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ON θ-$(1,2)$-SEMI-PREGENERALIZED CLOSED SETS

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Abstract

The aim of this paper is to introduce the notion of θ-$(1,2)$-semi-pregeneralized closed set in bitopological space and study its properties.

1 Introduction

In 1983, Abd El-Monsef et al.[1] defined β-open sets and Andrijevic [2] called these sets as semi-preopen sets. The notion of semi-pre-θ-open set was introduced by Noiri [6] in 2003. The concept of $(1,2)$-semi-preopen sets was defined and investigated by Raja Rajeswari and Lellis Thivagar [7] in 2005. The notion of $(1,2)$-semi-preirresolute function what we call as $(1,2)$-β-irresolute function, was introduced by Navalagi et al.[5]. The $(1,2)$-semi-pre-θ-open sets and the vividly $(1,2)$-β-irresolute function were introduced in [3].

In this paper, we introduce a new form of closed set called θ-$(1,2)$-semi-pregeneralized closed set in a bitopological space by utilizing the $(1,2)$-semipre-θ-closure operator. Moreover, the notions of θ-$(1,2)$-semi-pregeneralized -continuous function and θ-$(1,2)$-semi-pregeneralized-irresolute function are introduced and studied. We also define θ-$(1,2)$-semi-pregeneralized homeomorphism.

2 Preliminaries

The interior and the closure of a subset A of a topological space $(X,\ \tau)$ are denoted by $int(A)$ and $cl(A)$, respectively.

In the following sections by X, Y and Z, we mean a bitopological space $(X,\ \tau_1,\ \tau_2)$, $(Y,\ \sigma_1,\ \sigma_2)$ and $(Z,\ \varrho_1,\ \varrho_2)$, respectively.

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Definition 1 *A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open [4] if $A \in \tau_1 \cup \tau_2$ and $\tau_1\tau_2$-closed if its complement in X is $\tau_1\tau_2$-open. The $\tau_1\tau_2$-$cl(A)$ is the intersection of all the $\tau_1\tau_2$-closed sets containing A.*

Definition 2 *A subset A of a space X is said to be an $(1,2)$-semi-preopen set [7] if $A \subset \tau_1\tau_2$-$cl(\tau_1$- $int(\tau_1\tau_2$-$cl(A)))$ and $(1,2)$-semi-preclosed if its complement in X is $(1,2)$-semi-preopen.*

The family of all
(i) $(1,2)$-semi-preopen sets in X is denoted by $(1,2)$-$SPO(X)$.
(ii) $(1,2)$-semi-preopen sets containing $x \in X$ is denoted by $(1,2)$-$SPO(X, x)$.
(iii) $(1,2)$-semi-preclosed sets in X is denoted by $(1,2)$-$SPC(X)$.

Definition 3 *For any subset A of a bitopological space X, the $(1,2)$-semi-preclosure of A denoted by $(1,2)$-$spcl(A)$ [7] is the intersection of all the $(1,2)$-semi-preclosed sets containing A. The $(1,2)$-semi-preinterior of a subset A of X is the union of all the $(1,2)$-semi-preopen sets contained in A, and is denoted by $(1,2)$-$spint(A)$ and A is $(1,2)$-semi-preopen if $(1,2)$-$spint(A) = A$.*

Remark 4 *It was observed that a subset A of a bitopological space X is $(1,2)$-semi-preclosed if $(1,2)$-$spcl(A) = A$. If $A \subset B$, then$(1,2)$-$spcl(A) \subset (1,2)$-$spcl(B)$.*

Definition 5 *A function f:$X \to Y$ is called*
(i) $(1,2)$-β-irresolute [5] if $f^{-1}(V)$ is $(1,2)$-semi-preopen for every $(1,2)$-semi-preopen set V in Y.
(ii) vividly $(1,2)$-β-irresolute [3] if for each point $x \in X$ and each $V \in (1,2)$-$SPO(X, f(x))$, there exists a $U \in (1,2)$-$SPO(X, x)$ such that $f((1,2)$-$spcl(U)) \subset V$.

It is shown that every vividly $(1,2)$-β-irresolute function is $(1,2)$-β-irresolute but not the converse.

The $(1,2)$-semipre-θ-interior and $(1,2)$-semipre-θ-closure of a subset A of X are denoted by $(1,2)$-$spint_\theta(A)$ and $(1,2)$-$spcl_\theta(A)$ are defined as follows.
$(1,2)$-$spint_\theta(A) = \{x \in X : x \in U \subset (1,2)$-$spcl(U) \subset A$ for some $(1,2)$-semi-preopen set U of $X\}$ and
$(1,2)$-$spcl_\theta(A) = \{x \in X : (1,2)$-$spcl(U) \cap A \neq \emptyset$ for every $(1,2)$-semi-preopen set containing $x\}$.

Remark 6 *Let A be a subset of X. Then A is $(1,2)$-semipre-θ-open (briefly $(1,2)$-sp-θ-open)[3] if and only if $A = (1,2)$-$spint_\theta(A)$ and $(1,2)$-semipre-θ-closed (briefly $(1,2)$-sp-θ-closed) if and only if $A = (1,2)$-$spcl_\theta(A)$. $(1,2)$-$spint_\theta(A)$ is $(1,2)$-sp-θ-open and $(1,2)$-$spcl_\theta(A)$ is $(1,2)$-sp-θ-closed. It is observed that every $(1,2)$-sp-θ-open set is $(1,2)$-semi-preopen [3].*

It is shown in [3] that $X\setminus (1,2)$-$spint_\theta(A) = (1,2)$-$spcl_\theta(X \setminus A)$ and $(1,2)$-$spint_\theta(X \setminus A) = X\setminus (1,2)$-$spcl_\theta(A)$. If $A \subset B$, then $(1,2)$-$spcl_\theta(A) \subset (1,2)$-$spcl_\theta(B)$.

Definition 7 *A subset A of a space X is said to be $(1,2)$-semi-preregular (briefly $(1,2)$-sp-regular)[3] if it is both $(1,2)$-semi-preopen and $(1,2)$-semi-preclosed.*

The family of all $(1,2)$-semi-preregular sets in X is denoted by $(1,2)$-$SPR(X)$.

Definition 8 *A space X is said to be $(1,2)$-semi-preregular [3] if for each $(1,2)$-semi-preclosed set F and each point $x \in X \backslash F$, there exist disjoint $(1,2)$semi-preopen sets U, V such that $x \in U$ and $F \subset V$.*

Lemma 9 *For a space X the following properties are equivalent.*
(i) X is $(1,2)$-semi-preregular.
(ii) For each $U \in (1,2)$-$SPO(X)$ and each $x \in U$, there exists $V \in (1,2)$-$SPO(X)$ such that $x \in V \subset (1,2)$-$spcl(V) \subset U$.
(iii) For each $U \in (1,2)$-$SPO(X)$ and each $x \in U$, there exists $V \in (1,2)$-$SPR(X)$ such that $x \in V \subset U$.

3 θ-$(1,2)$-Semi-Pregeneralized Closed Sets

In this section we define the θ-$(1,2)$-semi-pregeneralized closed sets and study some properties.

Definition 10 *A subset A of a space X is called θ-$(1,2)$-semi-pregeneralized closed set(briefly θ-$(1,2)$-spg-closed set) if $(1,2)$-$spcl_\theta(A) \subset U$ whenever $A \subset U$ and U is $(1,2)$-semi-preopen in X.*

The complement of a θ-$(1,2)$-*spg*-closed set in X is called θ-$(1,2)$-semi-pregeneralized open (briefly θ-$(1,2)$-*spg*-open).

Lemma 11 *Every $(1,2)$-sp-θ-closed set is θ-$(1,2)$-spg-closed.*

Proof. The proof follows from the fact that for an $(1,2)$-*sp*-θ-closed set $(1,2)$-$spcl_\theta A = A$. ■

Remark 12 *The converse of Lemma 11 is not true as shown in the following example.*

Example 13 *Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a,c\}, X\}$. Then the set $\{b,c\}$ is $(1,2)$-spg-closed but not $(1,2)$-sp-θ-closed.*

Theorem 14 *A subset A of X is θ-$(1,2)$-spg-open if and only if $F \subset (1,2)$-$spint_\theta(A)$ whenever F is $(1,2)$-semi-preclosed in X and $F \subset A$.*

Proof. Necessity. Let A be θ-$(1,2)$-*spg*-open and $F \subset A$, where F is $(1,2)$-semi-preclosed. Then $X \setminus A \subset X \setminus F$ and $X \setminus F$ is $(1,2)$-semi-preopen. Therefore, $(1,2)$-$spcl_\theta(X \setminus A) \subset X \setminus F$. Hence $(1,2)$-$spcl_\theta\ (X \setminus A) = X \setminus ((1,2)$-$spint_\theta\ (A)) \subset X \setminus F$. Thus we have $F \subset (1,2)$-$spint_\theta(A)$.
Sufficiency. If F is $(1,2)$-semi-preclosed and $F \subset (1,2)$-$spint_\theta(A)$ whenever $F \subset A$, then $X \backslash A \subset X \backslash F$ and $X \backslash (1,2)$-$spint_\theta(A) \subset X \backslash F$. That is, $(1,2)$-$spcl_\theta(X \backslash A) \subset X \setminus F$. Therefore, $X \setminus A$ is $(1,2)$-*spg*-closed and hence A is θ-$(1,2)$-*spg*-open. ■

Definition 15 *A space X is said to be $(1,2)$-β-T_1 if for any two distinct points x, y of X, there exists $(1,2)$-semi-preopen sets U, V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.*

Theorem 16 *A bitopological space X is $(1,2)$-β-T_1 if and only if $\{x\}$ is $(1,2)$-semi-preclosed in X for every $x \in X$.*

Proof. If $\{x\}$ is $(1,2)$-semi-preclosed in X for every $x \in X$, for $x \neq y$, $X \setminus \{x\}$, $X \setminus \{y\}$ are $(1,2)$-semi-preopen sets such that $y \in X \setminus \{x\}$ and $x \in X\setminus \{y\}$. Therefore, X is $(1,2)$-β-T_1. Conversely, if X is $(1,2)$-β-T_1 and if $y \in X \setminus \{x\}$ then $x \neq y$. Therefore, there exist $(1,2)$-semi-preopen sets U_x, V_y in X such that $x \in U_x$ but $y \notin U_x$ and $y \in V_y$ but $x \notin V_y$. Let G be the union of all such V_y. Then G is an $(1,2)$-semi-preopen set and $G \subset X\setminus \{x\} \subset X$. Therefore, $X \setminus \{x\}$ is an $(1,2)$-semi-preopen set in X. ■

Lemma 17 *Let A be θ-$(1,2)$-spg-closed subset of X. Then*
(i) $(1,2)$-$spcl_\theta(A) \setminus A$ does not contain a nonempty $(1,2)$-semi-preclosed set.
(ii) $(1,2)$-$spcl_\theta(A) \setminus A$ is θ-$(1,2)$-spg-open.

Proof. (i). Let F be an $(1,2)$-semi-preclosed set contained in $(1,2)$-$spcl_\theta(A)\setminus A$. Then $X \setminus F$ is $(1,2)$-semi-preopen and $A \subset X \setminus F$, it follows that $(1,2)$-$spcl_\theta(A) \subset X \setminus F$. Thus we get $F \subset X \setminus (1,2)$-$spcl_\theta(A)$ and $F \subset (1,2)$-$spcl_\theta(A)$. Hence $F = \emptyset$.
(ii). If A is θ-$(1,2)$-*spg*-closed and F is an $(1,2)$-semi-preclosed set contained in $(1,2)$-$spcl_\theta(A) \setminus A$, then F is empty by (i). Therefore, $F \subset (1,2)$-$spint_\theta$ $((1,2)$-$spcl_\theta(A) \setminus A)$. By Theorem 14, $(1,2)$-$spcl_\theta(A) \setminus A$ is θ-$(1,2)$-*spg*-open. ■

Theorem 18 *In a $(1,2)$-β-T_1 space X, every θ-$(1,2)$-spg-closed set is $(1,2)$-sp-θ-closed.*

Proof. Let $A \subset X$ be θ-$(1,2)$-*spg*-closed and $x \in (1,2)$-$spcl_\theta(A)$. Since X is $(1,2)$-β-T_1, $\{x\}$ is $(1,2)$-semi-preclosed and by Lemma 17, $x \notin (1,2)$-$spcl_\theta(A) \setminus A$. This implies that $x \in A$ and hence $(1,2)$-$spcl_\theta(A) \subset A$ and hence A is $(1,2)$-*sp*-θ-closed. ■

Theorem 19 *[3] Let A be a subset of X. Then*
(i) $A \in (1,2)$-$SPO(X)$ if and only if $(1,2)$-$spcl(A) \in (1,2)$-$SPR(X)$.
(ii) $A \in (1,2)$-$SPC(X)$ if and only if $(1,2)$-$spint(A) \in (1,2)$-$SPR(X)$.

Theorem 20 *For any subset A of a space X, the following are equivalent.*
(i) $(1,2)$-$spcl_\theta(A) = \bigcap \{V{:}A \subset V$ and V is $(1,2)$-sp-θ-closed$\}$.
(ii) $(1,2)$-$spcl_\theta(A) = \bigcap \{V{:}A \subset V$ and $V \in (1,2)$-$SPR(X)\}$.

Proof. (i). If $x \notin (1,2)$-$spcl_\theta(A)$, then there exists $V \in (1,2)$-$SPO(X, x)$ such that $(1,2)$-$spcl(V) \cap A = \emptyset$. By Theorem 19, $X \setminus (1,2)$-$spcl(V)$ is $(1,2)$-semi-preregular. Hence, $X \setminus (1,2)$-$spcl(V)$ is an $(1,2)$-*sp*-θ-closed set containing A and $x \notin X \setminus (1,2)$-$spcl(V)$. Therefore, $x \notin \bigcap\{V{:}A \subset V$ and V is $(1,2)$-*sp*-θ-closed$\}$.

Conversely, if $x \notin \bigcap\{V{:}A \subset V$ and V is $(1,2)$-*sp*-θ-closed$\}$, then there exists an $(1,2)$-*sp*-θ-closed set V such that $A \subset V$ and $x \notin V$. Then there exists $U \in (1,2)$-$SPO(X)$ such that $x \in U \subset (1,2)$-$spcl(U) \subset X \setminus V$, Therefore, $(1,2)$-$spcl(U) \cap A \subset (1,2)$-$scl(U) \cap V = \emptyset$. Hence $x \notin (1,2)$-$spcl_\theta(A)$.

(ii). It can be proved in a similar manner. ■

Theorem 21 *Let A and B be subsets of X. Then the following properties hold.*

(i) If $A \subset B$, *then* $(1,2)$-$spcl_\theta(A) \subset (1,2)$-$spcl_\theta\ (B)$.

(ii) $(1,2)$-$spcl_\theta((1,2)$- $spcl_\theta(A)) = (1,2)$-$spcl_\theta(A)$.

Proof. (i). Proof is obvious.

(ii). $(1,2)$-$spcl_\theta A \subset (1,2)$-$spcl_\theta((1,2)$- $spcl_\theta(A))$, in general. If $x \notin (1,2)$-$spcl_\theta(A)$, then there exists $V \in (1,2)$-$SPR(X,x)$ such that $V \cap A = \emptyset$. Since $V \in (1,2)$-$SPR(X)$, $V \cap (1,2)$-$spcl_\theta(A) = \emptyset$ which shows that $x \notin (1,2)$-$spcl_\theta((1,2)$-$spcl_\theta(A))$. Therefore, $(1,2)$-$spcl_\theta((1,2)$-$spcl_\theta(A)) \subset (1,2)$-$spcl_\theta\ (A)$. ■

Lemma 22 *If A is a* θ-$(1,2)$-*spg-closed set of a space X such that* $A \subset B \subset (1,2)spcl_\theta(A)$, *then B is also* θ-$(1,2)$-*spg-closed in X.*

Proof. Let U be $(1,2)$-semi-preopen in X such that $B \subset U$. Then $A \subset U$. Since A is θ-$(1,2)$-*spg*-closed, $(1,2)$-$spcl_\theta(A) \subset U$ and by Theorem 21, $(1,2)$-$spcl_\theta$ $(B) \subset (1,2)$-$spcl_\theta((1,2)$-$spcl_\theta(A)) = (1,2)$-$spcl_\theta(A) \subset U$. Therefore, B is θ-$(1,2)$-*spg*-closed. ■

Definition 23 *For a subset A of a space X we define* $A_\theta^{\Lambda(1,2)sp}$ *as follows :* $A_\theta^{\Lambda(1,2)sp}$ $= \{x \in X{:}\ (1,2)$-$spcl_\theta(\{x\}) \cap A \neq \emptyset\}$

Proposition 24 $A_\theta^{\Lambda(1,2)sp} = \bigcap\ \{U{:}A \subset U,\ U$ *is* $(1,2)$-*sp*-θ-*open*$\}$ *for any subset A of X.*

Proof. Let $x \in A_\theta^{\Lambda(1,2)sp}$ and $x \notin \bigcap\{U{:}A \subset U,\ U$ is $(1,2)$-*sp*-θ-open$\}$. Then there exists an $(1,2)$-*sp*-θ-open set U containing A such that $x \notin U$. Let $y \in (1,2)$-$spcl_\theta(\{x\}) \cap A$. Thus $y \in U$ and $x \notin U$, a contradiction. If $x \in \bigcap\ \{U{:}A \subset U, U$ is $(1,2)$-*sp*-θ-open$\}$ and $x \notin A_\theta^{\Lambda(1,2)sp}$, then $(1,2)$-$spcl_\theta(\{x\}) \cap A = \emptyset$. Hence $x \notin X \setminus (1,2)$-$spcl_\theta(\{x\})$, where $X \setminus (1,2)$-$spcl_\theta\ (\{x\})$ is an $(1,2)$-*sp*-θ-open set containing A. But this is impossible since $x \in \bigcap\ \{U{:}A \subset U, U is (1,2)$-*sp*-$\theta$-open$\}$. Therefore, $x \in A_\theta^{\Lambda(1,2)sp}$. ■

Thus $A_\theta^{\Lambda(1,2)sp}$ is the intersection of all the $(1,2)$ -*sp*-θ-open sets containing A which is by the usual notation, $(1,2)$-$spker_\theta(A)$.

Lemma 25 *Let X be a topological space and* $x \in X$. *The following are equivalent.*

(i) $x \in (1,2)$-$spcl_\theta(\{y\})$.

(ii) $y \in (1,2)$-$spker_\theta(\{x\})$.

Proof. $(i) \Rightarrow (ii)$. If $y \notin (1,2)$-$spker_\theta(\{x\})$, then there exists an $(1,2)$-sp-θ-open set U containing x such that $x \notin (1,2)$-$spcl_\theta(\{y\})$.

$(ii) \Rightarrow (i)$. Proof is similar. ■

Lemma 26 *The following statements are equivalent for any two points x, y in a space X.*

(i) $(1,2)$-$spker_\theta(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$.

(ii) $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$.

Proof. $(i) \Rightarrow (ii)$. Let $(1,2)$-$spker_\theta$ $(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$. Then there exists a point z in X such that $z \in (1,2)$-$spker_\theta(\{x\})$ and $z \notin (1,2)$-$spker_\theta(\{y\})$. From $z \in (1,2)$-$spker_\theta(\{x\})$, it follows that $\{x\} \cap (1,2)$-$spcl_\theta(\{z\}) \neq \emptyset$. This implies that $x \in (1,2)$-$spcl_\theta(\{z\})$. From $z \notin (1,2)$-$spker_\theta(\{y\})$ it follows that $\{y\} \cap (1,2)$-$spcl_\theta(\{z\}) = \emptyset$. Since $x \in (1,2)$-$spcl_\theta(\{z\})$, $(1,2)$-$spcl_\theta(\{x\}) \subset (1,2)$-$spcl_\theta(\{z\})$ and $\{y\} \cap (1,2)$-$spcl_\theta(\{x\}) = \emptyset$. Hence $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$.

$(ii) \Rightarrow (i)$. Let $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$. Then there exists a point z in X such that $z \in (1,2)$-$spcl_\theta(\{x\})$ and $z \notin (1,2)$-$spcl_\theta(\{y\})$. Hence there exists an $(1,2)$-sp-θ-open set containing z and therefore, x but not y. Therefore, $y \notin (1,2)$-$spker_\theta(\{x\})$ and $(1,2)$- $spker_\theta(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$. ■

Definition 27 *A space X is said to be $(1,2)$-β-θ-R_0 if every $(1,2)$-sp-θ-open set contains the $(1,2)$-semipre-θ-closure of each of its singletons.*

Theorem 28 *A space X is $(1,2)$-β-θ-R_0 if and only if for any x and y in X, $(1,2)$-$spcl_\theta$ $(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$ implies $(1,2)$-$spcl_\theta(\{x\}) \cap$ $(1,2)$-$spcl_\theta(\{y\}) = \emptyset$.*

Proof. Necessity. If X is $(1,2)$-β-θ-R_0 and x, y in X such that $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$, then there exists $z \in (1,2)$-$spcl_\theta(\{x\})$ such that $z \notin (1,2)$-$spcl_\theta(\{y\})$, say. Therefore, there exists $V \in (1,2)$-$SPO(X)$ such that $y \notin V$ and $z \in V$ and hence $x \in V$. Thus we get $x \notin (1,2)$-$spcl_\theta(\{y\})$ and therefore, $x \in X \setminus (1,2)$-$spcl_\theta(\{y\})$. This implies that $(1,2)$-$spcl_\theta(\{x\}) \subset X \setminus$ $(1,2)$-$spcl_\theta(\{y\})$ and therefore, $(1,2)$-$spcl_\theta(\{x\}) \cap$ $(1,2)$-$spcl_\theta(\{y\}) = \emptyset$.

Sufficiency. Let V be $(1,2)$-sp-θ-open and $x \in V$. If $y \in X \setminus V$, then $x \neq y$ and $x \notin (1,2)$-$spcl_\theta(\{y\})$. This shows that $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta$ $(\{y\})$ and hence by our assumption, $(1,2)$-$spcl_\theta(\{x\}) \cap$ $(1,2)$-$spcl_\theta(\{y\}) = \emptyset$. Hence $y \notin (1,2)$-$spcl_\theta(\{x\})$. Therefore, $(1,2)$-$spcl_\theta(\{x\}) \subset V$ ■

Theorem 29 *A space X is $(1,2)$-β-θ-R_0 if and only if for any x and y in X, $(1,2)$-$spker_\theta(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$ implies $(1,2)$-$spker_\theta(\{x\}) \cap$ $(1,2)$-$spker_\theta(\{y\}) = \emptyset$.*

Proof. Suppose that X is $(1,2)$-β-θ-R_0 and if for any x and y in X, $(1,2)$-$spker_\theta(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$, then by Lemma 26, $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$. If $z \in (1,2)$-$spker_\theta(\{x\}) \cap (1,2)$- $spker_\theta(\{y\})$, then from $z \in (1,2)$-$spker_\theta(\{x\})$ and by Lemma 25, it follows that $x \in (1,2)$-$spcl_\theta(\{z\})$. Since $x \in (1,2)$-$spcl_\theta(\{x\})$, by Theorem 28, $(1,2)$-$spcl_\theta(\{x\}) = (1,2)$-$spcl_\theta(\{z\})$. Similarly, we have $(1,2)$-$spcl_\theta(\{y\}) = (1,2)$-$spcl_\theta$ $(\{z\})$, a contradiction. Therefore, $(1,2)$-$spker_\theta(\{x\}) \cap$ $(1,2)$-$spker_\theta(\{y\}) = \emptyset$.

Conversely, let x, y be any two points in X such that $(1,2)$-$spker_\theta(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$ implies $(1,2)$-$spker_\theta(\{x\}) \cap$ $(1,2)$-$spker_\theta(\{y\}) = \emptyset$. If $(1,2)$-$spcl_\theta(\{x\}) \neq (1,2)$-$spcl_\theta(\{y\})$, then by Lemma 26, $(1,2)$-$spker_\theta(\{x\}) \neq (1,2)$-$spker_\theta(\{y\})$. Hence $(1,2)$-$spker_\theta(\{x\}) \cap$ $(1,2)$-$spker_\theta(\{y\}) = \emptyset$ which implies that $(1,2)$-$spcl_\theta(\{x\}) \cap$ $(1,2)$-$spcl_\theta(\{y\}) = \emptyset$. For, if $z \in (1,2)$-$spcl_\theta(\{x\})$, then $x \in (1,2)$-$spker_\theta(\{z\})$ and therefore, $(1,2)$-$spker_\theta(\{x\}) \cap$ $(1,2)$-$spker_\theta(\{z\}) \neq \emptyset$. Therefore, by hypothesis, $(1,2)$-$spker_\theta(\{z\}) = (1,2)$-$spker_\theta(\{x\})$. Then $z \in (1,2)$-$spcl_\theta(\{x\}) \cap$ $(1,2)$-$spcl_\theta(\{y\})$ implies that $(1,2)$-$spker_\theta(\{x\}) = (1,2)$-$spker_\theta$ $(\{z\})$ $= (1,2)$-$spker_\theta(\{y\})$, a contradiction. Therefore, by Theorem 28, X is $(1,2)$-β-θ-R_0. ■

4 θ-$(1,2)$-β-Irresolute Functions

In this section we introduce the notion of θ-$(1,2)$-β-irresolute functions.

Definition 30 *A map $f{:}X \to Y$ is called θ-$(1,2)$-β-irresolute if for each $x \in X$ and each $V \in (1,2)$-$SPO(Y, f(x))$, there exists $U \in (1,2)$-$SPO(X, x)$ such that $f((1,2)$-$spcl(U)) \subset (1,2)$-$spcl(V)$.*

Theorem 31 *Every $(1,2)$-β-irresolute map is θ-$(1,2)$-β-irresolute.*

Proof. Let $x \in X$ and $V \in (1,2)$-$SPO(X, f(x))$. Since f is $(1,2)$-β-irresolute, $f^{-1}(V)$ is $(1,2)$-semi-preopen and $f^{-1}((1,2)$-$spcl(V))$ is $(1,2)$-semi-preclosed in X. Let $U = f^{-1}(V)$. Then $U \in (1,2)$-$SPO(X, x)$ and $(1,2)$-$spcl(U) \subset f^{-1}((1,2)$-$spcl(V))$. Therefore, $f((1,2)$-$spcl(U)) \subset (1,2)$-$spcl(V)$. Hence f is θ-$(1,2)$-β-irresolute. ■

Remark 32 *The converse of Theorem 31, is not true in general, as shown in the following example.*

Example 33 *Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X\}$, $\tau_2 = \{\emptyset, \{b, c\}, X\}$ and $Y = \{p, q, r\}$, $\sigma_1 = \{\emptyset, \{p\}, \{p, q\}, Y\}$ and $\sigma_2 = \{\emptyset, \{p\}, Y\}$. Define a function $f{:}X \to Y$ as $f(a) = p$, $f(b) = r$ and $f(c) = q$. Then f is θ-$(1,2)$-β-irresolute but not $(1,2)$-β-irresolute since $f^{-1}(\{p\}) = \{a, b\} \notin (1,2)$-$SPO(X)$.*

Remark 34 *Thus we have*

vividly $(1,2)$-β-irresolute $\Rightarrow$ $(1,2)$-β-irresolute $\Rightarrow$ θ-$(1,2)$-β-irresolute

and none of them is reversible.

Theorem 35 *For a function $f{:}X \to Y$ the following properties are equivalent.*
(i) f is θ-$(1,2)$-β-irresolute.
(ii) $(1,2)$-$spcl_\theta(f^{-1}(B)) \subset f^{-1}((1,2)$-$spcl_\theta(B))$ for every subset B of Y.
(iii) $f((1,2)$-$spcl_\theta(A)) \subset (1,2)$-$spcl_\theta(f(A))$ for every subset A of X.

Proof. $(i) \Rightarrow (ii)$.
Let B be any subset of Y. Suppose that $x \notin f^{-1}((1,2)\text{-}spcl_\theta(B))$. Then $f(x) \notin (1,2)$-$spcl_\theta(B)$ and there exists $V \in (1,2)$-$SPO(X, f(x))$ such that $(1,2)$-$spcl(V) \cap B = \emptyset$. Since f is θ-$(1,2)$-β-irresolute, there exists $U \in (1,2)$-$SPO(X,x)$ such that $f((1,2)$-$spcl(U)) \subset (1,2)$-$spcl(V)$. Therefore, $f((1,2)$-$spcl(U)) \cap B = \emptyset$ and $(1,2)$-$spcl(U)$ $\cap\ f^{-1}(B) = \emptyset$. Hence, $x \notin (1,2)$-$spcl_\theta(f^{-1}(B))$. Therefore, $(1,2)$-$spcl_\theta$ $(f^{-1}(B))$ $\subset f^{-1}((1,2)\text{-}spcl_\theta(B))$.

$(ii) \Rightarrow (iii)$. Let A be any subset of X. Then $(1,2)$-$spcl_\theta(A) \subset (1,2)$-$spcl_\theta$ $(f^{-1}(f(A))) \subset f^{-1}((1,2)\text{-}spcl_\theta(f(A)))$ and hence $f((1,2)$-$spcl_\theta(A)) \subset$ $(1,2)$-$spcl_\theta(f(A))$.

$(iii) \Rightarrow (ii)$. Let B be a subset of Y. By (iii), $f((1,2)$-$spcl_\theta(f^{-1}(B))) \subset (1,2)$-$spcl_\theta(f(f^{-1}(B))) \subset (1,2)$-$spcl_\theta(B)$ and $(1,2)$-$spcl_\theta(f^{-1}(B)) \subset f^{-1}((1,2)\text{-}spcl_\theta(B))$.

$(ii) \Rightarrow (i)$. Let $x \in X$ and $V \in (1,2)$-$SPO(Y, f(x))$. Then $(1,2)$-$spcl(V)$ and $Y \setminus (1,2)$-$spcl(V)$ are disjoint and $f(x) \notin (1,2)$-$spcl_\theta(Y \setminus (1,2)$-$spcl(V))$. Hence $x \notin f^{-1}((1,2)\text{-}spcl_\theta(Y \setminus (1,2)\text{-}spcl(V)))$ and by (ii), $x \notin (1,2)$-$spcl_\theta(f^{-1}(Y \setminus (1,2)$-$spcl(V)))$. Then there exists $U \in (1,2)$-$SPO(X,x)$ such that $(1,2)$-$spcl(U) \cap f^{-1}(Y \setminus (1,2)\text{-}spcl(V)) = \emptyset$ and then $f(1,2)$-$spcl(U) \subset (1,2)$-$spcl(V)$. Hence, f is θ-$(1,2)$-β-irresolute. ■

Theorem 36 *For a function $f{:}X \to Y$ the following properties are equivalent.*
(i) f is θ-$(1,2)$-β-irresolute.
(ii) $f^{-1}(V) \subset (1,2)$-$spint_\theta$ $(f^{-1}((1,2)\text{-}spcl(V)))$ for every $V \in (1,2)$-$SPO(Y)$.
(iii) $(1,2)$-$spcl_\theta(f^{-1}(V)) \subset f^{-1}((1,2)\text{-}spcl(V))$ for every $V \in (1,2)$-$SPO(Y)$.

Proof. $(i) \Rightarrow (ii)$. Let $V \in (1,2)$-$SPO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in (1,2)$-$SPO(X,x)$ such that $f((1,2)$-$spcl(U)) \subset (1,2)$-$spcl(V)$. Thus $x \in U \subset (1,2)$-$spcl(U) \subset f^{-1}((1,2)\text{-}spcl(V))$ and $x \in (1,2)$-$spint_\theta(f^{-1}((1,2)\text{-}spcl(V)))$. Hence $f^{-1}(V) \subset (1,2)$-$spint_\theta$ $(f^{-1}((1,2)\text{-}spcl(V)))$.

$(ii) \Rightarrow (iii)$. Let $V \in (1,2)$-$SPO(Y)$ and $x \notin f^{-1}((1,2)\text{-}spcl(V))$. Then $f(x) \notin (1,2)$-$spcl(V)$ and there exists $W \in (1,2)$-$SPO(Y, f(x))$ such that $W \cap V = \emptyset$ and $(1,2)$-$spcl(W) \cap V = \emptyset$. Then $f^{-1}((1,2)\text{-}spcl(W)) \cap f^{-1}(V) = \emptyset$. Now $x \in f^{-1}(W)$ and by (ii), $x \in (1,2)$-$spint_\theta(f^{-1}((1,2)\text{-}spcl(W)))$. There exists $U \in (1,2)$-$SPO(X,x)$ such that $(1,2)$-$spcl(U) \subset f^{-1}((1,2)\text{-}spcl(W))$. Thus $(1,2)$-$spcl(U) \cap f^{-1}(V) = \emptyset$ and hence x $\notin (1,2)$-$spcl_\theta(f^{-1}(V))$. Thus we get $(1,2)$-$spcl_\theta(f^{-1}(V))$ $\subset f^{-1}((1,2)\text{-}spcl(V))$.

$(iii) \Rightarrow (i)$ Let $x \in X$ and $V \in (1,2)$-$SPO(Y, f(x))$. Then $V \cap (Y \setminus (1,2)$-$spcl(V)) = \emptyset$ and $f(x) \notin (1,2)$-$spcl(Y \setminus (1,2)$-$spcl(V))$. Therefore, $x \notin f^{-1}((1,2)\text{-}spcl(Y \setminus (1,2)\text{-}spcl(V)))$ and by (iii), $x \notin (1,2) spcl_\theta(f^{-1}(Y \setminus (1,2)\text{-}spcl(V)))$. There exists $U \in (1,2)$-$SPO(X,x)$ such that $(1,2)$-$spcl(U) \cap f^{-1}(Y \setminus (1,2)\text{-}spcl(V)) = \emptyset$. Hence $f((1,2)$-$spcl(U)) \subset (1,2)$-$spcl(V)$ and hence f is θ-$(1,2)$-β-irresolute. ■

Theorem 37 *Let Y be an $(1,2)$-semi-preregular space. Then, for a function $f{:}X \to Y$ the following are equivalent.*
(i) f is vividly $(1,2)$-β-irresolute.
(ii) f is $(1,2)$-β-irresolute.
(iii) f is θ-$(1,2)$-β-irresolute.

Proof. $(i) \Rightarrow (ii)$ It is proved in [3].

$(ii) \Rightarrow (iii)$ By Theorem 31 it is obvious.

$(iii) \Rightarrow (i)$. If $x \in X$ and $V \in (1,2)$-$SPO(Y, f(x))$. Since Y is $(1,2)$-semi-preregular, by (ii) of Lemma 9, there exists $W \in (1,2)$-$SPO(Y)$ such that $f(x) \in W \subset (1,2)$-$spcl(W) \subset V$. Since f is θ-$(1,2)$-β-irresolute, there exists $U \in (1,2)$-$SPO(X, x)$ such that $f((1,2)$-$spcl(U)) \subset (1,2)$-$spcl(W) \subset V$. Therefore, f is vividly $(1,2)$-β-irresolute. ■

5 θ-$(1,2)$-Semi-pregeneralized Continuous Functions

Definition 38 *A function $f{:}X \to Y$ is called*

(i) θ-$(1,2)$-semi-pregeneralized continuous (briefly θ-$(1,2)$-spg-continuous) if $f^{-1}(F)$ is θ-$(1,2)$-spg-closed set in X for every $(1,2)$-semi-preclosed set of Y.

(ii) θ-$(1,2)$-semi-pregeneralized irresolute (briefly θ-$(1,2)$-spg-irresolute) if $f^{-1}(F)$ is θ-$(1,2)$-spg-closed in X for every θ-$(1,2)$-spg-closed set F of Y.

Recall that a function $f{:}X \to Y$ is vividly $(1,2)$-β-irresolute if and only if $f^{-1}(V)$ is $(1,2)$-*sp*-θ-closed in X for every $(1,2)$-semi-preclosed set in Y [3].

Theorem 39 *If a function $f{:}X \to Y$ is vividly $(1,2)$-β-irresolute, then it is θ-$(1,2)$-spg-continuous.*

Proof. If V is $(1,2)$-semi-preclosed in Y, then $f^{-1}(V)$ is $(1,2)$-*sp*-θ-closed in X. Therefore, by Lemma 11, $f^{-1}(V)$ is θ-$(1,2)$-*spg*-closed. ■

Remark 40 *The converse of the Theorem 39 is not true in general, as shown in the following example.*

Example 41 *Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{a, c\}, X\}$ and $Y = \{p, q\}$. $\sigma_1 = \{\emptyset, \{p\}, Y\}$ and $\sigma_2 = \{\emptyset, \{q\}, Y\}$. Define a function $f{:}X \to Y$ as $f(a) = p$, $f(b) = f(c) = q$. Then f is θ-$(1,2)$-spg-continuous but not vividly $(1,2)$-β-irresolute since for $a \in X$, there does not exist an $(1,2)$-semi-preopen set U such that $f((1,2)$-$spcl(U)) \subset \{p\}$.*

Definition 42 *A function $f{:}X \to Y$ is called always $(1,2)$-sp-θ-open (resp. always $(1,2)$-sp-θ-closed) if $f(U)$ is $(1,2)$-sp-θ-open (resp. $(1,2)$-sp-θ-closed) in Y for every $(1,2)$-sp-θ-open (resp.$(1,2)$-sp-θ-closed) set U of X.*

Theorem 43 *For a function $f{:}X \to Y$ the following are equivalent.*

(i) f is always $(1,2)$-sp-θ-closed.

(ii) For each $U \subset X$, $(1,2)$-$spcl_\theta(f(U)) \subset f((1,2)$-$spcl_\theta(U)$.

(iii) If $f^{-1}(V) \subset U$, where $V \subset Y$ and U is $(1,2)$-sp-θ-open in X, then there exists an $(1,2)$-sp-θ-open set $W \subset Y$ such that $V \subset W$ and $f^{-1}(W) \subset U$.

(iv) If $f^{-1}(y) \subset U$,where $y \in Y$ and U is $(1,2)$-sp-θ-open in X, then there exists an $(1,2)$-sp-θ-open set $W \subset Y$ such that $y \in W$ and $f^{-1}(W) \subset U$.

Theorem 44 *Let f:$X \to Y$ and g:$Y \to Z$ be two functions.*

(i) If f is θ-$(1,2)$-spg-irresolute and g is θ-$(1,2)$-spg-continuous, then $g \circ f$ is θ-$(1,2)$-spg-continuous.

(ii) If both f and g are θ-$(1,2)$-spg-irresolute, then $g \circ f$ is θ-$(1,2)$-spg-irresolute.

Definition 45 *A function f:$X \to Y$ is called a θ-$(1,2)$-spg-homeomorphism if*

(i) f is bijective.

(ii) f is θ-$(1,2)$-spg-irresolute.

(iii) f^{-1} is θ-$(1,2)$-spg- irresolute.

We denote the collection of all the θ-$(1,2)$-*spg*-homeomorhisms f:$X \to Y$ by θ $(1,2)$-*spgh*(X).

Theorem 46 *The collection $\theta(1,2)$-spgh(X) is a group .*

Proof. Define a binary operation $\star$ **:**$(1,2)$-*spgh*(X) $\times(1,2)$-*spgh*$(X) \to$ $(1,2)$-*spgh*(X) by $\star(f,g) = g \circ f$. Then $\star$ is well-defined and it is easily proved that under this binary operation $\theta(1,2)$-*spgh*(X) is a group. ∎

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