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ON CLASSES OF STARLIKE AND CONVEX MEROMORPHICALLY MULTIVALENT FUNCTIONS INVOLVING COEFFICIENT INEQUALITIES

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Abstract

Let $T_r^*(p)$ be the class of multivalent meromorphic functions $f(z)$ in a punctured disk $\mho_r^*$ with a simple pole of order p at the center of disk is defined. We considered the class of starlike multivalent meromorphic and convex multivalent meromorphic functions $S_\alpha^*(p), C_\alpha^*(p)$, respectively. The coefficient properties are obtained and the starlikeness and convexity of functions in $T_r^*(p)$ are also investigated with some other results.

1. Introduction

Let $T_r^*(p)$ denote the class of functions of the form

$$f(z) = ez^{-p} + {}_2F_1(a,b;c;z) - \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} \tag{1}$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n; \quad |z| < 1,$$

$a,b,c \in \mathbb{C}$ with $c \neq 0,-1,-2,\cdots,(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1,n-1), c > b > 0$, $c > a+b, t_{n-p+1} = \frac{(a,n-p+1)(b,n-p+1)}{(c,n-p+1)(n-p+1)!}, p \in I\!N$ and $e > 0$.

Also

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Thus

$$f(z) = ez^{-p} + \sum_{n=2p}^{\infty} t_{n-p+1} z^{n-p+1}, \quad |z| < 1, \tag{2}$$

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which are analytic and multivalent in $\mathcal{U}_r^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < r \leq 1\}$ having simple pole at the origin.

A function $f(z) \in T_r^*(p)$ is said to be multivalent meromorphically starlike of order α if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad 0 \leq \alpha < p, \quad z \in \mathcal{U}_r^* \tag{3}$$

denoted by $f(z) \in S_\alpha^*(p)$.

Furthermore, a function $f(z) \in T_r^*(p)$ is said to be multivalent meromorphically convex of order α if

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \quad 0 \leq \alpha < p, z \in \mathcal{U}_r^* \tag{4}$$

denoted by $f(z) \in C_\alpha^*(p)$. We know that

$$f(z) \in C_\alpha^*(p) \ \text{ iff } \ -zf'(z) \in S_\alpha^*(p). \tag{5}$$

There are many authors who have studied the various interesting properties of these classes, M. K. Aouf [1], Aouf and Srivastava [2], Kulkarni, Naik and Srivastava [3], Mogra [5], S. Owa and N. Pascu [6]. The present paper is essentially motivated by the paper due to Owa and Pascu [6], Ozaki [7] and T. Mathur [4].

2. Coefficient Inequalities for Functions

Theorem 2.1 : Let $f(z) \in T_r^*(p)$, then $f(z) \in S_\alpha^*(p)$ if

$$\sum_{n=2p}^{\infty} (k + n - p + 1 + |2\alpha - k + n - p + 1|)|t_{n-p+1}|r^{n+1} < 2e(p - \alpha) \tag{6}$$

for $0 \leq \alpha < p$ and $\alpha < k \leq p, p \in I\!N, z \in \mathcal{U}_r^*$.

Proof : Let $f(z) \in T_r^*(p)$, then we have

$$\begin{aligned}
&|zf'(z) + kf(z)| - |zf'(z) + (2\alpha - k)f(z)| \\
&= |(k-p)ez^{-p} + \sum_{n=2p}^{\infty}(k + n - p + 1)t_{n-p+1}z^{n-p+1}| \\
&-|(2\alpha - k - p)ez^{-p} + \sum_{n=2p}^{\infty}(2\alpha - k + n - p + 1)t_{n-p+1}z^{n-p+1}|.
\end{aligned}$$

Therefore, by (6) we get

$$\begin{aligned}
&r^p|zf'(z) + kf(z)| - r^p|zf'(z) + (2\alpha - k)f(z)| \\
&\leq 2e(\alpha - p) + \sum_{n=2p}^{\infty}(k + n - p + 1 + |2\alpha - k + n - p + 1|)|t_{n-p+1}|r^{n+1} \leq 0
\end{aligned}$$

then we have $\left|\frac{zf'(z)+kf(z)}{zf'(z)+(2\alpha-k)f(z)}\right| \leq 1$. Thus $f(z) \in S_\alpha^*(p)$.

By taking $k = p$ in Theorem 2.1, we have
Corollary 2.1 : If $f \in T_r^*(p)$ satisfies

$$\sum_{n=2p}^{\infty} (n-p+\alpha+1)|t_{n-p+1}|r^{n+1} \leq e(p-\alpha) \tag{7}$$

for some $\frac{p}{2} \leq \alpha < p$, then $f \in S_\alpha^*(p)$.
Corollary 2.2 : If $f \in T_r^*(p)$ and $t_{n-p+1} = |t_{n-p+1}|e^{-\frac{n+1}{2\pi}i}$, then $f(z) \in S_\alpha^*(p)$ if and only if

$$\sum_{n=2p}^{\infty} (n-p+\alpha+1)|t_{n-p+1}|r^{n+1} \leq e(p-\alpha) \tag{8}$$

where $\frac{p}{2} \leq \alpha < p$.
Proof : By Corollary 2.1, we have $f(z) \in S_\alpha^*(p)$. Conversely, assume that $f(z) \in S_\alpha^*(p)$, then

$$\begin{aligned} Re\left(-\frac{zf'(z)}{f(z)}\right) &= Re\left(-\frac{-pez^{-p} + \sum\limits_{n=2p}^{\infty}(n-p+1)t_{n-p+1}z^{n-p+1}}{ez^{-p} + \sum\limits_{n=2p}^{\infty} t_{n-p+1}z^{n-p+1}}\right) \\ &= Re\left(\frac{pe - \sum\limits_{n=p}^{\infty}(n-p+1)t_{n-p+1}z^{n+1}}{e + \sum\limits_{n=2p}^{\infty} t_{n-p+1}z^{n+1}}\right) > \alpha. \end{aligned}$$

Let $z = re^{\frac{1}{2\pi}i}$, then we have $t_{n-p+1}z^{n+1} = |t_{n-p+1}|r^{n+1}$. Therefore

$$pe - \sum_{n=2p}^{\infty}(n-p+1)|t_{n-p+1}|r^{n+1} \geq \alpha e + \alpha \sum_{n=2p}^{\infty} |t_{n-p+1}|r^{n+1}.$$

Thus, the last inequality is equivalent to (8).
Example 2.1 : Let

$$f(z) = \frac{e}{z^p} + t_{p+1}z^{p+1} + \left(\frac{e(p-\alpha) - (p+\alpha+1)|t_{p+1}|}{n-p+\alpha+1}\right) e^{i\theta} z^{n-p+1}$$

where $t_{p+1} = \frac{(a,p+1)(b,p+1)}{(c,p+1)(p+1)!}$, for all a, b, c defined in (1) thus

$$\begin{aligned} f(z) &= \frac{e}{z^p} + \frac{(a,p+1)(b,p+1)}{(c,p+1)(p+1)!} z^{p+1} && (1) \\ &\quad + \left(\frac{e(p-\alpha) - (p+\alpha+1)}{n-p+\alpha+1}\left|\frac{(a,p+1)(b,p+1)}{(c,p+1)(p+1)!}\right|\right) e^{i\theta} z^{n-p+1} && (2) \end{aligned}$$

for some real θ, with $\frac{p}{2} \leq a < p$, then $f(z) \in S_\alpha^*(p)$.

Remark 2.1 : If $f(z) \in T_r^*(p)$ with $a = b = 0$, then Corollary 2.2 holds true for $0 \leq \alpha < p$.
Corollary 2.3 : If $f(z) \in T_r^*(p)$ given by (2) with $t_{n-p+1} \geq 0$, then $f(z) \in S_\alpha^*(p)$ if and only if

$$\sum_{n=2p}^{\infty} (n-p+\alpha+1) t_{n-p+1} r^{n+1} \leq e(p-\alpha) \tag{10}$$

for some $\frac{p}{2} \leq \alpha < p$.
Theorem 2.2 : Let $f(z) \in T_r^*(p)$ defined by (2). If $f(z)$ satisfies

$$\sum_{n=2p}^{\infty} (n-p+1)(n-p+1+\alpha)|t_{n-p+1}| r^{n+1} \leq e(p-\alpha) \tag{11}$$

then $f(z) \in C_\alpha^*(p)$, for $\frac{p}{2} \leq \alpha < p, p \in I\!N, z \in \mathcal{U}_r^*$.
Proof : Since $-zf'(z) \in S_\alpha^*(p)$ if and only if $f(z) \in C_\alpha^*(p)$ then

$$\begin{aligned} -zf'(z) &= -\left[-pez^{-p} + \sum_{n=2p}^{\infty} (n-p+1) t_{n-p+1} z^{n-p+1}\right] \\ &= p\left[\frac{e}{z^p} + \sum_{n=2p}^{\infty} \frac{(p-n-1)}{p} t_{n-p+1} z^{n-p+1}\right]. \end{aligned}$$

By (7), we get $-zf'(z) \in S_\alpha^*(p)$, if

$$\sum_{n=2p}^{\infty} (n-p+\alpha+1)(n-p+1)|t_{n-p+1}| r^{n+1} \leq e(p-\alpha).$$

Thus, $f(z) \in C_\alpha^*(p)$.
Corollary 2.4 : If $f(z) \in T_r^*(p)$ be given by (2) with $t_{n-p+1} = |t_{n-p+1}| e^{-\frac{n+1}{2\pi} i}$, then $f(z) \in C_\alpha^*(p)$ if and only if

$$\sum_{n=2p}^{\infty} (n-p+1)(n-p+1+\alpha)|t_{n-p+1}| r^{n+1} \leq e(p-\alpha) \tag{12}$$

for $0 \leq \alpha < p$.
Corollary 2.5 : If $f(z) \in T_r^*(p)$ be given by (2) with $t_{n-p+1} \geq 0$, then $f(z) \in C_\alpha^*(p)$ if and only if

$$\sum_{n=2p}^{\infty} (n-p+1)(n-p+1+\alpha) t_{n-p+1} r^{n+1} \leq e(p-\alpha). \tag{13}$$

Example 2.2 : Let

$$\begin{aligned} f(z) &= \frac{e}{z^p} + \frac{(a,p+1)(b,p+1)}{(c,p+1)(p+1)!} z^{p+1} \\ &\quad + \left(\frac{e(p-\alpha)-(p+\alpha+1)}{(n-p+1)(n-p+1+\alpha)} \left|\frac{(a,p+1)(b,p+1)}{(c,p+1)(p+1)!}\right|\right) e^{i\theta} z^{n-p+1} \end{aligned}$$

for some real θ, with $0 \leq \alpha < p$, then $f(z) \in C^*_\alpha(p)$.
Remark 2.2 : If $f(z) \in T^*_r(p)$ given by (2) with $a = b = 0$, then Corollary 2.4 holds true for $0 \leq \alpha < p$.

3. Starlikeness and Convexity of Functions

Theorem 3.1 : Let $f(z) \in T^*_r(p)$, then $f(z) \in S^*_\alpha(p)$ for $0 \leq r < r_0$, where r_0 is the smallest positive roof of the equation

$$(p+1+\alpha)|t_{p+1}|r^{2p+3} - \delta(\sqrt{p+2})r^{2p+2} - (p+1+\alpha)|r^{2p+1} - e(p-\alpha)r^2 + e(p-\alpha) = 0 \tag{14}$$

where

$$\delta = \sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)|t_{n-p+1}|^2} + \alpha\sqrt{\sum_{n=2p+1}^{\infty}\frac{1}{n-p+1}|t_{n-p+1}|^2}. \tag{15}$$

Proof : We know that

$$\sum_{n=2p}^{\infty}(n-p+1+\alpha)|t_{n-p+1}|r^{n+1} = (p+1+\alpha)|t_{p+1}|r^{2p+1} + \sum_{n=2p+1}^{\infty}(-p+1+\alpha)|t_{n-p+1}|r^{n+1}$$

So, by Cauchy inequality, we have

$$\sum_{n=2p}^{\infty}(n-p+1+\alpha)|t_{n-p+1}|r^{n+1} \leq (p+1+\alpha)|t_{p+1}|r^{2p+1}$$

$$+\sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)|t_{n-p+1}|^2}\sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)r^{2n+2}}$$

$$+\alpha\sqrt{\sum_{n=2p+1}^{\infty}\frac{1}{n-p+1}|t_{n-p+1}|^2}\sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)r^{2n+2}}$$

$$\leq (p+1+\alpha)|t_{p+1}|r^{2p+1} + \sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)r^{2n+2}}$$

$$\left(\sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)|t_{n-p+1}|^2} + \alpha\sqrt{\sum_{n=2p+1}^{\infty}\frac{1}{n-p+1}|t_{n-p+1}|^2}\right)$$

$$\leq (p+1+\alpha)|t_{p+1}|r^{2p+1} + \sqrt{\frac{(p+2)r^{2(2p+2)}}{(1-r^2)^2}}\delta$$

where δ defined above

$$\leq (p+1+\alpha)|t_{p+1}|r^{2p+1} + \frac{\sqrt{p+2}r^{2p+2}}{1-r^2}\delta < e(p-\alpha)$$

by using Corollary 2.2.

Thus, $f(z) \in S^*_\alpha(p)$ for $0 \leq r < r_0$.

Putting $t_{p+1} = 0$ we obtain the following

Corollary 3.1 : A function $f(z) \in T^*_r(p)$ with $t_{p+1} = 0$ belongs to the class $S^*_\alpha(p)$ for $0 \leq r < r_0$ where r_0 is the smallest positive root of the equation

$$\delta(\sqrt{p+2}r^{2p+2}) + e(p-\alpha)r^2 = e(p-\alpha) \tag{16}$$

where δ is given by (15).

Next we consider the problems of radius for convexity of functions $f(z) \in T^*(p)$.

Theorem 3.2 ; Let $f(z) \in T^*_r(p)$, then $f(z) \in C^*_\alpha(p)$ for $0 \leq r < r_1$, where r_1 is the smallest root of the equation

$$(p+1+\alpha)|t_{p+1}|r^{2p+3} - \sigma(\sqrt{p+2})r^{2p+2} - (p+1+\alpha)|t_{p+1}|r^{2p+1} \tag{3}$$
$$-e(p-\alpha)r^2 + e(p-\alpha) = 0 \tag{4}$$

where

$$\sigma = \sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)^3|t_{n-p+1}|} + \alpha\sqrt{\sum_{n=2p+1}^{\infty}(n-p+1)|t_{n-p+1}|^2}. \tag{18}$$

Proof : By using Theorem 3.1 and Theorem 2.2

So we can omit the details.

Putting $t_{p+1} = 0$ in Theorem 3.2 we can obtain the following

Corollary 3.2 : Let $f(z) \in T^*_(p)$ with $t_{p+1} = 0$ such that $f(z) \in C^*_\alpha(p)$ for $0 \leq r < r_1$, where r_1 is the smallest positive root of the equation

$$\sigma(\sqrt{p+2}r^{2p+2}) + e(p-\alpha)r^2 = e(p-\alpha) \tag{19}$$

where σ is given by (18).

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