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WEAKLY COMPATIBLE MAPPINGS AND ALTMAN TYPE CONTRACTION

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Abstract

In this paper, a common fixed point theorem for two pairs of weakly compatible mappings satisfying Altman type contraction in metric space is proved. Our result extends and improves several known results.

1. Introduction

Let A and S be two self-maps of a metric space (X, d). Sessa [14] defined A and S to be *weakly commuting* if

$$d(ASx, SAx) \leq d(Ax, Sx), \tag{1}$$

for all $x \in X$.

Jungck [5] defined A and S to be *compatible* if

$$lim_n d(ASx_n, SAx_n) = 0, \tag{2}$$

whenever there exists a sequence $\{x_n\}$ in X such that $lim_n Sx_n = lim_n Ax_n = t$, for some $t \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible but neither implication is reversible (see, for instance, Example 1 of Sessa and Fisher [15] and Example 2.2 of Jungck [5]).

In 1993, Jungck, Murthy and Cho [7] defined A and S to be *compatible of type (A)* if

$$lim_n d(ASx_n, SSx_n) = lim_n d(SAx_n, AAx_n) = 0, \tag{3}$$

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whenever there exists a sequence $\{x_n\}$ in X such that $lim_n Sx_n = lim_n Ax_n = t$, for some $t \in X$. Clearly, weakly commuting mappings are compatible of type (A) but this implication is not reversible (see, for instance, Examples 2.1 and 2.2 of Jungck, Murthy and Cho [7]). It follows from [7] that the notions of compatible maps and compatible of type (A) are independent to each other (see also Examples 4.8 and 4.9 below).

In [9], the compatible maps of type (P) was introduced and compared with compatible maps and compatible maps of type (A). The mappings A and S of a metric space (X, d) are called *compatible of type (P)* if

$$lim_n d(AAx_n, SSx_n) = 0,$$

whenever there exists a sequence $\{x_n\}$ in X such that $lim_n Sx_n = lim_n Ax_n = t$, for some $t \in X$.

It is easy to verify that compatible maps of type (P) is also independent to compatible maps and compatible maps of type (A) (see examples in Pathak et. al. [9]).

In 1998, Jungck and Rhoades [6] defined A and S to be *weakly compatible* if

$$SAx = ASx \quad \text{whenever} \quad Ax = Sx. \tag{4}$$

The example of Popa [10, p.34] shows that weakly compatible maps need not be compatible or compatible of type (A) or compatible of type (P) (see also Example 4.7 below).

The following Lemma asserts that the concept of weakly compatible mapping is more general than the concepts of compatibility and compatibility of type (A) and (P). So we will use the weakly compatible mapping in our theorems.

Lemma 1.1 [5] (resp. [7], [9]). *Let A and S be compatible (resp. compatible of type (A), compatible of type (P)) self-mappings of a metric space (X, d). If $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.*

Thus $Ax = Sx$, for some $x \in X$ with compatibility (compatible of type (A) or compatible of type (P)) implies that

$$ASx = SSx = SAx = AAx. \tag{5}$$

2. Altman condition

In 1975, Altman [1] introduced a generalized contraction. Let (X, d) be a metric space and $f : X \to X$. Then f is called a *generalized contraction* if, for all $x, y \in X$,

$$d(fx, fy) \leq G(d(x, y)), \tag{6}$$

where G is a real-valued non-decreasing function satisfying the following conditions:
(a) $0 < G(t) < t$, for all $t > 0$, $G(0) = 0$,
(b) $g(t) = \frac{t}{t-G(t)}$ is non-increasing on $(0, \infty)$,
(c) $\int_0^{t_1} g(t)dt < +\infty$ for each $t_1 > 0$.
Henceforth, we shall denote by $\mathbb{R}, R_+$ and $\mathbb{N}$ the set of real numbers, the set of nonnegative real numbers and the set of natural numbers, respectively. Let $\mathcal{G}_0$ denotes the family of real-valued functions G on set $D - \{0\}$, where $D = \text{cl(ran } d)$, that is,

$$\mathcal{G}_0 = \{G : G(t) \in \mathbb{R}, \quad \forall\, t \in D - \{0\}\}.$$

The family $\mathcal{G}_0$ of $G's$ defined above is same as in Carbone et. al. [3].

After Altman's theorem on metric space, Carbone and Singh [2], Rhoades and Watson [12], Watson, Meade and Norris [16] etc. proved fixed point theorems for generalized contractions. We will use more general contraction condition than above.

3. Preliminaries

The following theorem was proved by Sahu and Dewangan [13].

Theorem A. *Let S and T be self-mappings of a complete metric space (X, d). Let $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ be sequences of self-maps on X satisfying the following conditions:*

(i) $A_iX \subseteq TX$, $B_iX \subseteq SX$,

(ii) $d(A_ix, B_iy) \leq G(m(x, y))$,

for all $x, y \in X$, where $G \in \mathcal{G}_0$, the family of real-valued functions G, and

$m(x, y) = max\{d(Sx, Ty), d(A_ix, Sx), d(B_iy, Ty), \frac{1}{2}[d(B_iy, Sx) + d(A_ix, Ty)]\}$,

(iii) one of A_i, B_i, S or T is continuous and

(iv) A_i and S and B_i and T are compatible of type (A).

Then each A_i, B_i, S and T have a unique common fixed point in X.

Let $\mathcal{F}$ is the set of all functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that
(*) f is isotone, i.e., if $t_1 \leq t_2$ then $f(t_1) \leq f(t_2)$, for all $t_1, t_2 \in \mathbb{R}_+$,
(**) f is upper semi-continuous,
(***) $f(t) < t$, for each $t > 0$.

In the light of above notation the following theorem was proved by Popa and Pathak [11].

Theorem B. *Let A, B, S and T be self-mappings of a complete metric space (X, d) satisfying the conditions:*

(i) $AX \subseteq TX, \quad BX \subseteq SX,$

(ii) the inequality

$$[1 + p\, d(Sx, Ty)]d(Ax, By) \leq p\, max\{d(Ax, Sx)d(By, Ty),\ d(Ax, Ty)d(By, Sx)\}$$

$$+ f\big(max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \tfrac{1}{2}[d(By, Sx) + d(Ax, Ty)]\}\big),$$

holds for all $x, y \in X$, where $p \geq 0$ and $f \in \mathcal{F}$,

(iii) one of A, B, S or T is continuous, and

(iv) A and S are compatible of type (A) and B and T are compatible of type (A).

Then A, B, S and T have common fixed point z. Further, z is the unique common fixed point of A and S and of B and T.

Our aim of this paper is to prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying Altman type contraction condition and to derive few known results as corollaries. In our main result we have dropped the completeness of whole space X in Theorem B, by choosing the range space of one of the four mappings complete; relaxed the duality of conditions on mappings in compatibility of type (A) by taking weakly compatible mappings and dropped requirement of the continuity of one of the four mappings.

4. Main Results

We now state and prove our main theorem.

Theorem 4.1. *Let A, B, S and T be four self-mappings of a metric space (X, d) satisfying the following conditions:*

(i) $AX \subseteq TX, \quad BX \subseteq SX,$

(ii) $[1 + p\, d(Sx, Ty)]d(Ax, By) \leq$

$$p\, max\{d(Ax, Sx)d(By, Ty),\ d(Ax, Ty)d(By, Sx)\} + G\big(m(x, y)\big),$$

for all $x, y \in X$*, where* $p \geq 0$,

$m(x,y) = max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}$,

and $G \in \mathcal{G}_0$ *satisfies the Altman type conditions (a)-(c).*

If one of AX*,* BX*,* SX *or* TX *is a complete subspace of* X*, then*

(iii) (A,S) *have a coincidence point.*

(iv) (B,T) *have a coincidence point.*

Moreover, if both the pairs (A,S) *and* (B,T) *are weakly compatible then* A*,* B*,* S *and* T *have a unique common fixed point.*

Proof. Pick $x_0 \in X$, then by condition (i) we can choose a sequence $\{x_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \ \text{ and } \ Bx_{2n+1} = Sx_{2n+2} = y_{2n+1},$$

for all $n = 0, 1, 2....$

We now show that the sequence $\{y_n\}$ defined above is Cauchy in X.

Let us denote $d(y_n, y_{n+1})$ by d_n, for each $n = 0, 1, 2....$ First we will show that $d_{n+1} \leq G(d_n)$ and then we claim that

$$lim_{n\to\infty} d_n = 0 \tag{7}$$

and then show that $\{y_n\}$ is a Cauchy sequence in X.

For this, putting x_{2n+2} for x and x_{2n+1} for y in (ii) we obtain

$$[1+p\,d_{2n}]d_{2n+1} \leq p\,max\{d_{2n+1}d_{2n},\ 0\} + G(max\{d_{2n}, d_{2n+1}, d_{2n}, \tfrac{1}{2}d(y_{2n}, y_{2n+2})\}).$$

But, from the triangle inequality for metric d, we have

$$\begin{aligned}\tfrac{1}{2}d(y_{2n}, y_{2n+2}) &\leq \tfrac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] = \tfrac{1}{2}[d_{2n} + d_{2n+1}] \\ &\leq max\{d_{2n}, d_{2n+1}\}.\end{aligned}$$

Using this in above, we obtain

$$[1+p\,d_{2n}]d_{2n+1} \leq p\,d_{2n}d_{2n+1} + G(max\{d_{2n},\ d_{2n+1}\}).$$

If we choose d_{2n+1} as "max" in above then we have

$$d_{2n+1} \leq G(d_{2n+1}) < d_{2n+1},$$

a contradiction. Hence,

$$d_{2n+1} \leq G(d_{2n}). \tag{8}$$

Similarly, by setting x_{2n+2} for x and x_{2n+3} for y in (ii) we obtain

$$[1 + p\, d_{2n+1}]d_{2n+2} \leq p\, max\{d_{2n+1}d_{2n+2},\ 0\} + G(max\{d_{2n+1}, d_{2n+1}, d_{2n+2}, \tfrac{1}{2}d(y_{2n+1}, y_{2n+3})\}),$$

i.e., $d_{2n+2} \leq G(max\{d_{2n+1},\ d_{2n+1},\ d_{2n+2},\ \frac{1}{2}d(y_{2n+1}, y_{2n+3})\}) = G(d_{2n+1})$ whence

$$d_{2n+2} \leq G(d_{2n+1}). \tag{9}$$

Unifying (8) and (9) we obtain

$$d_{n+1} \leq G(d_n), \tag{10}$$

for all $n = 0, 1, 2....$

Next, define a sequence $\{t_n\}$ by $t_{n+1} = G(t_n)$ with $t_1 = d_0 = d(y_0, y_1)$. It then follows by assumption (a) that, $0 < G(t_n) = t_{n+1} < t_n < t_1, \forall n \geq 1$.

Furthermore, by induction we will show that $d_n \leq t_{n+1}$. If $n = 1$; then by putting x_2 for x and x_1 for y in condition (ii), we have

$$[1 + p\, d(y_1, y_0)]d(y_2, y_1) \leq$$

$$p\, max\{d(y_2, y_1)d(y_1, y_0),\ 0\} + G\big(max\{d(y_1, y_0), d(y_2, y_1), d(y_1, y_0), \tfrac{1}{2}d(y_2, y_0)\}\big),$$

whence

$$d_1 = d(y_1, y_2) \leq G(max\{d(y_0, y_1), d(y_1, y_2), \tfrac{1}{2}d(y_2, y_0)\}$$

$$\leq G(max\{d(y_0, y_1), d(y_1, y_2)\}) = G(d(y_0, y_1)) = G(d_0) = G(t_1) = t_2;$$

because choosing of $d(y_1, y_2)$ as "max" gives $d_1 \leq G(d_1) < d_1$, which is a contradiction.

Thus for $n = 1$, we observe that $d_1 \leq t_2$.

Assume for some fixed n that, $d_n \leq t_{n+1}$ is true. Then for induction; we have, since G is non-decreasing,

$$d_{n+1} \leq G(d_n) \leq G(t_{n+1}) = t_{n+2}.$$

This follows that $d_n \leq t_{n+1}$, for all $n \in \mathbb{N}$.

Now, by conditions (a)-(c) and $d_n \leq t_{n+1} = G(t_n),\ n \in \mathbb{N}$ which shows that $lim_{n\to\infty} t_n = lim_{n\to\infty} d_n = 0$, it follows that $\{y_n\}$ is a Cauchy sequence.

In fact, if $m, n \in \mathbb{N}$ with $m \geq n$, then

$$d(y_m, y_n) \leq \sum_{k=n}^{m-1} d_k \leq \sum_{k=n}^{m-1} t_{k+1} = \sum_{k=n+1}^{m} t_k$$

$$= \sum_{k=n+1}^{m} \frac{t_k(t_k - t_{k+1})}{t_k - G(t_k)} \leq \sum_{k=n+1}^{m} \int_{t_{k+1}}^{t_k} g(t)dt \leq \int_{t_{n+2}}^{t_m} g(t)dt.$$

Since the last term tends to zero as $n \to \infty$, the sequence $\{t_n\}$ is convergent and $\int_0^{t_1} g(t)dt < +\infty$ for each $t_1 \in D - \{0\}$ and hence $\{y_n\}$ is a Cauchy sequence in X.

Now, we suppose that the range of one of the four mappings is complete.

Case I. Suppose that TX is a complete subspace of X, then the subsequence $\{y_{2n+1}\} = \{Tx_{2n+1}\}$ is Cauchy in TX and hence converges to a limit (say z) in X. Since $\{y_n\}$ is Cauchy and its subsequence $\{y_{2n+1}\}$ is convergent to z, so $\{y_n\}$ also converges to z. Hence its subsequence $\{y_{2n+2}\}$ is also convergent to z. Thus we have

$$lim_{n\to\infty} Tx_{2n+1} = lim_{n\to\infty} Bx_{2n+1} = lim_{n\to\infty} Ax_{2n} = lim_{n\to\infty} Sx_{2n} = z.$$

Let $v \in T^{-1}z$ then $Tv = z$. We claim that $Bv = z$. For this, setting $x = x_{2n}$ and $y = v$ in (ii) we have

$$[1 + p\, d(Sx_{2n}, Tv)]d(Ax_{2n}, Bv) \leq$$

$$p\, max\{d(Ax_{2n}, Sx_{2n})d(Bv, Tv), d(Ax_{2n}, Tv)d(Bv, Sx_{2n})\} + G\big(m(x_{2n}, v)\big),$$

where $m(x_{2n}, v) = max\{d(Sx_{2n}, Tv),\ d(Ax_{2n}, Sx_{2n}),\ d(Bv, Tv),$
$\frac{1}{2}[d(Ax_{2n}, Tv) + d(Bv, Sx_{2n})]\}.$

Letting $n \to \infty$ it yields

$$d(z, Bv) \leq G\big(d(z, Bv)\big) < d(z, Bv),$$

a contradiction. Thus $d(Bv, z) = 0$, so that $Bv = z$. Hence $z = Bv = Tv$, showing that v is a coincidence point of (B, T).

Further, since $BX \subseteq SX$, $Bv = z$ implies that $z \in SX$. Let $u = S^{-1}z$, then $Su = z$. Now we claim that $Au = z$. For this, putting $x = u$ and $y = v$ in (ii) we have

$$[1 + p.0]d(Au, z) \leq p\, max\{0,\ 0\} + G(max\{0,\ d(Au, z),\ 0, \tfrac{1}{2}d(Au, z)\}),$$

i.e., $d(Au,z) \le G(d(Au,z)) < d(Au,z)$
a contradiction. Thus $Au = z$. Hence $z = Au = Su$, showing that u is a coincidence point of (A,S).

Case II. If we assume SX a complete subspace of X, then analogous arguments establishes the earlier conclusion. The remaining two cases are essentially the same as the previous cases. Indeed, if AX is complete, then by (i), $z \in AX \subseteq TX$. Similarly if BX is complete, then $z \in BX \subseteq SX$. Thus pairs (A,S) and (B,T) have coincidence points. Hence in all we have

$$z = Au = Su = Bv = Tv. \tag{11}$$

This proves our assertions (iii) and (iv).

Now, the weak compatibility of (A,S) gives $Az = ASu = SAu = Sz$; i.e.,

$$Az = Sz. \tag{12}$$

Similarly, the weak compatibility of (B,T) gives $Bz = BTv = TBv = Tz$; i.e.,

$$Bz = Tz. \tag{13}$$

To show that z is a coincidence point of A, B, S and T we have to show that $Az = Bz$. For this, putting $x = z$ and $y = z$ in (ii) we have

$$[1 + pd(Sz,Tz)]d(Az,Bz) \le p\,max\{d(Az,Sz)d(Bz,Tz), d(Az,Tz)d(Bz,Sz)\}$$
$$+G\Big(max\{d(Sz,Tz),\ d(Az,Sz),\ d(Bz,Tz),\ \tfrac{1}{2}[d(Az,Tz)+d(Bz,Sz)]\}\Big).$$

Using (12) and (13), we obtain

$$d(Az,Bz) \le G(d(Az,Bz)) < d(Az,Bz),$$

which is a contradiction. Thus $Az = Bz$. Hence from (12) and (13),

$$Az = Sz = Bz = Tz. \tag{14}$$

To show that z is a common fixed point, putting $x = z$ and $y = v$ in (ii) we have

$$[1 + pd(Sz,Tv)]d(Az,Bv) \le p\,max\{d(Az,Sz)d(Bv,Tv),\ d(Az,Tv)d(Bv,Sz)\}$$
$$+G\big(max\{d(Sz,Tv),\ d(Az,Sz),\ d(Bv,Tv),\ \tfrac{1}{2}[d(Az,Tv)+d(Bv,Sz)]\}\big).$$

Using (11) it yields

$$d(Az,z) \le G\big(d(Az,z)\big) < d(Az,z),$$

a contradiction. Thus we obtain

$$z = Az = Bz = Sz = Tz. \tag{15}$$

Uniqueness of common fixed point z follows easily by (ii), as

$$[1+pd(z,z')]d(z,z') \le p\, max\{0,\ d(z,z')d(z,z')\}+ G\Big(max\{d(z,z'),\ d(z,z), d(z',z'),\ \tfrac{1}{2}[d(z,z')+d(z,z')]\}\Big).$$

i.e., $$d(z,z') \le G(d(z,z')) < d(z,z'),$$

a contradiction. Thus z is the unique common fixed point of A, B, S and T. This completes the proof.

If $p=0$, then our Theorem 4.1 reduces to the following Corollary.

Corollary 4.2. *Let A, B, S and T be four self-mappings of a metric space (X,d) satisfying the following conditions:*

(i) $AX \subseteq TX, \quad BX \subseteq SX,$

(ii) $d(Ax,By) \le G(m(x,y))$, *for all* $x,y \in X$, *where*

$$m(x,y) = max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \tfrac{1}{2}[d(Ax,Ty)+d(By,Sx)]\}$$

and $G \in \mathcal{G}_0$ *satisfies the Altman type conditions (a)-(c).*

If one of AX, BX, SX or TX is a complete subspace of X, then

(iii) (A,S) *have a coincidence point,*

(iv) (B,T) *have a coincidence point.*

Moreover, if both the pairs (A,S) and (B,T) are weakly compatible then A, B, S and T have a unique common fixed point.

Remark 4.3. If $\{A_i\}_{i\in\mathbb{N}}$, S and T be self-mappings of a metric space (X,d) then we have the following Corollary as a generalization of Popa and Pathak [11].

Corollary 4.4. *Let $\{A_i\}_{i\in\mathbb{N}}$, S and T be self-mappings of a metric space (X,d) such that*

(i) $A_1X \subseteq TX, \quad A_2X \subseteq SX,$

(ii) $$[1+pd(Sx,Ty)]d(A_ix,A_{i+1}y) \le p\, max\{d(A_ix,Sx)d(A_{i+1}y,Ty), (A_ix,Ty)d(A_{i+1}y,Sx)\} + G\big(m(x,y)\big),$$

for all $x, y \in X$, where $p \geq 0$, $G \in \mathcal{G}_0$ satisfies the Altman's conditions and

$$m(x,y) = max\{d(Sx,Ty), d(A_ix,Sx), d(A_{i+1}y,Ty), \tfrac{1}{2}[d(A_ix,Ty) + d(A_{i+1}y,Sx)]\}.$$

If one of A_iX, SX or TX is a complete subspace of X, and if the pairs (A_1, S) and (A_2, T) are weakly compatible then $\{A_i\}_{i\in\mathbb{N}}$, S and T have a unique common fixed point.

Remark 4.5. If we take sequences $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ instead of A and B in Theorem 4.1, then we get the following Corollary as a generalization of Theorem A[13], in which the completeness of X and compatibility of type (A) are relaxed by completeness of one subspace and weak compatibility.

Corollary 4.6. *Let S and T be self-maps of a metric space (X, d). Let $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ be two sequences of a metric space (X, d) satisfying the conditions:*

(i) $A_iX \subseteq TX, \quad B_iX \subseteq SX,$

(ii) $$[1 + pd(Sx,Ty)]d(A_ix,B_iy) \leq p\,max\{d(A_ix,Sx)d(B_iy,Ty), (A_ix,Ty)d(B_iy,Sx)\} + G(m(x,y)),$$

where $p \geq 0$, $G \in \mathcal{G}_0$ satisfies the Altman type conditions (a)-(c) and

$$m(x,y) = max\{d(Sx,Ty), d(A_ix,Sx), d(B_iy,Ty), \tfrac{1}{2}[d(A_ix,Ty) + d(B_iy,Sx)]\}.$$

If one of A_iX, B_iX, SX or TX is a complete subspace of X, then

(iii) (A_i, S) and (B_i, T) have coincidence points.

Moreover, if both the pairs $(A_i,\ S)$ and $(B_i,\ T)$ are weakly compatible then A_i, B_i, S and T have a unique common fixed point.

Now we give some examples to show the relative strength for various types of compatible mappings and to validate for our main Theorem 4.1.

The following example shows that weakly compatible maps need not imply compatible, compatible of type (A) and compatible of type (P).

Example 4.7. Let $X = \mathbb{R}$, with the usual metric d. Define $A,\ S : \mathbb{R} \to \mathbb{R}$ by:

$$Ax = [x],\ \forall x \in \mathbb{R} \qquad \text{and} \qquad Sx = \begin{cases} -2, & \text{if } x \leq 0, \\ 0, & \text{if } 0 < x < 2, \\ 2, & \text{if } x \geq 2, \end{cases}$$

where $[x]$ denotes the integral part of x.

In the above if we put $\{x_n\} = \{\frac{1}{n}\}$ then $Ax_n = 0 = Sx_n$ and $SAx_n = -2 = SSx_n \neq ASx_n = AAx_n = 0$. Showing that pair (A, S) is neither compatible nor compatible of type (A) nor compatible of type (P), but it is weakly compatible as (A, S) commutes at their coincidence points $x = \pm 2$.

The following example shows that compatibility need not imply compatibility of type (A) and type (P).

Example 4.8. Let $X = \mathbb{R}$, with the usual metric d. Define $A,\ S : \mathbb{R} \to \mathbb{R}$ by:

$$Ax = x,\ \forall x \in \mathbb{R} \quad \text{and} \quad Sx = \begin{cases} 0, & \text{if } x \text{ is an integer}, \\ 1, & \text{if } x \text{ is not an integer}. \end{cases}$$

Then, for the sequences $\{x_n\} = \{1 \pm \frac{1}{n+1}\}$, we see that $Ax_n \to 1,\ AAx_n \to 1,\ Sx_n = ASx_n = SAx_n = 1$ but $SSx_n = 0$, as $n \to \infty$. Showing that pair (A, S) is compatible but neither compatible of type (A) nor type (P). However $x = 0$ is the point of weak compatibility.

The following example shows that compatible mappings of type (A) need not be compatible and compatible of type (P).

Example 4.9. Let $X = \mathbb{R}$, with the usual metric d. Define $A,\ S : \mathbb{R} \to \mathbb{R}$ by

$$Ax = \begin{cases} 2, & \text{if } x = 2 \text{ or } x > 5, \\ 6, & \text{if } 2 < x \leq 5, \end{cases} \quad \text{and } Sx = \begin{cases} 2, & \text{if } x = 2, \\ 6, & \text{if } 2 < x < 5, \\ x - 3, & if, x \geq 5. \end{cases}$$

Then, for the sequence $\{x_n\} = \{5 + \frac{1}{n}\}$ we see that $Sx_n \to 2,\ Ax_n = 2 = SAx_n = AAx_n$ and $ASx_n = 6 = SSx_n$, as $n \to \infty$. Thus pair (A, S) is compatible of type (A) but neither compatible nor compatible of type (P). However $x = 2$ is the point of weak compatibility.

Now we give an example to show the validity of our main Theorem 4.1.

Example 4.10. Let A, B, S and T be four self-mappings of a metric space X, endowed with the usual metric d. Let $X = [0, \frac{3}{2}]$. Define the mappings $A, B, S, T : X \to X$ by:

$$Ax = 1, \quad Sx = x, \quad Bx = 1 \ \text{ and } \ Tx = \tfrac{1}{2}(1 + x), \quad \forall x \in X.$$

Let $G : [0, \infty) \to \mathbb{R}$ be defined by $G(t) = \frac{t}{3}$. Then we observe that:

(i) $AX = \{1\} \subseteq TX = \left[\frac{1}{2}, \frac{5}{4}\right] \subseteq X$ and $BX = \{1\} \subseteq SX = [0, \frac{3}{2}] \subseteq X$;

(ii) Since, $d(Ax, By) = 0,\ d(Sx, Ty) = \frac{1}{2}|2x - y - 1|,\ d(Ax, Sx) = |1 - x|,\ d(By, Ty) = \frac{1}{2}|1 - y| = d(Ax, Ty)$ and $d(By, Sx) = |1 - x|, \ \forall x, y \in X$; we have for condition (ii) that,

$$[1+pd(Sx,Ty)]d(Ax,By) \leq p\,max\{d(Ax,Sx)d(By,Ty), d(Ax,Ty)d(By,Sx)\} + G(m(x,y))$$

or, $0 \leq \frac{p}{2}|1-x|.|1-y| + G(m(x,y))$,

where $m(x,y) = max\{\frac{1}{2}|2x-y-1|,\ |1-x|,\ \frac{1}{2}|1-y|,\ \frac{1}{2}(\frac{1}{2}|1-y|+|1-x|)\} \geq 0$.

Thus condition (ii) is true for all $x, y \in X$ and $p \geq 0$.

Further, we see that $m(x,y) = 0$, if and only if,

$$\frac{1}{2}|2x-y-1| = 0 = |1-x| = \frac{1}{2}|1-y| = \frac{1}{2}(\frac{1}{2}|1-y|+|1-x|)$$

i.e., $x = 1$, $y = 1$. Thus $m(1,1) = 0$ and therefore $G(0) = 0$.

We observe that SX and TX are complete subspaces of X. Further, we have $g(t) = \frac{3}{2}$, so that $\int_0^{t_1} g(t)dt = \frac{3}{2}t_1 < +\infty$ where $t_1 \in (0, \frac{5}{4}]$.

We notice that the pairs (A,S) and (B,T) have the coincidence point $x = 1$ where they commutes. So that (A,S) and (B,T) are weakly compatible. Thus all the conditions of Theorem 4.1 are satisfied. The only common fixed point of A, B, S and T is $x = 1$. This validates Theorem 4.1.

The following example also shows the validity of our main Theorem 4.1.

Example 4.11. Let A, B, S and T be four self-mappings on $X = [-1,1]$ with $|.|$ is the usual metric. Let $\mathcal{G}_0$ denotes the family of real-valued functions G with $0 < G(t) < t$. Suppose that $G(t) = t/2$ for all $t > 0$ and $G(0) = 0$. So that $g(t) = \frac{t}{t-G(t)} = 2$ for all $t > 0$, and so $\int_0^{t_1} g(t)dt = 2t_1 < +\infty$ where $t_1 \in (0,1]$.

Let us define the four mappings by:

$Ax = 1,\ Sx = \frac{1}{4}(3+x)$, $Bx = 1$ and $Tx = \frac{1}{2}(1+x)$, $\forall x \in [-1,1] = X$.

Then we observe that:

(i) $AX = \{1\} \subseteq TX = [0,1]$ and $BX = \{1\} \subseteq SX = [\frac{1}{2}, 1]$.

(ii) Now, $d(Sx,Ty) = \frac{1}{4}|1+x-2y|$, $d(Ax,By) = 0$, $d(Ax,Sx) = \frac{1}{4}|1-x| = d(By,Sx)$ and $d(By,Ty) = \frac{1}{2}|1-y| = d(Ax,Ty)$, for all $x, y \in X$. Then condition (ii) is easily satisfied. Further,

$$m(x,y) = 0 \iff max\{\tfrac{1}{4}|1+x-2y|, \tfrac{1}{4}|1-x|, \tfrac{1}{2}|1-y|, \tfrac{1}{2}[\tfrac{1}{2}|1-y| + \tfrac{1}{4}|1-x|]\} = 0$$

$$\Longleftrightarrow \text{each of the values } \tfrac{1}{4}|1+x-2y|, \tfrac{1}{4}|1-x|, \tfrac{1}{2}|1-y| \text{ and } [\tfrac{1}{4}|1-y| + \tfrac{1}{8}|1-x|] \text{ must be equal to zero,}$$

$$\Longleftrightarrow x = 1, y = 1.$$

i.e., $G(0) = G(m(1,1)) = 0$.

Thus all the conditions of Theorem 4.1 are satisfied. The point $x = 1$ is a coincidence point of pairs (A, S) and (B, T) and that $x = 1$ is the only common fixed point of A, B, S and T in $X = [-1, 1]$. This validates Theorem 4.1.

Remark 4.12. Our Theorem 4.1 is remain true if the pairs (A, S) and (B, T) are R-weakly commuting [8] instead of weakly compatible.

Remark 4.13. The implicit relation of Popa [10] can also be applied to our theorems instead of inequality (ii) as in Imdad, Kumar and Khan [4].

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References

[1] M. Altman, *An integral test for series and generalized contractions*, Amer. Math. Monthly 82(1975), 827-829.

[2] A. Carbone and S. P. Singh, *Fixed points for Altman type mappings*, Indian J. Pure Appl. Math. 18(1987), 1082-1087.

[3] A. Carbone, B. E. Rhodes and S. P. Singh, *A fixed point theorem for generalized map*, Indian J. Pure Appl. Math. 20(1989), 543-548.

[4] M. Imdad, A. S. Kumar and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Radovi Mat. 11(2002), 135-143.

[5] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. 9(1986), 771-779.

[6] G. Jungck and B. E. Rhoades, *Fixed points for set-valued functions without continuity*, Indian J. Pure Appl. Math. 29(3)(1998), 227-238.

[7] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japon. 38(2)(1993), 381-390.

[8] R. P. Pant, *Common fixed points for non-commuting mappings*, J. Math. Anal. Appl. 188(1994), 436-440.

[9] H. K. Pathak, Y. J. Cho, S. M. Kang and B. S. Lee, *Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming*, Le Mathematische, 50(1995), 15-33.

[10] V. Popa, *On general common fixed point theorem of Meir-Keeler type for non-continuous weak compatible mappings*, Filomat(Nis) 18(2004), 33-40.

[11] V. Popa and H. K. Pathak, *Common fixed point theorems for compatible mappings and compatible mappings of type (A)*, Mat. Vesnik 49(1997), 109-114.

[12] B. E. Rhoades and B. Watson, *Generalized contractions and fixed points in metric spaces*, Math. Japon 6(1989), 975-982.

[13] D. R. Sahu and C. L. Dewangan, *Compatible of type (A) and common fixed points*, Banyan Math. Jour. 1(1994), 49-57.

[14] S. Sessa, *On fixed points of weak commutativity condition in a fixed point consideration*, Publ. Inst. Math. 32(46)(1982), 149-153.

[15] S. Sessa and B. Fisher, *Common fixed points of weakly-commuting mappings*, Bull. Acad. Polon. Ser. Math. 35(1987), 341-349.

[16] B. Watson, B. A. Meade and C. W. Norris, *A note on theorem of Altman*, Indian J. Pure Appl. Math. 17(1986), 1082-1087.

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