ON $\theta$-$(1,2)$-SEMI-PREGENERALIZED CLOSED SETS

S. Athisaya Ponmani, R. Raja Rajeswari, M. Lellis Thivagar
and Erdal Ekici

Abstract

The aim of this paper is to introduce the notion of $\theta$-$(1,2)$-semi-pregeneralized closed set in bitopological space and study its properties.

1 Introduction

In 1983, Abd El-Monsef et al. [1] defined $\beta$-open sets and Andrijevic [2] called these sets as semi-preopen sets. The notion of semi-pre-$\theta$-open set was introduced by Noiri [6] in 2003. The concept of $(1,2)$-semi-preopen sets was defined and investigated by Raja Rajeswari and Lellis Thivagar [7] in 2005. The notion of $(1,2)$-semi-preirresolute function what we call as $(1,2)$-$\beta$-irresolute function, was introduced by Navalagi et al. [5]. The $(1,2)$-semi-pre-$\theta$-open sets and the vividly $(1,2)$-$\beta$-irresolute function were introduced in [3].

In this paper, we introduce a new form of closed set called $\theta$-$(1,2)$-semi-pregeneralized closed set in a bitopological space by utilizing the $(1,2)$-semipre-$\theta$-closure operator. Moreover, the notions of $\theta$-$(1,2)$-semi-pregeneralized -continuous function and $\theta$-$(1,2)$-semi-pregeneralized irresolute function are introduced and studied. We also define $\theta$-$(1,2)$-semi-pregeneralized homeomorphism.

2 Preliminaries

The interior and the closure of a subset $A$ of a topological space $(X, \tau)$ are denoted by $int(A)$ and $cl(A)$, respectively.

In the following sections by $X$, $Y$ and $Z$, we mean a bitopological space $(X, \tau_1, \tau_2)$, $(Y, \sigma_1, \sigma_2)$ and $(Z, \varrho_1, \varrho_2)$, respectively.

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**Definition 1** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1\tau_2$-open $[4]$ if $A \in \tau_1 \cup \tau_2$ and $\tau_1\tau_2$-closed if its complement in $X$ is $\tau_1\tau_2$-open. The $\tau_1\tau_2$-$cl(A)$ is the intersection of all the $\tau_1\tau_2$-closed sets containing $A$.

**Definition 2** A subset $A$ of a space $X$ is said to be an $(1,2)$-semi-preopen set $[7]$ if $A \subset \tau_1\tau_2$-$cl(\tau_1 \cdot \text{int}(\tau_1\tau_2$-$cl(A)))$ and $(1,2)$-semi-preclosed if its complement in $X$ is $(1,2)$-semi-preopen.

The family of all
(i) $(1,2)$-semi-preopen sets in $X$ is denoted by $(1,2)$-$SPO(X)$.
(ii) $(1,2)$-semi-preopen sets containing $x \in X$ is denoted by $(1,2)$-$SPO(X, x)$.
(iii) $(1,2)$-semi-preclosed sets in $X$ is denoted by $(1,2)$-$SPC(X)$.

**Definition 3** For any subset $A$ of a bitopological space $X$, the $(1,2)$-semi-preclosure of $A$ denoted by $(1,2)$-$spcl(A)$ $[7]$ is the intersection of all the $(1,2)$-semi-preclosed sets containing $A$. The $(1,2)$-semi-preinterior of a subset $A$ of $X$ is the union of all the $(1,2)$-semi-preopen sets contained in $A$, and is denoted by $(1,2)$-$spint(A)$ and $A$ is $(1,2)$-semi-preopen if $(1,2)$-$spint(A) = A$.

**Remark 4** It was observed that a subset $A$ of a bitopological space $X$ is $(1,2)$-semi-preclosed if $(1,2)$-$spcl(A) = A$. If $A \subset B$, then $(1,2)$-$spcl(A) \subset (1,2)$-$spcl(B)$.

**Definition 5** A function $f: X \rightarrow Y$ is called
(i) $(1,2)$-$\beta$- irresolute $[5]$ if $f^{-1}(V)$ is $(1,2)$-semi-preopen for every $(1,2)$-semi-preopen set $V$ in $Y$.
(ii) vividly $(1,2)$-$\beta$- irresolute $[3]$ if for each point $x \in X$ and each $V \in (1,2)$-$SPO(X, f(x))$, there exists a $U \in (1,2)$-$SPO(X, x)$ such that $f((1,2)$-$spcl(U)) \subset V$.

It is shown that every vividly $(1,2)$-$\beta$- irresolute function is $(1,2)$-$\beta$- irresolute but not the converse.

The $(1,2)$-semipre-$\theta$-interior and $(1,2)$-semipre-$\theta$-closure of a subset $A$ of $X$ are denoted by $(1,2)$-$spint_\theta(A)$ and $(1,2)$-$spcl_\theta(A)$ are defined as follows.

$(1,2)$-$spint_\theta(A) = \{x \in X : x \in U \in (1,2)$-$spcl(U) \subset A \text{ for some } (1,2)$-semi-preopen set $U \text{ of } X\}$

$(1,2)$-$spcl_\theta(A) = \{x \in X : (1,2)$-$spcl(U) \cap A \neq \emptyset \text{ for every } (1,2)$-semi-preopen set containing $x\}$

**Remark 6** Let $A$ be a subset of $X$. Then $A$ is $(1,2)$-semipre-$\theta$-open (briefly $(1,2)$-$sp$-$\theta$-open) $[3]$ if and only if $A = (1,2)$-$spint_\theta(A)$ and $(1,2)$-semipre-$\theta$-closed (briefly $(1,2)$-$sp$-$\theta$-closed) if and only if $A = (1,2)$-$spcl_\theta(A)$. $(1,2)$-$spint_\theta(A)$ is $(1,2)$-$sp$-$\theta$-open and $(1,2)$-$spcl_\theta(A)$ is $(1,2)$-$sp$-$\theta$-closed. It is observed that every $(1,2)$-$sp$-$\theta$-open set is $(1,2)$-semi-preopen $[3]$.

It is shown in $[3]$ that $X \setminus (1,2)$-$spint_\theta(A) = (1,2)$-$spcl_\theta(X \setminus A)$ and $(1,2)$-$spint_\theta(X \setminus A) = X \setminus (1,2)$-$spcl_\theta(A)$. If $A \subset B$, then $(1,2)$-$spcl_\theta(A) \subset (1,2)$-$spcl_\theta(B)$. 

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Definition 7 A subset A of a space X is said to be (1,2)-semi-preregular (briefly (1,2)-sp-regular) if it is both (1,2)-semi-preopen and (1,2)-semi-preclosed. The family of all (1,2)-semi-preregular sets in X is denoted by \((1,2)-\text{SPR}(X)\).

Definition 8 A space X is said to be \((1,2)-\text{semi-preregular}\) if for each \((1,2)\)-semi-preclosed set \(F\) and each point \(x \in X \setminus F\), there exist disjoint \((1,2)\) semi-preopen sets \(U, V\) such that \(x \in U\) and \(F \subseteq V\).

Lemma 9 For a space \(X\) the following properties are equivalent.
(i) \(X\) is \((1,2)\)-semi-preregular.
(ii) For each \(U \in (1,2)-\text{SO}(X)\) and each \(x \in U\), there exists \(V \in (1,2)-\text{SPO}(X)\) such that \(x \in V \subseteq (1,2)-\text{spcl}(V) \subseteq U\).
(iii) For each \(U \in (1,2)-\text{SO}(X)\) and each \(x \in U\), there exists \(V \in (1,2)-\text{SPR}(X)\) such that \(x \in V \subseteq U\).

3 \(\theta-(1,2)\)-Semi-Pregeneralized Closed Sets

In this section we define the \(\theta-(1,2)\)-semi-pregeneralized closed sets and study some properties.

Definition 10 A subset \(A\) of a space \(X\) is called \(\theta-(1,2)\)-semi-pregeneralized closed set/briefly \(\theta-(1,2)\)-spg-closed set) if \((1,2)\)-spcl\(_{\theta}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \((1,2)\)-semi-preopen in \(X\).

The complement of a \(\theta-(1,2)\)-spg-closed set in \(X\) is called \(\theta-(1,2)\)-semi-pregeneralized open (briefly \(\theta-(1,2)\)-spg-open).

Lemma 11 Every \((1,2)\)-sp-\(\theta\)-closed set is \(\theta-(1,2)\)-spg-closed.

Proof. The proof follows from the fact that for an \((1,2)\)-sp-\(\theta\)-closed set \((1,2)\)-spcl\(_{\theta}A = A\).

Remark 12 The converse of Lemma 11 is not true as shown in the following example.

Example 13 Let \(X = \{a, b, c\}\), \(\tau_1 = \emptyset, \{a\}, \{a, b\}, X\) and \(\tau_2 = \emptyset, \{a\}, \{a, c\}, X\). Then the set \(\{b, c\}\) is \((1,2)\)-spg-closed but not \((1,2)\)-sp-\(\theta\)-closed.

Theorem 14 A subset \(A\) of \(X\) is \(\theta-(1,2)\)-spg-open if and only if \(F \subseteq (1,2)\)-semi-preclosed in \(X\) and \(F \subseteq A\).

Proof. Necessity. Let \(A\) be \(\theta-(1,2)\)-spg-open and \(F \subseteq A\), where \(F\) is \((1,2)\)-semi-preclosed. Then \(X \setminus A \subseteq X \setminus F\) and \(X \setminus F\) is \((1,2)\)-semi-preopen. Therefore, \((1,2)\)-spcl\(_{\theta}(X \setminus A) \subseteq X \setminus F\). Hence \((1,2)\)-spcl\(_{\theta}(X \setminus A) = X \setminus ((1,2)\)-spint\(_{\theta}(A)) \subseteq X \setminus F\). Thus we have \(F \subseteq (1,2)\)-spint\(_{\theta}(A)\).

Sufficiency. If \(F\) is \((1,2)\)-semi-preclosed and \(F \subseteq (1,2)\)-spint\(_{\theta}(A)\) whenever \(F \subseteq A\), then \(X \setminus A \subseteq X \setminus F\) and \((1,2)\)-spint\(_{\theta}(A) \subseteq X \setminus F\). That is, \((1,2)\)-spcl\(_{\theta}(X \setminus A) \subseteq X \setminus F\). Therefore, \(X \setminus A\) is \((1,2)\)-spg-closed and hence \(A\) is \(\theta-(1,2)\)-spg-open. ■
Definition 15 A space $X$ is said to be $(1,2)$-$\beta$-$T_1$ if for any two distinct points $x, y$ of $X$, there exists $(1,2)$-semi-preopen sets $U$, $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Theorem 16 A bitopological space $X$ is $(1,2)$-$\beta$-$T_1$ if and only if $\{x\}$ is $(1,2)$-semi-precloased in $X$ for every $x \in X$.

Proof. If $\{x\}$ is $(1,2)$-semi-preclosed in $X$ for every $x \in X$, for $x \neq y$, $X \setminus \{x\}$, $X \setminus \{y\}$ are $(1,2)$-semi-preopen sets such that $y \in X \setminus \{x\}$ and $x \in X \setminus \{y\}$. Therefore, $X$ is $(1,2)$-$\beta$-$T_1$. Conversely, if $X$ is $(1,2)$-$\beta$-$T_1$ and if $y \in X \setminus \{x\}$ then $x \neq y$. Therefore, there exist $(1,2)$-semi-preopen sets $U_x, V_y$ in $X$ such that $x \in U_x$ but $y \notin U_x$ and $y \in V_y$ but $x \notin V_y$. Let $G$ be the union of all such $V_y$. Then $G$ is an $(1,2)$-semi-preopen set and $G \subset X \setminus \{x\} \subset X$. Therefore, $X \setminus \{x\}$ is an $(1,2)$-semi-preopen set in $X$. ■

Lemma 17 Let $A$ be $\theta$-$(1,2)$-spg-closed subset of $X$. Then

(i) $(1,2)$-$\mbox{spcl}_\theta(A) \setminus A$ does not contain a nonempty $(1,2)$-semi-precloased set.
(ii) $(1,2)$-$\mbox{spcl}_\theta(A) \setminus A$ is $\theta$-$(1,2)$-spg-open.

Proof. (i). Let $F$ be an $(1,2)$-semi-precloased set contained in $(1,2)$-$\mbox{spcl}_\theta(A) \setminus A$. Then $X \setminus F$ is $(1,2)$-semi-preopen and $A \subset X \setminus F$, it follows that $(1,2)$-$\mbox{spcl}_\theta(A) \subset X \setminus F$. Thus we get $F \subset X \setminus (1,2)$-$\mbox{spcl}_\theta(A)$ and $F \subset (1,2)$-$\mbox{spcl}_\theta(A)$. Hence $F = \emptyset$.

(ii). If $A$ is $\theta$-$(1,2)$-spg-closed and $F$ is an $(1,2)$-semi-precloased set contained in $(1,2)$-$\mbox{spcl}_\theta(A) \setminus A$, then $F$ is empty by (i). Therefore, $F \subset (1,2)$-$\mbox{spint}_\theta((1,2)$-$\mbox{spcl}_\theta(A) \setminus A)$. By Theorem 14, $(1,2)$-$\mbox{spcl}_\theta(A) \setminus A$ is $\theta$-$(1,2)$-spg-open. ■

Theorem 18 In a $(1,2)$-$\beta$-$T_1$ space $X$, every $\theta$-$(1,2)$-spg-closed set is $(1,2)$-sp-$\theta$-closed.

Proof. Let $A \subset X$ be $\theta$-$(1,2)$-spg-closed and $x \in (1,2)$-$\mbox{spcl}_\theta(A)$. Since $X$ is $(1,2)$-$\beta$-$T_1$, $\{x\}$ is $(1,2)$-semi-precloased and by Lemma 17, $x \notin (1,2)$-$\mbox{spcl}_\theta(A) \setminus A$. This implies that $x \in A$ and hence $(1,2)$-$\mbox{spcl}_\theta(A) \subset A$ and hence $A$ is $(1,2)$-sp-$\theta$-closed. ■

Theorem 19 [3] Let $A$ be a subset of $X$. Then

(i) $A \in (1,2)$-$\mbox{SP}(X)$ if and only if $(1,2)$-$\mbox{spcl}(A) \in (1,2)$-$\mbox{SP}(X)$.
(ii) $A \in (1,2)$-$\mbox{SP}(X)$ if and only if $(1,2)$-$\mbox{spint}(A) \in (1,2)$-$\mbox{SP}(X)$.

Theorem 20 For any subset $A$ of a space $X$, the following are equivalent.

(i) $(1,2)$-$\mbox{spcl}(A) = \bigcap \{V: A \subset V \text{ and } V \text{ is } (1,2)-\text{sp}-\theta-\text{closed}\}$;
(ii) $(1,2)$-$\mbox{spcl}(A) = \bigcap \{V: A \subset V \text{ and } V \in (1,2)$-$\mbox{SP}(X)\}$.

Proof. (i). If $x \notin (1,2)$-$\mbox{spcl}(A)$, then there exists $V \in (1,2)$-$\mbox{SP}(X,x)$ such that $(1,2)-\mbox{spcl}(V) \cap A = \emptyset$. By Theorem 19, $X \setminus (1,2)$-$\mbox{spcl}(V)$ is $(1,2)$-semi-prerregular. Hence, $X \setminus (1,2)$-$\mbox{spcl}(V)$ is an $(1,2)$-sp-$\theta$-closed set containing $A$ and $x \notin X \setminus (1,2)$-$\mbox{spcl}(V)$. Therefore, $x \notin \bigcap \{V: A \subset V \text{ and } V \in (1,2)$-$\mbox{SP}(X)\}$. ■
Conversely, if \( x \notin \bigcap \{ V : A \subset V \text{ and } V \text{ is } (1,2)\text{-sp-}\theta\text{-closed} \} \), then there exists an \((1,2)\)-sp-\(\theta\)-closed set \( V \) such that \( A \subset V \) and \( x \notin V \). Then there exists \( U \in (1,2)\)-SPO\(\text{pr}(X) \) such that \( x \in U \subset (1,2)\)-spcl\(\theta(U) \subset X \setminus V \). Therefore, \((1,2)\)-spcl\(\theta(U) \cap A \subset (1,2)\)-scl\(\theta(U) \cap V = \emptyset \). Hence \( x \notin (1,2)\)-spcl\(\theta(A) \).

(ii). It can be proved in a similar manner. □

**Theorem 21** Let \( A \) and \( B \) be subsets of \( X \). Then the following properties hold.

(i) If \( A \subset B \), then \((1,2)\)-spcl\(\theta(A) \subset (1,2)\)-spcl\(\theta(B) \).

(ii) \((1,2)\)-spcl\(\theta((1,2)\)-spcl\(\theta(A) = (1,2)\)-spcl\(\theta(A) \).

**Proof**. (i). Proof is obvious.

(ii). \((1,2)\)-spcl\(\theta(A \subset (1,2)\)-spcl\(\theta((1,2)\)-spcl\(\theta(A) \), in general. If \( x \notin (1,2)\)-spcl\(\theta(A) \), then there exists \( V \in (1,2)\)-SPO\(\text{pr}(X, x) \) such that \( V \cap A = \emptyset \). Since \( V \in (1,2)\)-SPO\(\text{pr}(X, V \cap (1,2)\)-spcl\(\theta(A) = \emptyset \) which shows that \( x \notin (1,2)\)-spcl\(\theta((1,2)\)-spcl\(\theta(A) \). Therefore, \((1,2)\)-spcl\(\theta((1,2)\)-spcl\(\theta(A) \subset (1,2)\)-spcl\(\theta(A) \). □

**Lemma 22** If \( A \) is a \( \theta(1,2)\)-spg-closed set of a space \( X \) such that \( A \subset B \subset (1,2)\)-spcl\(\theta(A) \), then \( B \) is also \( \theta(1,2)\)-spg-closed in \( X \).

**Proof**. Let \( U \) be \((1,2)\)-semi-preopen in \( X \) such that \( B \subset U \). Then \( A \subset U \).

Since \( A \) is \( \theta(1,2)\)-spg-closed, \((1,2)\)-spcl\(\theta(A) \subset U \) and by Theorem 21, \((1,2)\)-spcl\(\theta(B) \subset (1,2)\)-spcl\(\theta((1,2)\)-spcl\(\theta(A) \) \( = (1,2)\)-spcl\(\theta(A) \subset U \). Therefore, \( B \) is \( \theta(1,2)\)-spg-closed. □

**Definition 23** For a subset \( A \) of a space \( X \) we define \( A_{\theta}^{(1,2)sp} \) as follows: \( A_{\theta}^{(1,2)sp} = \{ x \in X : (1,2)\)-spcl\(\theta(\{ x \}) \cap A \neq \emptyset \} \)

**Proposition 24** \( A_{\theta}^{(1,2)sp} = \bigcap \{ U : A \subset U, \text{ U is } (1,2)\text{-sp-}\theta\text{-open} \} \) for any subset \( A \) of \( X \).

**Proof**. Let \( x \in A_{\theta}^{(1,2)sp} \) and \( x \notin \bigcap \{ U : A \subset U, U \text{ is } (1,2)\text{-sp-}\theta\text{-open} \} \). Then there exists an \((1,2)\)-sp-\(\theta\)-open set \( U \) containing \( A \) such that \( x \notin U \). Let \( y \in (1,2)\)-spcl\(\theta(\{ x \}) \cap U \). Thus \( y \in U \) and \( x \notin U \), a contradiction. If \( x \in \bigcap \{ U : A \subset U, U \text{ is } (1,2)\text{-sp-}\theta\text{-open} \} \) and \( x \notin A_{\theta}^{(1,2)sp} \), then \((1,2)\)-spcl\(\theta(\{ x \}) \cap A = \emptyset \). Hence \( x \notin X \setminus (1,2)\)-spcl\(\theta(\{ x \} \cap A \neq \emptyset \). Therefore, \( x \in A_{\theta}^{(1,2)sp} \). □

Thus \( A_{\theta}^{(1,2)sp} \) is the intersection of all the \( (1,2)\)-sp-\(\theta\)-open sets containing \( A \) which is by the usual notation, \((1,2)\)-spker\(\theta(A) \).

**Lemma 25** Let \( X \) be a topological space and \( x \in X \). The following are equivalent.

(i) \( x \in (1,2)\)-spcl\(\theta(\{ y \}) \).

(ii) \( y \in (1,2)\)-spker\(\theta(\{ x \}) \).
The following statements are equivalent for any two points \(x, y\) hence by our assumption, (1). Suppose that (1) holds. This implies that (1), since \(x \neq y\). Therefore, (1) holds. Hence, (1) holds. Proof is similar.

**Lemma 26** The following statements are equivalent for any two points \(x, y\) in a space \(X\).

(i) \((1,2)\)-spker\(_0\)\(\{x\}\) \(\neq \) \((1,2)\)-spker\(_0\)\(\{y\}\).

(ii) \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\).

**Proof.** (i) \(\Rightarrow\) (ii). Let \((1,2)\)-spker\(_0\)\(\{x\}\) \(\neq \) \((1,2)\)-spker\(_0\)\(\{y\}\). Then there exists a point \(z\) in \(X\) such that \(z \in (1,2)\)-spker\(_0\)\(\{x\}\) and \(z \notin (1,2)\)-spker\(_0\)\(\{y\}\).

From \(z \in (1,2)\)-spker\(_0\)\(\{x\}\), it follows that \(\{x\} \cap (1,2)\)-spcl\(_\theta\)\(\{z\}\) \(\neq \) \(\emptyset\). This implies that \(x \in (1,2)\)-spcl\(_\theta\)\(\{x\}\). Therefore, \(\{x\} \cap (1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \(\emptyset\).

Hence, \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\).

(ii) \(\Rightarrow\) (i). Let \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\). Then there exists a point \(z\) in \(X\) such that \(z \in (1,2)\)-spcl\(_\theta\)\(\{x\}\) and \(z \notin (1,2)\)-spcl\(_\theta\)\(\{y\}\). Hence there exists an \((1,2)\)-sp\(\theta\)-open set containing \(z\) and therefore, \(x\) but not \(y\). Therefore, \(y \notin (1,2)\)-spker\(_0\)\(\{x\}\) and \((1,2)\)-spker\(_0\)\(\{x\}\) \(\neq \) \((1,2)\)-spker\(_0\)\(\{y\}\).

**Definition 27** A space \(X\) is said to be \((1,2)\)-\(\beta\thetaR_0\) if every \((1,2)\)-sp\(\theta\)-open set contains the \((1,2)\)-semipre-\(\theta\)-closure of each of its singletons.

**Theorem 28** A space \(X\) is \((1,2)\)-\(\beta\thetaR_0\) if and only if for any \(x, y\) in \(X\), \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\) implies \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\cap \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\) \(= \) \(\emptyset\).

**Proof.** Necessity. If \(X\) is \((1,2)\)-\(\beta\theta\)-R\(_0\) and \(x, y\) in \(X\) such that \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\), then there exists \(z \in (1,2)\)-spcl\(_\theta\)\(\{x\}\) such that \(z \notin (1,2)\)-spcl\(_\theta\)\(\{y\}\), say. Therefore, there exists \(V \in (1,2)\)-SPO\(_\theta\)(\(X\)) such that \(y \notin V\) and \(y \in V\) and hence \(x \in V\). Thus we get \(x \notin (1,2)\)-spcl\(_\theta\)\(\{y\}\) and therefore, \(x \in X \setminus (1,2)\)-spcl\(_\theta\)\(\{y\}\). This implies that \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\subset \) \(X \setminus (1,2)\)-spcl\(_\theta\)\(\{y\}\) and therefore, \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\cap \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\) \(= \) \(\emptyset\).

Sufficiency. Let \(V \in (1,2)\)-sp\(\theta\)-open and \(x, y\) in \(V\). If \(y \notin X \setminus V\), then \(x \neq y\) and \(x \notin (1,2)\)-spcl\(_\theta\)\(\{y\}\). This shows that \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\) and hence by our assumption, \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\cap \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\) \(= \) \(\emptyset\). Hence \(y \notin (1,2)\)-spcl\(_\theta\)\(\{x\}\). Therefore, \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\subset \) \(V\).

**Theorem 29** A space \(X\) is \((1,2)\)-\(\beta\theta\)-R\(_0\) if and only if for any \(x, y\) in \(X\), \((1,2)\)-skker\(_0\)\(\{x\}\) \(\neq \) \((1,2)\)-skker\(_0\)\(\{y\}\) implies \((1,2)\)-skker\(_0\)\(\{x\}\) \(\cap \) \((1,2)\)-skker\(_0\)\(\{y\}\) \(= \) \(\emptyset\).

**Proof.** Suppose that \(X\) is \((1,2)\)-\(\beta\theta\)-R\(_0\) and if for any \(x, y\) in \(X\), \((1,2)\)-skker\(_0\)\(\{x\}\) \(\neq \) \((1,2)\)-skker\(_0\)\(\{y\}\), then by Lemma 26, \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(\neq \) \((1,2)\)-spcl\(_\theta\)\(\{y\}\). If \(z \in (1,2)\)-skker\(_0\)\(\{x\}\) \(\cap \) \((1,2)\)-skker\(_0\)\(\{y\}\), then from \(z \in (1,2)\)-skker\(_0\)\(\{x\}\) and by Lemma 25, it follows that \(x \in (1,2)\)-spcl\(_\theta\)\(\{z\}\). Since \(x \in (1,2)\)-spcl\(_\theta\)\(\{x\}\), by Theorem 28, \((1,2)\)-spcl\(_\theta\)\(\{x\}\) \(= \) \((1,2)\)-spcl\(_\theta\)\(\{z\}\). Similarly,
we have \((1,2)\)-\(\text{spcl}_\theta(\{y\}) = (1,2)\)-\(\text{spcl}_\theta(\{z\})\), a contradiction. Therefore, \((1,2)\)-\(\text{spker}_\theta(\{x\}) \cap (1,2)\)-\(\text{spker}_\theta(\{y\}) = \emptyset\).

Conversely, let \(x, y\) be any two points in \(X\) such that \((1,2)\)-\(\text{spker}_\theta(\{x\}) \neq (1,2)\)-\(\text{spker}_\theta(\{y\})\) implies \((1,2)\)-\(\text{spker}_\theta(\{x\}) \cap (1,2)\)-\(\text{spker}_\theta(\{y\}) = \emptyset\). If \((1,2)\)-\(\text{spcl}_\theta(\{x\}) \neq (1,2)\)-\(\text{spcl}_\theta(\{y\})\), then by Lemma 26, \((1,2)\)-\(\text{spker}_\theta(\{x\}) \neq (1,2)\)-\(\text{spker}_\theta(\{y\})\). Hence \((1,2)\)-\(\text{spker}_\theta(\{x\}) \cap (1,2)\)-\(\text{spker}_\theta(\{y\}) = \emptyset\) which implies that \((1,2)\)-\(\text{spcl}_\theta(\{x\}) \cap (1,2)\)-\(\text{spcl}_\theta(\{y\}) = \emptyset\). Therefore, by hypothesis, \((1,2)\)-\(\text{spker}_\theta(\{z\}) = (1,2)\)-\(\text{spker}_\theta(\{x\})\). Then \(z \in (1,2)\)-\(\text{spcl}_\theta(\{x\}) \cap (1,2)\)-\(\text{spcl}_\theta(\{y\})\) implies that \((1,2)\)-\(\text{spker}_\theta(\{x\}) = (1,2)\)-\(\text{spker}_\theta(\{z\}) = (1,2)\)-\(\text{spker}_\theta(\{y\})\), a contradiction. Therefore, by Theorem 28, \(X\) is \((1,2)\)-\(\theta\)-\(R_0\).

4 \(\theta\)-(1,2)-\(\beta\)-Irresolute Functions

In this section we introduce the notion of \(\theta\)-(1,2)-\(\beta\)-irresolute functions.

Definition 30 A map \(f:X \to Y\) is called \(\theta\)-(1,2)-\(\beta\)-irresolute if for each \(x \in X\) and each \(V \in (1,2)\)-\(\text{SPO}(Y, f(x))\), there exists \(U \in (1,2)\)-\(\text{SPO}(X, x)\) such that \(f((1,2)\)-\(\text{spcl}(U)) \subset (1,2)\)-\(\text{spcl}(V)\).

Theorem 31 Every \((1,2)\)-\(\beta\)-irresolute map is \(\theta\)-(1,2)-\(\beta\)-irresolute.

Proof. Let \(x \in X\) and \(V \in (1,2)\)-\(\text{SPO}(X, f(x))\). Since \(f\) is \((1,2)\)-\(\beta\)-irresolute, \(f^{-1}(V)\) is \((1,2)\)-semi-preopen and \(f^{-1}((1,2)\)-\(\text{spcl}(V))\) is \((1,2)\)-semi-preclosed in \(X\). Let \(U = f^{-1}(V)\). Then \(U \in (1,2)\)-\(\text{SPO}(X, x)\) and \((1,2)\)-\(\text{spcl}(U) \subset f^{-1}((1,2)\)-\(\text{spcl}(V))\). Therefore, \(f((1,2)\)-\(\text{spcl}(U)) \subset (1,2)\)-\(\text{spcl}(V)\). Hence \(f\) is \(\theta\)-(1,2)-\(\beta\)-irresolute.

Remark 32 The converse of Theorem 31, is not true in general, as shown in the following example.

Example 33 Let \(X = \{a, b, c\}, \tau_1 = \{\emptyset, X\}, \tau_2 = \{\emptyset, \{b, c\}, X\}\) and \(Y = \{p, q, r\}, \sigma_1 = \{\emptyset, \{p\}, \{p, q\}, Y\}\) and \(\sigma_2 = \{\emptyset, \{p\}, Y\}\). Define a function \(f:X \to Y\) as \(f(a) = p, f(b) = r\) and \(f(c) = q\). Then \(f\) is \(\theta\)-(1,2)-\(\beta\)-irresolute but not \((1,2)\)-\(\beta\)-irresolute since \(f^{-1}(\{p\}) = \{a, b\} \notin (1,2)\)-\(\text{SPO}(X)\).

Remark 34 Thus we have

vividly \((1,2)\)-\(\beta\)-irresolute \(\Rightarrow\) \((1,2)\)-\(\beta\)-irresolute \(\Rightarrow\) \(\theta\)-(1,2)-\(\beta\)-irresolute

and none of them is reversible.

Theorem 35 For a function \(f:X \to Y\) the following properties are equivalent.

(i) \(f\) is \(\theta\)-(1,2)-\(\beta\)-irresolute.

(ii) \((1,2)\)-\(\text{spcl}_\theta(f^{-1}(B)) \subset f^{-1}((1,2)\)-\(\text{spcl}_\theta(B))\) for every subset \(B\) of \(Y\).

(iii) \(f((1,2)\)-\(\text{spcl}_\theta(A)) \subset (1,2)\)-\(\text{spcl}_\theta(f(A))\) for every subset \(A\) of \(X\).
Proof. (i) ⇒ (ii).
Let \( B \) be any subset of \( Y \). Suppose that \( x \notin f^{-1}((1,2)\text{-}\text{spcl}_\theta(B)) \). Then \( f(x) \notin (1,2)\text{-}\text{spcl}_\theta(B) \) and there exists \( V \in (1,2)\text{-}\text{SPO}(X, f(x)) \) such that \((1,2)\text{-}\text{spcl}(V) \cap B = \emptyset \). Since \( f \) is \( \theta-(1,2)\)-\text{irresolute}, there exists \( U \in (1,2)\text{-}\text{SPO}(X, x) \) such that \((1,2)\text{-}\text{spcl}(U) \) \( \subseteq \) \((1,2)\text{-}\text{spcl}(V) \). Therefore, \( f((1,2)\text{-}\text{spcl}(U)) \cap B = \emptyset \) and \((1,2)\text{-}\text{spcl}(U) \) \( \cap \) \( f^{-1}(B) = \emptyset \). Hence, \( x \notin (1,2)\text{-}\text{spcl}_\theta(f^{-1}(B)) \). Therefore, \((1,2)\text{-}\text{spcl}_\theta(f^{-1}(B)) \) \( \subseteq \) \((1,2)\text{-}\text{spcl}_\theta(B) \).

(ii) ⇒ (iii). Let \( \text{A} \) be any subset of \( X \). Then \((1,2)\text{-}\text{spcl}_\theta(A) \subseteq (1,2)\text{-}\text{spcl}_\theta(f^{-1}(f(A))) \) and hence \( f((1,2)\text{-}\text{spcl}_\theta(A)) \subseteq (1,2)\text{-}\text{spcl}_\theta(f(A)) \).

(iii) ⇒ (ii). Let \( \text{B} \) be a subset of \( Y \). By (iii), \( f((1,2)\text{-}\text{spcl}_\theta(f^{-1}(B))) \subseteq (1,2)\text{-}\text{spcl}_\theta(f^{-1}(B)) \subseteq (1,2)\text{-}\text{spcl}_\theta(f^{-1}(f^{-1}((1,2)\text{-}\text{spcl}_\theta(B)))). \)

Proof. (i) ⇒ (ii).
Let \( V \in (1,2)\text{-}\text{SPO}(Y) \) and \( x \notin f^{-1}(V) \). Then \( f(x) \in V \) and there exists \( U \in (1,2)\text{-}\text{SPO}(X, f(x)) \) such that \((1,2)\text{-}\text{spcl}(U) \subseteq (1,2)\text{-}\text{spcl}(V) \). Thus \( x \in U \subseteq (1,2)\text{-}\text{spcl}(U) \subseteq f^{-1}((1,2)\text{-}\text{spcl}(V)) \) and \( x \in (1,2)\text{-}\text{spint}_\theta(f^{-1}((1,2)\text{-}\text{spcl}(V))) \). Hence \( f^{-1}(V) \subseteq (1,2)\text{-}\text{spint}_\theta(f^{-1}(((1,2)\text{-}\text{spcl}(V))) \).

(ii) ⇒ (iii).
Let \( V \in (1,2)\text{-}\text{SPO}(Y) \) and \( x \notin f^{-1}((1,2)\text{-}\text{spcl}(V)) \). Then \( f(x) \notin (1,2)\text{-}\text{spcl}(V) \) and there exists \( W \in (1,2)\text{-}\text{SPO}(Y, f(x)) \) such that \( W \cap V = \emptyset \) and \((1,2)\text{-}\text{spcl}(W) \) \( \cap \) \( V = \emptyset \). Then \( f^{-1}((1,2)\text{-}\text{spcl}(W)) \) \( \cap \) \( f^{-1}(V) = \emptyset \). Now \( x \in f^{-1}(W) \) and by (ii), \( x \in (1,2)\text{-}\text{spint}_\theta(f^{-1}(((1,2)\text{-}\text{spcl}(W))) \). There exists \( U \in (1,2)\text{-}\text{SPO}(X, x) \) such that \((1,2)\text{-}\text{spcl}(U) \subseteq f^{-1}((1,2)\text{-}\text{spcl}(W)) \). Thus \((1,2)\text{-}\text{spcl}(U) \) \( \cap \) \( f^{-1}(V) = \emptyset \) and hence \( x \notin (1,2)\text{-}\text{spcl}_\theta(f^{-1}(V)) \). Thus we get \((1,2)\text{-}\text{spcl}_\theta(f^{-1}(V)) \subseteq f^{-1}(((1,2)\text{-}\text{spcl}(V))) \).

(iii) ⇒ (i).
Let \( x \in X \) and \( V \in (1,2)\text{-}\text{SPO}(Y, f(x)) \). Then \( V \cap (Y \setminus (1,2)\text{-}\text{spcl}(V)) = \emptyset \) and \( f(x) \notin (1,2)\text{-}\text{spcl}(Y \setminus (1,2)\text{-}\text{spcl}(V)) \). Therefore, \( x \notin f^{-1}((1,2)\text{-}\text{spcl}(Y \setminus (1,2)\text{-}\text{spcl}(V))) \) and by (iii), \( x \notin (1,2)\text{-}\text{spcl}_\theta(f^{-1}(Y \setminus (1,2)\text{-}\text{spcl}(V)) \). There exists \( U \in (1,2)\text{-}\text{SPO}(X, x) \) such that \((1,2)\text{-}\text{spcl}(U) \subseteq f^{-1}(Y \setminus (1,2)\text{-}\text{spcl}(V)) \) \( = \emptyset \). Hence \( f((1,2)\text{-}\text{spcl}(U)) \subseteq (1,2)\text{-}\text{spcl}(V) \) and hence \( f \) is \( \theta-(1,2)\)-\text{irresolute}.

Theorem 36 For a function \( f:X \rightarrow Y \) the following properties are equivalent.

(i) \( f \) is \( \theta-(1,2)\)-\text{irresolute}.

(ii) \( f^{-1}(V) \subseteq (1,2)\text{-}\text{spint}_\theta(f^{-1}((1,2)\text{-}\text{spcl}(V))) \) for every \( V \in (1,2)\text{-}\text{SPO}(Y) \).

(iii) \((1,2)\text{-}\text{spcl}_\theta(f^{-1}(V)) \subseteq f^{-1}(((1,2)\text{-}\text{spcl}(V))) \) for every \( V \in (1,2)\text{-}\text{SPO}(Y) \).

Theorem 37 Let \( Y \) be an \( (1,2)\)-semi-preregular space. Then, for a function \( f:X \rightarrow Y \) the following are equivalent.

(i) \( f \) is vividly \( (1,2)\)-\text{irresolute}.

(ii) \( f \) is \( (1,2)\)-\text{irresolute}.

(iii) \( f \) is \( \theta-(1,2)\)-\text{irresolute}.
Proof. (i) ⇒ (ii) It is proved in [3].
(ii) ⇒ (iii) By Theorem 31 it is obvious.
(iii) ⇒ (i). If \(x \in X\) and \(V \in (1,2)\)-SPO\((Y, f(x))\). Since \(Y\) is \((1,2)\)-semi-preregular, by (ii) of Lemma 9, there exists \(W \in (1,2)\)-SPO\((Y)\) such that \(f(x) \in W \subset (1,2)\)-spcl\((W) \subset V\). Since \(f\) is \(\theta-(1,2)\)-\(\beta\)-irresolute, there exists \(U \in (1,2)\)-SPO\((X, x)\) such that \(f((1,2)\)-sp\(cl(U)) \subset (1,2)\)-spcl\((W) \subset V\). Therefore, \(f\) is vividly \((1,2)\)-\(\beta\)-irresolute.

5 \(\theta-(1,2)\)-Semi-pregeneralized Continuous Functions

Definition 38 A function \(f:X \rightarrow Y\) is called

(i) \(\theta-(1,2)\)-semi-pregeneralized continuous (briefly \(\theta-(1,2)\)-spg-continuous) if \(f^{-1}(F)\) is \(\theta-(1,2)\)-spg-closed set in \(X\) for every \((1,2)\)-semi-preclosed set of \(Y\).

(ii) \(\theta-(1,2)\)-semi-pregeneralized irresolute (briefly \(\theta-(1,2)\)-spg-irresolute) if \(f^{-1}(F)\) is \(\theta-(1,2)\)-spg-closed in \(X\) for every \(\theta-(1,2)\)-spg-closed set \(F\) of \(Y\).

Recall that a function \(f:X \rightarrow Y\) is vividly \((1,2)\)-\(\beta\)-irresolute if and only if \(f^{-1}(V)\) is \((1,2)\)-sp-\(\theta\)-closed in \(X\) for every \((1,2)\)-semi-preclosed set in \(Y\) [3].

Theorem 39 If a function \(f:X \rightarrow Y\) is vividly \((1,2)\)-\(\beta\)-irresolute, then it is \(\theta-(1,2)\)-spg-continuous.

Proof. If \(V\) is \((1,2)\)-semi-preclosed in \(Y\), then \(f^{-1}(V)\) is \((1,2)\)-sp-\(\theta\)-closed in \(X\). Therefore, by Lemma 11, \(f^{-1}(V)\) is \(\theta-(1,2)\)-spg-closed.

 Remark 40 The converse of the Theorem 39 is not true in general, as shown in the following example.

Example 41 Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, X\}\), \(\tau_2 = \{\emptyset, \{a, c\}, X\}\) and \(Y = \{p, q\}\). \(\sigma_1 = \{\emptyset, \{p\}, Y\}\) and \(\sigma_2 = \{\emptyset, \{q\}, Y\}\). Define a function \(f:X \rightarrow Y\) as \(f(a) = p\), \(f(b) = f(c) = q\). Then \(f\) is \(\theta-(1,2)\)-spg-continuous but not vividly \((1,2)\)-\(\beta\)-irresolute since for \(a \in X\), there does not exist an \((1,2)\)-semi-preopen set \(U\) such that \(f((1,2)\)-sp\(cl(U)) \subset \{p\}\).

Definition 42 A function \(f:X \rightarrow Y\) is called always \((1,2)\)-sp-\(\theta\)-open (resp. always \((1,2)\)-sp-\(\theta\)-closed) if \(f(U)\) is \((1,2)\)-sp-\(\theta\)-open (resp. \((1,2)\)-sp-\(\theta\)-closed) in \(Y\) for every \((1,2)\)-sp-\(\theta\)-open (resp. \((1,2)\)-sp-\(\theta\)-closed) set \(U\) of \(X\).

Theorem 43 For a function \(f:X \rightarrow Y\) the following are equivalent.

(i) \(f\) is always \((1,2)\)-sp-\(\theta\)-closed.

(ii) For each \(U \subset X\), \((1,2)\)-sp\(cl(f(U)) \subset f((1,2)\)-sp\(cl(U))\).

(iii) If \(f^{-1}(V) \subset U\), where \(V \subset Y\) and \(U\) is \((1,2)\)-sp-\(\theta\)-open in \(X\), then there exists an \((1,2)\)-sp-\(\theta\)-open set \(W \subset Y\) such that \(V \subset W\) and \(f^{-1}(W) \subset U\).

(iv) If \(f^{-1}(y) \subset U\), where \(y \in Y\) and \(U\) is \((1,2)\)-sp-\(\theta\)-open in \(X\), then there exists an \((1,2)\)-sp-\(\theta\)-open set \(W \subset Y\) such that \(y \in W\) and \(f^{-1}(W) \subset U\).
Theorem 44 Let \( f: X \to Y \) and \( g: Y \to Z \) be two functions.

(i) If \( f \) is \( \theta-(1,2) \)-spg-irresolute and \( g \) is \( \theta-(1,2) \)-spg-continuous, then \( g \circ f \) is \( \theta-(1,2) \)-spg-continuous.

(ii) If both \( f \) and \( g \) are \( \theta-(1,2) \)-spg-irresolute, then \( g \circ f \) is \( \theta-(1,2) \)-spg-irresolute.

Definition 45 A function \( f: X \to Y \) is called a \( \theta-(1,2) \)-spg-homeomorphism if

(i) \( f \) is bijective.

(ii) \( f \) is \( \theta-(1,2) \)-spg-irresolute.

(iii) \( f^{-1} \) is \( \theta-(1,2) \)-spg-irresolute.

We denote the collection of all the \( \theta-(1,2) \)-spg-homeomorphisms \( f: X \to Y \) by \( \theta-(1,2) \)-spgh(X).

Theorem 46 The collection \( \theta-(1,2) \)-spgh(X) is a group.

Proof. Define a binary operation \( \star: (1,2) \)-spgh(X) \( \times (1,2) \)-spgh(X) \( \to (1,2) \)-spgh(X) by \( \star(f,g) = g \circ f \). Then \( \star \) is well-defined and it is easily proved that under this binary operation \( \theta-(1,2) \)-spgh(X) is a group.

References


On $\theta-(1,2)$-Semi-Pregeneralized Closed Sets

Address
S. Athisaya Ponmani:
Department of Mathematics, Jayaraj Annapackiam College for Women, Periyakulam, Theni (Dt.)-625601, Tamilnadu, India.
E-mail: athisayaponmani@yahoo.co.in

R. Raja Rajeswari
Department of Mathematics, Sri Parasakthi College, Courtalam, Tirunelveli (Dt.)-627802, Tamilnadu, India.
E-mail: raji_arul2000@yahoo.co.in

M. Lellis Thivagar
Department of Mathematics, Arul Anandar College, Karumathur, Madurai (Dt.)-625514, Tamilnadu, India.
E-mail: mlthivagar@yahoo.co.in
Erdal Ekici
Department of Mathematics, Canakkale Onsekiz Mart University, Terzioglu Campus, 17020 Canakkale, Turkey.
E-mail: eekici@comu.edu.tr