Also, we define compatible maps of type (I) for multivalued mappings and prove a common fixed point theorem for this type mappings. We use an implicit relation to prove our main theorems.

1 Introduction and preliminaries

In this paper (X,d) denotes a metric space and $\mathcal{B}(X)$ stands for the set of all bounded subsets of X. The function δ of $\mathcal{B}(X)\times\mathcal{B}(X)$ into $[0,\infty)$ is defined as

$$\begin{aligned}\delta(A,B) &= \sup\{d(a,b): a\in A, b\in B\},\\ D(A,B) &= \inf\{d(a,b): a\in A, b\in B\}\end{aligned}$$

for all A,B in $\mathcal{B}(X)$. If $A=\{a\}$ is singleton, we write $\delta(A,B)=\delta(a,B)$ and if $B=\{b\}$, then we put $\delta(A,B)=\delta(a,b)=d(a,b)$. It is easily seen that

$$\begin{aligned}\delta(A,B) &= \delta(B,A)\geq 0,\\ \delta(A,B) &\leq \delta(A,C)+\delta(C,B),\\ \delta(A,A) &= \operatorname{diam} A,\\ \delta(A,B) &= 0 \text{ implies } A=B=\{a\}\end{aligned}$$

for all A,B,C in $\mathcal{B}(X)$. We recall some definitions and basic lemmas of Fisher [4] and Imdad et al. [5]. Let $\{A_n: n=1,2,...\}$ be a sequence of subsets of X. We say that the sequence $\{A_n\}$ converges to a subset A of X if each point a in A is the limit of a convergent sequence $\{a_n\}$ with a_n in A_n for $n=1,2,\ldots$ and if for any $\varepsilon>0$, there exists an integer N such that $A_n\subseteq A_\varepsilon$ for $n>N$, A_ε being the union of all open spheres with centers in A and radius ε. The following lemmas hold.

2000 *Mathematics Subject Classifications.* 54H25, 47H10.
Key words and Phrases. Fixed point, weakly compatible mappings, compatible mappings of type (I), implicit relation.
Received: September 12, 2007
Communicated by Dragan S. Djordjević

Lemma 1 ([4]). *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of (X,d) which converge to the bounded subsets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Lemma 2 ([5]). *If $\{A_n\}$ is a sequence of bounded sets in the complete metric space (X,d) and if $\lim_{n\to\infty}\delta(A_n,\{y\})=0$ for some $y\in X$, then $\{A_n\}\to\{y\}$.*

A set-valued mapping F of X into $\mathcal{B}(X)$ is continuous at the point x in X if whenever $\{x_n\}$ is a sequence of points of X converging to x, the sequence $\{Fx_n\}$ in $\mathcal{B}(X)$ converges to Fx. F is said to be continuous in X if it is continuous at each point x in X. We say that z is a fixed point of F if z is in Fz.

The following definition is given by Jungck and Rhoades [7].

Definition 1. *Let $A:X\to X$ and $F:X\to\mathcal{B}(X)$ two mappings. The pair (A,F) is weakly compatible if A and F are commute at coincidence points, i. e., for each point u in X such that $Fu=\{Au\}$, we have $FAu=AFu$. (Note that the equation $Fu=\{Au\}$ implies that Fu is singleton).*

Now we introduce the following definition.

Definition 2. *Let $A:X\to X$ and $F:X\to\mathcal{B}(X)$ two mappings. The pair (A,F) is compatible of type (I) if*

$$d(u,Au)\le\overline{\lim_{n\to\infty}}\,\delta(u,FAx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $FAx_n\in\mathcal{B}(X)$, $Fx_n\to\{u\}$, $Ax_n\to u$ for some $u\in X$.

The above definition is given by Pathak et al. [9] for single-valued mappings in 1999.

Proposition 1. *Let $A:X\to X$ and $F:X\to\mathcal{B}(X)$ two mappings. If (A,F) is compatible of type (I) and $\{Ap\}=Fp$ for some $p\in X$, then $\delta(Fp,AAp)\le\delta(Fp,FAp)$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n=p$ for $n=1,2,3,...$ and $\{Ap\}=Fp$ for some $p\in X$. Then we have $Ax_n\to Ap$ and $Fx_n\to\{Ap\}$. Since the pair (A,F) is compatible of type (I) we have

$$\delta(Fp,AAp)=d(Ap,AAp)\le\overline{\lim_{n\to\infty}}\,\delta(Ap,FAx_n)=\delta(Ap,FAp)=\delta(Fp,FAp).$$

□

There are two examples in [10] such that the concepts of weakly compatible maps and compatible maps of type (I) are independent from each other for single valued mappings. The following example shows that (A,F) is compatible of type (I) but not weakly compatible.

Example 1. *Let $X = [0, \infty)$ be with the usual metric. Define $A : X \to X$ and $F : X \to B(X)$ by*

$$Ax = \begin{cases} 2 & \text{if } x \in [0,2] \\ 2+x & \text{if } x \in (2,\infty) \end{cases} \quad \text{and} \quad Fx = \begin{cases} [2, 2+x] & \text{if } x \in [0,2) \\ [3+x, 4+x] & \text{if } x \in [2,\infty) \end{cases}.$$

Note that 2 is a fixed point of A, then (A, F) is compatible of type (I). On the other hand, coincidence point of A and F is only 0 and these mappings are not commuting at 0. Thus (A, F) is not weakly compatible.

2 Implicit relation

Implicit relation on metric spaces have been used in many articles (see [2], [3], [6], [11], [12], [13], [14]).

Let R_+ denote the nonnegative real numbers and let $T : R_+^6 \to R$ be a continuous mapping. We define the following properties:

$T_1 : T(t_1, ..., t_6)$ is non-increasing in variables $t_2, ..., t_6$.

T_2 : there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0, f(t) < t$ for $t > 0$, such that for $u \geq 0$,

$$T(u, v, v, u, u+v, 0) \leq 0$$

or

$$T(u, v, u, v, 0, u+v) \leq 0$$

implies $u \leq f(v)$.

$T_3 : (a)\ T(u, u, 0, u, u, u) > 0$ and $(b)\ T(u, u, u, 0, u, u) > 0, \forall u > 0$.

$T_4 : T(u, u, 0, 0, u, u) > 0, \forall u > 0$.

Note that T_1 and $T_3(a)$ or $T_3(b)$ implies T_4.

Example 2. $T(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1-\alpha)[at_5 + bt_6]$, *where* $0 \leq \alpha < 1, 0 \leq a < \frac{1}{2}, 0 \leq b < \frac{1}{2}$.

T_1 : *Obviously.*

T_2 : *Let $u > 0$ and $T(u, v, v, u, u+v, 0) = u - \alpha \max\{u, v\} - (1-\alpha)a(u+v) \leq 0$. If $u \geq v$, then $(1-a)u \leq av$ which implies $a \geq \frac{1}{2}$, a contradiction. Thus $u < v$ and $u \leq \frac{\alpha+(1-\alpha)a}{1-(1-\alpha)a} v = \beta v$, Similarly, let $u > 0$ and $T(u, v, u, v, 0, u+v) \leq 0$ imply $u \leq \frac{\alpha+(1-\alpha)b}{1-(1-\alpha)b} v = \gamma v$. If $u = 0$ then $u \leq \gamma v$. Thus T_2 is satisfying with $f(t) = \max\{\beta, \gamma\}t$.*

$T_3 : T(u, u, 0, u, u, u) = T(u, u, u, 0, u, u) = (1-\alpha)(1-a-b)u > 0, \forall u > 0$.

Example 3. $T(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, *where* $k \in (0, 1)$.

T_1 : *Obviously.*

T_2 : *Let $u > 0$ and $T(u, v, v, u, u+v, 0) = u - k \max\{u, v\} \leq 0$. If $u \geq v$, then $u \leq ku$, which is a contradiction. Thus $u < v$ and $u \leq kv$. Similarly, let $u > 0$ and $T(u, v, u, v, 0, u+v) \leq 0$ then we have $u \leq kv$. If $u = 0$, then $u \leq kv$. Thus T_2 is satisfying with $f(t) = kt$.*

$T_3 : T(u, u, 0, u, u, u) = T(u, u, u, 0, u, u) = u - ku > 0$, $\forall u > 0$.

Example 4. *$T(t_1,...,t_6)=t_1-\psi(\max\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\})$, where $\psi:R_+\to R_+$ increasing and $\psi(0)=0,\psi(t)<t$ for $t>0$.*

T_1 : Obviously.

T_2 : Let $u>0$ and $T(u,v,v,u,u+v,0)=u-\psi(\max\{u,v\})\le 0$. If $u\ge v$, then $u-\psi(u)\le 0$, which is a contradiction. Thus $u<v$ and $u\le\psi(v)$. Similarly, let $u>0$ and $T(u,v,u,v,0,u+v)\le 0$ then we have $u\le\psi(v)$. If $u=0$ then $u\le\psi(v)$. Thus T_2 is satisfying with $f=\psi$.

T_3 : $T(u,u,0,u,u,u)=T(u,u,u,0,u,u)=u-\psi(u)>0,\forall u>0$.

Example 5. *$T(t_1,...,t_6)=t_1^2-t_1(at_2+bt_3+ct_4)-dt_5t_6$, where $a>0$, $b,c,d\ge 0$, $a+b+c<1$, $a+b+d<1$ and $a+c+d<1$.*

T_1 : Obviously.

T_2 : Let $u>0$ and $T(u,v,v,u,u+v,0)=u^2-u(av+bv+cu)\le 0$. Then $u\le(\frac{a+b}{1-c})v=h_1v$. Similarly, let $u>0$ and $T(u,v,u,v,0,u+v)\le 0$ then we have $u\le(\frac{a+c}{1-b})v=h_2v$. If $u=0$, then $u\le h_2v$. Thus T_2 is satisfying with $f(t)=\max\{h_1,h_2\}t$.

T_3 : $T(u,u,0,u,u,u)=u^2-u(au+cu)-du^2=u^2(1-a-c-d)>0,\forall u>0$ and $T(u,u,u,0,u,u)=u^2(1-a-b-d)>0,\forall u>0$.

3 Common fixed point theorems

We need the following lemma for the proof of our main theorems.

Lemma 3 ([15]). *For any $t>0, f(t)<t$ if and only if $\lim_{n\to\infty}f^n=0$, where f^n denotes the composition of f $n-$times with itself.*

Now we give one of the our main theorem.

Theorem 1. *Let A,B be mappings of a metric space (X,d) into itself and F,G be mappings from X into $\mathcal{B}(X)$ such that*

$$F(X)\subseteq B(X) \text{ and } G(X)\subseteq A(X). \tag{3.1}$$

Also, the mappings A,B,F and G are satisfying the following inequality

$$T(\delta(Fx,Gy),d(Ax,By),\delta(Ax,Fx),\delta(By,Gy),D(Ax,Gy),D(By,Fx))\le 0 \tag{3.2}$$

where T satisfies conditions T_1,T_2 and T_4. Suppose that any one of $A(X)$ or $B(X)$ is complete. If both pairs (A,F) and (B,G) are weakly compatible, then there exists a unique $z\in X$ such that $\{z\}=\{Az\}=\{Bz\}=Fz=Gz$.

Proof. Let x_0 be an arbitrary point in X. From (3.1), we choose a point x_1 in X such that $Bx_1\in Fx_0=Z_0$. For this point x_1 there exists a point x_2 in X such that $Ax_2\in Gx_1=Z_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$Bx_{2n+1}\in Fx_{2n}=Z_{2n},\ Ax_{2n+2}\in Gx_{2n+1}=Z_{2n+1} \tag{3.3}$$

for $n = 0, 1, 2, ...$For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$, for $n = 0, 1, 2, ...$ From (3.2) and (3.3), we have

$$T(\delta(Fx_{2n}, Gx_{2n+1}), d(Ax_{2n}, Bx_{2n+1}), \delta(Ax_{2n}, Fx_{2n}), \\ \delta(Bx_{2n+1}, Gx_{2n+1}), D(Ax_{2n}, Gx_{2n+1}), D(Bx_{2n+1}, Fx_{2n}) \le 0$$

and so we have

$$T(V_{2n}, V_{2n-1}, V_{2n-1}, V_{2n}, V_{2n-1} + V_{2n}, 0) \le 0.$$

From T_2, there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that

$$V_{2n} \le f(V_{2n-1}). \tag{3.4}$$

Similarly

$$T(\delta(Fx_{2n+2}, Gx_{2n+1}), d(Ax_{2n+2}, Bx_{2n+1}), \delta(Ax_{2n+2}, Fx_{2n+2}), \\ \delta(Bx_{2n+1}, Gx_{2n+1}), D(Ax_{2n+2}, Gx_{2n+1}), D(Bx_{2n+1}, Fx_{2n+2}) \le 0$$

and so we have

$$T(V_{2n+1}, V_{2n}, V_{2n+1}, V_{2n}, 0, V_{2n} + V_{2n+1}) \le 0.$$

From T_2, we have

$$V_{2n+1} \le f(V_{2n}). \tag{3.5}$$

From (3.4) and (3.5) we have, $V_n \le f^n(V_0)$ and from Lemma 3, we have $\lim_{n\to\infty} V_n = 0$.

Thus, if z_n is an arbitrary point in the set Z_n for $n = 0, 1, 2, ...$, it follows that

$$d(z_n, z_{n+1}) \le \delta(Z_n, Z_{n+1}) = V_n \to 0$$

as $n \to \infty$. Therefore the sequence $\{z_n\}$ and hence any subsequence thereof, is a Cauchy sequence in X.

Now suppose $B(X)$ is complete. Let $\{x_n\}$ be the sequence defined by (3.3). Since $Bx_{2n+1} \in Fx_{2n} = Z_{2n}$, for $n = 0, 1, 2, ...$, we have

$$d(Bx_{2m+1}, Bx_{2n+1}) \le \delta(Z_{2m}, Z_{2n}) < \varepsilon$$

for $m, n \ge n_0, n_0 = 1, 2, 3, ...$ Therefore by the above, the sequence $\{Bx_{2n+1}\}$ is Cauchy and hence $Bx_{2n+1} \to p = Bq \in B(X)$ for some $q \in X$. But $Ax_{2n} \in Gx_{2n-1} = Z_{2n-1}$ by (3.3), so that we have

$$d(Ax_{2n}, Bx_{2n+1}) \le \delta(Z_{2n-1}, Z_{2n}) = V_{2n-1} \to 0$$

as $n \to \infty$. Consequently $Ax_{2n} \to p$. Moreover, we have for $n = 1, 2, 3, ...$

$$\delta(Fx_{2n}, p) \le \delta(Fx_{2n}, Ax_{2n}) + d(Ax_{2n}, p) = V_{2n} + d(Ax_{2n}, p).$$

Therefore, $\delta(Fx_{2n}, p) \to 0$. In like manner it follows that $\delta(Gx_{2n-1}, p) \to 0$.

Since, for $n = 1, 2, 3, ...$

$$T(\delta(Fx_{2n}, Gq), d(Ax_{2n}, Bq), \delta(Ax_{2n}, Fx_{2n}), \delta(Bq, Gq), D(Ax_{2n}, Gq), D(Bq, Fx_{2n}) \le 0$$

and by T_1 we have

$$T(\delta(Fx_{2n}, Gq), d(Ax_{2n}, Bq), \delta(Ax_{2n}, Fx_{2n}), \delta(Bq, Gq), \delta(Ax_{2n}, Gq), \delta(Bq, Fx_{2n}) \le 0 .$$

We get as $n \to \infty$

$$T(\delta(p, Gq), 0, 0, \delta(p, Gq), \delta(p, Gq), 0) \le 0$$

and by T_2 ,there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0, f(t) < t$ for $t > 0$, we have $\delta(p, Gq) \le f(0) = 0$ and $\{p\} = Gq = \{Bq\}$.

But $G(X) \subseteq A(X)$, so $r \in X$ exists such that $\{Ar\} = Gq = \{Bq\}$. Now if $Fr \neq Gq$, $\delta(Fr, Gq) \neq 0$ so that we have

$$T(\delta(Fr, Gq), d(Ar, Bq), \delta(Ar, Fr), \delta(Bq, Gq), D(Ar, Gq), D(Bq, Fr)) \le 0$$

so we have

$$T(\delta(Fr, p), 0, \delta(Fr, p), 0, 0, d(Fr, p)) \le 0$$

and by T_2 we have $\delta(Fr, p) \le f(0) = 0$. It follows that $Fr = \{p\} = Gq = \{Ar\} = \{Bq\}$.

Since $Fr = \{Ar\}$ and the pair (A, F) is weakly compatible, we obtain $Fp = FAr = AFr = Ap$. Now using (3.2) we have

$$T(\delta(Fp, Gq), d(Ap, Bq), \delta(Ap, Fp), \delta(Bq, Gq), D(Ap, Gq), D(Bq, Fp)) \le 0$$

and so

$$T(\delta(Fp, p), d(Fp, p), 0, 0, \delta(Fp, p), \delta(Fp, p)) \le 0$$

which is a contradiction to T_4. Thus, $\delta(Fp, p) = 0$ and $Fp = \{p\} = \{Ap\}$. Similarly, $\{p\} = Gp = \{Bp\}$ if the pair (B, G) is weakly compatible. Therefore we obtain $\{p\} = \{Ap\} = \{Bp\} = Fp = Gp$.

To see the p is unique, suppose that $\{p'\} = \{Ap'\} = \{Bp'\} = Fp' = Gp'$ for some $p' \in X$, then

$$T(\delta(Fp, Gp'), d(Ap, Bp'), \delta(Ap, Fp), \delta(Bp', Gp'), D(Ap, Gp'), D(Bp', Fp)) \le 0$$

and so

$$T(\delta(p, p'), d(p, p'), 0, 0, \delta(p, p'), \delta(p, p')) \le 0$$

which is a contradiction to T_4. Thus we have $p = p'$. □

Remark 1. *If we use Example 2 and Theorem 1, we get Theorem 2. 1 of Ahmed [1]. If we choose A, B, F and G are single valued mappings in Theorem 1 with Example 3, we get an improved version of Theorem 3. 1 of Kang and Kim [8]. Similarly, many results can obtain by Theorem 1 and some examples.*

Now we give the other our main theorem.

Theorem 2. *Let A, B be mappings of a complete metric space (X, d) into itself and F, G be mappings from X into $\mathcal{B}(X)$ such that (3.1) holds. Also, the mappings A, B, F and G are satisfying the following inequality*

$$T(\delta(Fx, Gy), d(Ax, By), \frac{1}{2}\delta(Ax, Fx), \frac{1}{2}\delta(By, Gy), D(Ax, Gy), D(By, Fx)) \leq 0 \tag{3.6}$$

where T satisfies conditions T_1, T_2 and T_3. Suppose that the pairs (A, F) and (B, G) are compatible of type (I) and A or B is continuous, then there exists $z \in X$ such that $\{z\} = \{Az\} = \{Bz\} = Fz = Gz$.

Proof. Let the sequence $\{x_n\}$ is defined by (3.3). Similar operation in proof of Theorem 1, we have $Ax_{2n}, Bx_{2n+1}, Fx_{2n}, Gx_{2n+1} \to p$ for some $p \in X$ since X is complete.

Now suppose that B is continuous. Then the pair (B, G) is compatible of type (I), we have

$$d(p, Bp) \leq \overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}) \tag{3.7}$$

and $BBx_{2n+1} \to Bp$. Setting $x = x_{2n}$ and $y = Bx_{2n+1}$ in (3.6) we have

$$T(\delta(Fx_{2n}, GBx_{2n+1}), d(Ax_{2n}, BBx_{2n+1}), \frac{1}{2}\delta(Ax_{2n}, Fx_{2n}),$$
$$\frac{1}{2}\delta(BBx_{2n+1}, GBx_{2n+1}), D(Ax_{2n}, GBx_{2n+1}), D(BBx_{2n+1}, Fx_{2n}) \leq 0$$

then taking limit superior we have

$$T(\overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}), d(p, Bp), 0, \frac{1}{2}\overline{\lim_{n\to\infty}} \delta(Bp, GBx_{2n+1}),$$
$$\overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}), d(p, Bp)) \leq 0$$

and so

$$T(\overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}), \overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}), 0, \overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}),$$
$$\overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}), \overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}) \leq 0$$

which is a contradiction to T_3 if $\overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}) \neq 0$. Thus we have $\overline{\lim_{n\to\infty}} \delta(p, GBx_{2n+1}) = 0$ and so from (3.7) $p = Bp$.

Again setting $x = x_{2n}$ and $y = p$ in (3.6) and allowing $n \to \infty$ we have

$$T(\delta(p, Gp), 0, 0, \frac{1}{2}\delta(p, Gp), \delta(p, Gp), 0) \leq 0$$

with T_1 and T_2, there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0, f(t) < t$ for $t > 0$, we have $\delta(p, Gp) \leq f(0) = 0$, thus $\{p\} = \{Bp\} = Gp$.

Since $G(X) \subseteq A(X)$, there exists a point $q \in X$ such that $\{p\} = \{Bp\} = Gp = \{Aq\}$. Setting $x = q$ and $y = p$ in (3.6), we have

$$T(\delta(Fq, p), 0, \frac{1}{2}\delta(p, Fq), 0, 0, \delta(p, Fq)) \leq 0$$

implies from T_1 and T_2 we have $\delta(p, Fq) \leq f(0) = 0$ and so $\{p\} = Fq$.

Since (A, F) is compatible of type (I) and $\{Aq\} = \{Fq\} = p$, then using Preposition 1, we have $\delta(Fq, AAq) \leq \delta(Fq, FAq)$ and so

$$d(p, Ap) \leq \delta(p, Fp). \tag{3.8}$$

Again setting $x = p = y$ in (3.6) we have

$$T(\delta(Fp, p), \delta(Fp, p), \frac{1}{2}\delta(Fp, p), 0, \delta(Fp, p), \delta(Fp, p)) \leq 0$$

which is a contradiction to T_3 if $\delta(Fp, p) \neq 0$. Thus we have $Fp = \{p\}$ and from (3.8) we have $p = Ap$ so $\{p\} = \{Ap\} = \{Bp\} = Fp = Gp$.

The other case, A is continuous, can be disposed of following a similar argument as above. □

Acknowledgemets. The authors are grateful to the referees for their valuable comments in modifying the first version of this paper.

References

[1] M. A. Ahmed, *Common fixed point theorems for weakly compatible mappings*, Rocky Mountain J. Math. 33 (2) (2003), 1189-1203.

[2] I. Altun and D. Turkoglu, *Fixed point and homotopy result for mappings satisfying an implicit relation*, Discuss. Math. Differ. Incl. Control Optim., accepted.

[3] I. Altun, H. A. Hancer and D. Turkoglu, *A fixed point theorem for multimaps satisfying an implicit relation on metrically convex metric spaces*, Math. Commun. 11 (2006), 17-23.

[4] B. Fisher, *Common fixed point of mappings and set-valued mappings*, Rostock Math. Kolloq. 18 (1981), 69-77.

[5] M. Imdad, M. S. Khan and S. Sessa, *On some weak conditions of commutativity in common fixed point theorems*, Int. J. Math. Math. Sci. 11 (2) (1988), 289-296.

[6] M. Imdad, S. Kumar and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Math. 11 (1) (2002), 135-143.

[7] G. Jungck and B. E. Rhoades, *Fixed points for set-valued functions without continuity*, Indian J. Pure Appl. Math. 16 (1998), 227-238.

[8] S. M. Kang and Y. P. Kim, *Common fixed point theorems, Math. Japonica*, 37 (6) (1992), 1031-1039.

[9] H. K. Pathak, S. N. Mishra and A. K. Kalinde, *Common fixed point theorems with applications to nonlinear integral equations*, Demonstratio Math. 3 (1999), 547-564.

[10] H. K. Pathak, M.S. Khan and R. Tiwari, *A common fixed point theorem and its application to nonlinear integral equations*, Comp. Math. Appl. 53 (6) (2007), 961-971.

[11] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cerc. St. Ser. Math. Univ. Bacau 7 (1997), 127-133.

[12] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. 32 (1) (1999), 157-163.

[13] R. A. Rashwan and M. A. Ahmed, *Common fixed points for weakly δ-compatible mappings*, Ital. J. Pure Appl. Math. 8 (2000), 33-44.

[14] S. Sharma and B. Desphande, *On compatible mappings satisfying an implicit relation in common fixed point consideration*, Tamkang J. Math. 33 (3) (2002), 245-252.

[15] S. P. Singh and B. A. Meade, *A common fixed point theorem*, Bull. Austral. Math. Soc., 16 (1977), 49-53.

Address

Ishak Altun: Department of mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey
E-mail: ialtun@kku.edu.tr, ishakaltun@yahoo.com

Duran Turkoglu: Department of mathematics, Faculty of Science and Arts, Gazi University, 06500 Teknikokullar, Ankara, Turkey
E-mail: dturkoglu@gazi.edu.tr