SOME FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MULTIVALUED MAPPINGS SATISFYING AN IMPLICIT RELATION

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Abstract

In this paper, we prove a common fixed point theorem for multivalued mappings under the condition of weak compatibility. Also, we define compatible maps of type (I) for multivalued mappings and prove a common fixed point theorem for this type mappings. We use an implicit relation to prove our main theorems.

1 Introduction and preliminaries

In this paper $(X, d)$ denotes a metric space and $\mathcal{B}(X)$ stands for the set of all bounded subsets of $X$. The function $\delta$ of $\mathcal{B}(X) \times \mathcal{B}(X)$ into $[0, \infty)$ is defined as

$$
\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},
$$

$$
D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}
$$

for all $A, B$ in $\mathcal{B}(X)$. If $A = \{a\}$ is singleton, we write $\delta(A, B) = \delta(a, B)$ and if $B = \{b\}$, then we put $\delta(A, B) = \delta(a, b) = d(a, b)$. It is easily seen that

$$
\delta(A, B) = \delta(B, A) \geq 0,
$$

$$
\delta(A, B) \leq \delta(A, C) + \delta(C, B),
$$

$$
\delta(A, A) = \text{diam} A,
$$

$$
\delta(A, B) = 0 \text{ implies } A = B = \{a\}
$$

for all $A, B, C$ in $\mathcal{B}(X)$. We recall some definitions and basic lemmas of Fisher [4] and Imdad et al. [5]. Let $\{A_n : n = 1, 2, \ldots\}$ be a sequence of subsets of $X$. We say that the sequence $\{A_n\}$ converges to a subset $A$ of $X$ if each point $a$ in $A$ is the limit of a convergent sequence $\{a_n\}$ with $a_n$ in $A_n$ for $n = 1, 2, \ldots$ and if for any $\varepsilon > 0$, there exists an integer $N$ such that $A_n \subseteq A_\varepsilon$ for $n > N$, $A_\varepsilon$ being the union of all open spheres with centers in $A$ and radius $\varepsilon$. The following lemmas hold.
Lemma 1 ([4]). If \{A_n\} and \{B_n\} are sequences of bounded subsets of \((X, d)\) which converge to the bounded subsets \(A\) and \(B\) respectively, then the sequence \{δ(A_n, B_n)\} converges to \(δ(A, B)\).

Lemma 2 ([5]). If \{A_n\} is a sequence of bounded sets in the complete metric space \((X, d)\) and if \(\lim_{n \to \infty} δ(A_n, \{y\}) = 0\) for some \(y \in X\), then \(\{A_n\} \rightarrow \{y\}\).

A set-valued mapping \(F\) of \(X\) into \(B(X)\) is continuous at the point \(x\) in \(X\) if whenever \(\{x_n\}\) is a sequence of points of \(X\) converging to \(x\), the sequence \(\{Fx_n\}\) in \(B(X)\) converges to \(Fx\). \(F\) is said to be continuous in \(X\) if it is continuous at each point \(x\) in \(X\). We say that \(z\) is a fixed point of \(F\) if \(z\) is in \(Fz\).

Definition 1. Let \(A : X \rightarrow X\) and \(F : X \rightarrow B(X)\) two mappings. The pair \((A, F)\) is weakly compatible if \(A\) and \(F\) are commute at coincidence points, i.e., for each point \(u\) in \(X\) such that \(Fu = \{Au\}\), we have \(FAu = AFu\). (Note that the equation \(F u = \{Au\}\) implies that \(Fu\) is singleton).

Now we introduce the following definition.

Definition 2. Let \(A : X \rightarrow X\) and \(F : X \rightarrow B(X)\) two mappings. The pair \((A, F)\) is compatible of type \((I)\) if

\[d(u, Au) \leq \lim_{n \to \infty} δ(u, FAx_n)\]

whenever \(\{x_n\}\) is a sequence in \(X\) such that \(FAx_n \in B(X)\), \(Fx_n \to \{u\}\), \(Ax_n \to u\) for some \(u \in X\).

The above definition is given by Pathak et al. [9] for single-valued mappings in 1999.

Proposition 1. Let \(A : X \rightarrow X\) and \(F : X \rightarrow B(X)\) two mappings. If \((A, F)\) is compatible of type \((I)\) and \(\{Ap\} = Fp\) for some \(p \in X\), then \(δ(Fp, AAP) \leq δ(Fp, FAp)\).

Proof. Let \(\{x_n\}\) be a sequence in \(X\) defined by \(x_n = p\) for \(n = 1, 2, 3, \ldots\) and \(\{Ap\} = Fp\) for some \(p \in X\). Then we have \(Ax_n \to Ap\) and \(Fx_n \to \{Ap\}\). Since the pair \((A, F)\) is compatible of type \((I)\) we have

\[δ(Fp, AAP) = d(Ap, AAP) \leq \lim_{n \to \infty} δ(Ap, FAx_n) = δ(Ap, FAp) = δ(Fp, FAp)\].

There are two examples in [10] such that the concepts of weakly compatible maps and compatible maps of type \((I)\) are independent from each other for single valued mappings. The following example shows that \((A, F)\) is compatible of type \((I)\) but not weakly compatible.
Example 1. Let $X = [0, \infty)$ be with the usual metric. Define $A : X \to X$ and $F : X \to B(X)$ by

$$Ax = \begin{cases} 2 & \text{if } x \in [0, 2] \\ 2 + x & \text{if } x \in (2, \infty) \end{cases} \quad \text{and} \quad Fx = \begin{cases} [2, 2 + x] & \text{if } x \in [0, 2) \\ [3 + x, 4 + x] & \text{if } x \in [2, \infty) \end{cases}.$$  

Note that $2$ is a fixed point of $A$, then $(A, F)$ is compatible of type $(I)$. On the other hand, coincidence point of $F$ implies $T$.

Example 2. If $u$ then $F$ implies $u$ or $\{1, 2, 3, 4\}$. Note that $T$.

Example 3. If $u$ then $F$ implies $u$ or $[1, 2, 3, 4]$. Note that $T$.

2 Implicit relation

Implicit relation on metric spaces have been used in many articles (see [2], [3], [6], [11], [12], [13], [14]).

Let $R_+$ denote the nonnegative real numbers and let $T : R_+^n \to R$ be a continuous mapping. We define the following properties:

1. $T_1 : T(t_1, \ldots, t_6)$ is non-increasing in variables $t_2, \ldots, t_6$.
2. $T_2 :$ there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+, f(0) = 0, f(t) < t$ for $t > 0$, such that for $u \geq 0$,

$$T(u, v, u, v, u + v, 0) \leq 0$$

or

$$T(u, v, u, v, 0, u + v) \leq 0$$

implies $u \leq f(v)$.

1. $T_3 : (a)$ $T(u, u, 0, u, u) > 0$ and $(b)$ $T(u, u, u, 0, u) > 0, \forall u > 0$.
2. $T_4 : T(u, u, 0, 0, u) > 0, \forall u > 0$.

Note that $T_1$ and $T_3(a)$ or $T_3(b)$ implies $T_4$.

Example 2. $T(t_1, \ldots, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)\alpha t_5 + bt_6$, where $0 \leq \alpha < 1, 0 \leq a < \frac{1}{2}, 0 \leq b < \frac{1}{2}$.

1. $T_1 :$ Obviously.

2. $T_2 :$ Let $u > 0$ and $T(u, v, u, u, u + v, 0) = u - \alpha \max\{u, v\} - (1 - \alpha)\alpha u + v \leq 0$.

If $u \geq v$, then $(1 - \alpha)u \leq av$ which implies $u \geq \frac{1}{2}$, a contradiction. Thus $u < v$ and $u \leq \frac{\alpha + (1 - \alpha)a}{1 - \alpha}v = \beta v$. Similarly, let $u > 0$ and $T(u, v, u, v, 0, u + v) \leq 0$ imply $u \leq \frac{\alpha + (1 - \alpha)b}{1 - \alpha}v = \gamma v$. If $u = 0$ then $u \leq \gamma v$. Thus $T_2$ is satisfying with $f(t) = \max\{\beta, \gamma\}t$.

3. $T_3 : T(u, u, 0, u, u) = T(u, u, u, 0, u) = (1 - \alpha)(1 - a - b)u > 0, \forall u > 0$.

Example 3. $T(t_1, \ldots, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $k \in (0, 1)$.

1. $T_1 :$ Obviously.

2. $T_2 :$ Let $u > 0$ and $T(u, v, u, u, u + v, 0) = u - k \max\{u, v\} \leq 0$. If $u \geq v$, then $u \leq ku$, which is a contradiction. Thus $u < v$ and $u \leq kv$. Similarly, let $u > 0$ and $T(u, v, u, v, 0, u + v) \leq 0$ then we have $u \leq kv$. If $u = 0$, then $u \leq kv$. Thus $T_2$ is satisfying with $f(t) = kt$.

3. $T_3 : T(u, u, 0, u, u) = T(u, u, u, 0, u) = u - ku > 0, \forall u > 0$. 

Example 4. \( T(t_1, \ldots, t_6) = t_1 - \psi(\max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}) \), where \( \psi : R_+ \to R_+ \) increasing and \( \psi(0) = 0, \psi(t) < t \) for \( t > 0 \).

\( T_1 \): Obviously.

\( T_2 \): Let \( u > 0 \) and \( T(u, v, u, u + v, 0) = u - \psi(\max\{u, v\}) \leq 0 \). If \( u \geq v \), then \( u - \psi(u) \leq 0 \), which is a contradiction. Thus \( u < v \) and \( u \leq \psi(v) \). Similarly, let \( u > 0 \) and \( T(u, v, u, 0, u + v) \leq 0 \) then we have \( u \leq \psi(v) \). If \( u = 0 \) then \( u \leq \psi(v) \).

Thus \( T_2 \) is satisfying with \( f = \psi \).

\( T_3 \): \( T(u, u, 0, u, u) = T(u, u, 0, 0, u) = u - \psi(u) > 0, \forall u > 0 \).

Example 5. \( T(t_1, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6 \), where \( a > 0, b, c, d \geq 0 \), \( a + b + c < 1 \), and \( a + b + d < 1 \).

\( T_1 \): Obviously.

\( T_2 \): Let \( u > 0 \) and \( T(u, v, u, u + v, 0) = u^2 - u(au + bv + cu) \leq 0 \). Then \( u \leq (\frac{b+c}{a+b})v = h_1v \). Similarly, let \( u > 0 \) and \( T(u, v, u, v, 0, u + v) \leq 0 \) then we have \( u \leq (\frac{b+c}{a+b})v = h_2v \). If \( u = 0 \), then \( u \leq h_2v \). Thus \( T_2 \) is satisfying with \( f(t) = \max\{h_1, h_2\}t \).

\( T_3 \): \( T(u, u, 0, u, u) = u^2 - u(au + cu) - du^2 = u^2(1 - a - c - d) > 0, \forall u > 0 \) and \( T(u, u, 0, u, u) = u^2(1 - a - b - d) > 0, \forall u > 0 \).

### 3 Common fixed point theorems

We need the following lemma for the proof of our main theorems.

**Lemma 3 ([15])**. For any \( t > 0, f(t) < t \) if and only if \( \lim_{n \to \infty} f^n = 0 \), where \( f^n \) denotes the composition of \( f \) \( n \)-times with itself.

Now we give one of the our main theorem.

**Theorem 1.** Let \( A, B \) be mappings of a metric space \( (X, d) \) into itself and \( F, G \) be mappings from \( X \) into \( B(X) \) such that

\[
F(X) \subseteq B(X) \quad \text{and} \quad G(X) \subseteq A(X).
\]

Also, the mappings \( A, B, F \) and \( G \) are satisfying the following inequality

\[
T(\delta(Fx, Gy), d(Ax, By), \delta(Ax, Fx), \delta(By, Gx), D(Ax, Gy), D(By, Fx)) \leq 0
\]

where \( T \) satisfies conditions \( T_1, T_2 \) and \( T_3 \). Suppose that any one of \( A(X) \) or \( B(X) \) is complete. If both pairs \( (A, F) \) and \( (B, G) \) are weakly compatible, then there exists a unique \( z \in X \) such that \( \{z\} = \{Az\} = \{Bz\} = Fz = Gz \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). From (3.1), we choose a point \( x_1 \) in \( X \) such that \( Bx_1 \in Fx_0 = Z_0 \). For this point \( x_1 \) there exists a point \( x_2 \) in \( X \) such that \( Ax_2 \in Gx_1 = Z_1 \), and so on. Continuing in this manner we can define a sequence \( \{x_n\} \) as follows

\[
Bx_{2n+1} \in Fx_{2n} = Z_{2n}, \quad Ax_{2n+2} \in Gx_{2n+1} = Z_{2n+1}
\]
for $n = 0, 1, 2, ...$. For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$, for $n = 0, 1, 2, ...$. From (3.2) and (3.3), we have

$$T(\delta(Fx_{2n}, Gx_{2n+1}), d(Ax_{2n}, Bx_{2n+1}), \delta(Ax_{2n}, Fx_{2n}),$$
$$\delta(Bx_{2n+1}, Gx_{2n+1}), D(Ax_{2n}, Gx_{2n+1}), D(Bx_{2n+1}, Fx_{2n}) \leq 0$$

and so we have

$$T(V_{2n}, V_{2n-1}, V_{2n-1}, V_{2n}, V_{2n} + V_{2n}, 0) \leq 0.$$  

From $T_2$, there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that

$$V_{2n} \leq f(V_{2n-1}). \quad (3.4)$$

Similarly

$$T(\delta(Fx_{2n+2}, Gx_{2n+1}), d(Ax_{2n+2}, Bx_{2n+1}), \delta(Ax_{2n+2}, Fx_{2n+2}),$$
$$\delta(Bx_{2n+1}, Gx_{2n+1}), D(Ax_{2n+2}, Gx_{2n+1}), D(Bx_{2n+1}, Fx_{2n+2}) \leq 0$$

and so we have

$$T(V_{2n+1}, V_{2n}, V_{2n+1}, V_{2n}, V_{2n} + V_{2n+1}) \leq 0.$$  

From $T_2$, we have

$$V_{2n+1} \leq f(V_{2n}). \quad (3.5)$$

From (3.4) and (3.5) we have, $V_n \leq f^n(V_0)$ and from Lemma 3, we have $\lim_{n \to \infty} V_n = 0$.

Thus, if $z_n$ is an arbitrary point in the set $Z_n$ for $n = 0, 1, 2, ...$, it follows that

$$d(z_n, z_{n+1}) \leq \delta(Z_n, Z_{n+1}) = V_n \to 0$$

as $n \to \infty$. Therefore the sequence $\{z_n\}$ and hence any subsequence thereof, is a Cauchy sequence in $X$.

Now suppose $B(X)$ is complete. Let $\{x_n\}$ be the sequence defined by (3.3). Since $Bx_{2n+1} \in Fx_{2n} = Z_{2n}$, for $n = 0, 1, 2, ...$, we have

$$d(Bx_{2n+1}, Bx_{2n+1}) \leq \delta(Z_{2m}, Z_{2n}) < \epsilon$$

for $m, n \geq n_0, n_0 = 1, 2, 3, ...$. Therefore by the above, the sequence $\{Bx_{2n+1}\}$ is Cauchy and hence $Bx_{2n+1} \to p = Bq \in B(X)$ for some $q \in X$. But $Ax_{2n} \in Gx_{2n-1} = Z_{2n-1}$ by (3.3), so that we have

$$d(Ax_{2n}, Bx_{2n+1}) \leq \delta(Z_{2n-1}, Z_{2n}) = V_{2n-1} \to 0$$

as $n \to \infty$. Consequently $Ax_{2n} \to p$. Moreover, we have for $n = 1, 2, 3, ...$

$$\delta(Fx_{2n}, p) \leq \delta(Fx_{2n}, Ax_{2n}) + d(Ax_{2n}, p) = V_{2n} + d(Ax_{2n}, p).$$
Therefore, \( \delta(Fx_{2n}, p) \to 0 \). In like manner it follows that \( \delta(Gx_{2n-1}, p) \to 0 \).

Since, for \( n = 1, 2, 3, ... \)

\[
T(\delta(Fx_{2n}, Gq), d(Ax_{2n}, Bq), \delta(Ax_{2n}, Fx_{2n}), \\
\delta(Bq, Gq), D(Ax_{2n}, Gq), D(Bq, Fx_{2n}) \leq 0
\]

and by \( T_1 \) we have

\[
T(\delta(Fx_{2n}, Gq), d(Ax_{2n}, Bq), \delta(Ax_{2n}, Fx_{2n}), \\
\delta(Bq, Gq), \delta(Ax_{2n}, Gq), \delta(Bq, Fx_{2n}) \leq 0 .
\]

We get as \( n \to \infty \)

\[
T(\delta(p, Gq), 0, 0, \delta(p, Gq), 0, 0, \delta(p, Gq)) \leq 0
\]

and by \( T_2 \) there exist an upper semicontinuous and non-decreasing function \( f : R_+ \to R_+ \), \( f(0) = 0, f(t) < t \) for \( t > 0 \), we have \( \delta(p, Gq) \leq f(0) = 0 \) and \( \{p\} = Gq = \{Bq\} \).

But \( G(X) \subseteq A(X) \), so \( r \in X \) exists such that \( \{Ar\} = Gq = \{Bq\} \). Now if \( Fr \neq Gq, \delta(Fr, Gq) \neq 0 \) so that we have

\[
T(\delta(Fr, Gq), d(Ar, Bq), \delta(Ar, Fr), \delta(Bq, Gq), D(Ar, Gq), D(Bq, Fr)) \leq 0
\]

so we have

\[
T(\delta(Fr, p), 0, \delta(Fr, p), 0, 0, d(Fr, p)) \leq 0
\]

and by \( T_2 \) we have \( \delta(Fr, p) \leq f(0) = 0 \). It follows that \( Fr = \{p\} = Gq = \{Ar\} = \{Bq\} \).

Since \( Fr = \{Ar\} \) and the pair \( (A, F) \) is weakly compatible, we obtain \( Fp = FAr = AFr = Ap \).

Now using (3.2) we have

\[
T(\delta(Fp, Gq), d(Ap, Bq), \delta(Ap, Fp), \delta(Bq, Gq), D(Ap, Gq), D(Bq, Fp)) \leq 0
\]

and so

\[
T(\delta(Fp, p), d(Fp, p), 0, 0, \delta(Fp, p), \delta(Fp, p)) \leq 0
\]

which is a contradiction to \( T_4 \). Thus, \( \delta(Fp, p) = 0 \) and \( Fp = \{p\} = \{Ap\} \). Similarly, \( \{p\} = Gp = \{Bp\} \) if the pair \( (B, G) \) is weakly compatible. Therefore we obtain \( \{p\} = \{Ap\} = \{Bp\} = Fp = Gp \).

To see the \( p \) is unique, suppose that \( \{p'\} = \{Ap'\} = \{Bp'\} = Fp' = Gp' \) for some \( p' \in X \), then

\[
T(\delta(Fp, Gp'), d(Ap, Bp'), \delta(Ap, Fp), \delta(Bp', Gp'), D(Ap, Gp'), D(Bp', Fp)) \leq 0
\]

and so

\[
T(\delta(p, p'), d(p, p'), 0, 0, \delta(p, p'), \delta(p, p')) \leq 0
\]

which is a contradiction to \( T_4 \). Thus we have \( p = p' \).
Remark 1. If we use Example 2 and Theorem 1, we get Theorem 2. 1 of Ahmed [1]. If we choose $A, B, F$ and $G$ are single valued mappings in Theorem 1 with Example 3, we get an improved version of Theorem 3. 1 of Kang and Kim [8]. Similarly, many results can obtain by Theorem 1 and some examples.

Now we give the other our main theorem.

**Theorem 2.** Let $A, B$ be mappings of a complete metric space $(X, d)$ into itself and $F, G$ be mappings from $X$ into $\mathcal{B}(X)$ such that (3.1) holds. Also, the mappings $A, B, F$ and $G$ are satisfying the following inequality

$$T(\delta(Fx, Gy), d(Ax, By), \frac{1}{2}\delta(Ax, Fx), \frac{1}{2}\delta(By, Gy), D(Ax, Gy), D(By, Fx))$$

$$\leq 0 \quad (3.6)$$

where $T$ satisfies conditions $T_1, T_2$ and $T_3$. Suppose that the pairs $(A, F)$ and $(B, G)$ are compatible of type $(I)$ and $A$ or $B$ is continuous, then there exists $z \in X$ such that $\{z\} = \{Az\} = \{Bz\} = Fz = Gz$.

**Proof.** Let the sequence $\{x_n\}$ is defined by (3.3). Similar operation in proof of Theorem 1, we have $Ax_{2n}, Bx_{2n+1}, Fx_{2n}, Gx_{2n+1} \to p$ for some $p \in X$ since $X$ is complete.

Now suppose that $B$ is continuous. Then the pair $(B, G)$ is compatible of type $(I)$, we have

$$d(p, Bp) \leq \lim_{n \to \infty} \delta(p, GBx_{2n+1})$$

and $BBx_{2n+1} \to Bp$. Setting $x = x_{2n}$ and $y = Bx_{2n+1}$ in (3.6) we have

$$T(\delta(Fx_{2n}, GBx_{2n+1}), d(Ax_{2n}, BBx_{2n+1}), \frac{1}{2}\delta(Ax_{2n}, Fx_{2n}),$$

$$\frac{1}{2}\delta(BBx_{2n+1}, GBx_{2n+1}), D(Ax_{2n}, GBx_{2n+1}), D(BBx_{2n+1}, Fx_{2n}) \leq 0$$

then taking limit superior we have

$$T(\lim_{n \to \infty} \delta(p, GBx_{2n+1}), d(p, Bp), 0, \frac{1}{2}\lim_{n \to \infty} \delta(Bp, GBx_{2n+1}),)$$

$$\lim_{n \to \infty} \delta(p, GBx_{2n+1}), d(p, Bp)) \leq 0$$

and so

$$T(\lim_{n \to \infty} \delta(p, GBx_{2n+1}), \lim_{n \to \infty} \delta(p, GBx_{2n+1}), 0, \lim_{n \to \infty} \delta(p, GBx_{2n+1}),)$$

$$\lim_{n \to \infty} \delta(p, GBx_{2n+1}), \lim_{n \to \infty} \delta(p, GBx_{2n+1}) \leq 0$$

which is a contradiction to $T_3$ if $\lim_{n \to \infty} \delta(p, GBx_{2n+1}) \neq 0$. Thus we have

$$\lim_{n \to \infty} \delta(p, GBx_{2n+1}) = 0$$

and so from (3.7) $p = Bp$. 


Again setting $x = x_{2n}$ and $y = p$ in (3.6) and allowing $n \to \infty$ we have

$$T(\delta(p, Gp), 0, 0, \frac{1}{2} \delta(p, Gp), \delta(p, Gp), 0) \leq 0$$

with $T_1$ and $T_2$, there exist an upper semicontinuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, we have $\delta(p, Gp) \leq f(0) = 0$, thus $\{p\} = \{Bp\} = Gp$.

Since $G(X) \subseteq A(X)$, there exists a point $q \in X$ such that $\{p\} = \{Bp\} = Gp = \{Aq\}$.

Setting $x = q$ and $y = p$ in (3.6), we have

$$T(\delta(Fq, p), 0, \frac{1}{2} \delta(p, Fq), 0, 0, \delta(p, Fq)) \leq 0$$

implies from $T_1$ and $T_2$ we have $\delta(p, Fq) \leq f(0) = 0$ and so $\{p\} = Fq$.

Since $(A, F)$ is compatible of type (I) and $\{Aq\} = \{Fq\} = p$, then using Proposition 1, we have $\delta(Fq, AAq) \leq \delta(Fq, FAq)$ and so

$$d(p, Ap) \leq \delta(p, Fp). \quad (3.8)$$

Again setting $x = p = y$ in (3.6) we have

$$T(\delta(Fp, p), \delta(Fp, p), \frac{1}{2} \delta(Fp, p), 0, \delta(Fp, p), \delta(Fp, p)) \leq 0$$

which is a contradiction to $T_3$ if $\delta(Fp, p) \neq 0$. Thus we have $Fp = \{p\}$ and from (3.8) we have $p = Ap$ so $\{p\} = \{Ap\} = \{Bp\} = Fp = Gp$.

The other case, $A$ is continuous, can be disposed of following a similar argument as above.

**Acknowledgements.** The authors are grateful to the referees for their valuable comments in modifying the first version of this paper.

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