Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **22:1** (2008), 69–77

A BIVARIATE MARSHALL AND OLKIN EXPONENTIAL MINIFICATION PROCESS

Miroslav M. Ristić, Biljana Č. Popović, Aleksandar Nastić and Miodrag Đorđević

Abstract

In this paper we present a stationary bivariate minification process with Marshall and Olkin exponential distribution. The process is given by

$$\begin{aligned} X_n &= K\min(X_{n-1}, Y_{n-1}, \eta_{n1}), \\ Y_n &= K\min(X_{n-1}, Y_{n-1}, \eta_{n2}), \end{aligned}$$

where $\{(\eta_{n1}, \eta_{n2}), n \geq 1\}$ is a sequence of independent and identically distributed random vectors, the random vectors (X_m, Y_m) and (η_{n1}, η_{n2}) are independent for $m < n$ and $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$, $K > (\lambda_1 + \lambda_2 + \lambda_{12})/\lambda_{12}$. The innovation distribution, the joint distribution of random vectors (X_n, Y_n) and (X_{n-j}, Y_{n-j}), $j > 0$, the autocovariance and the autocorrelation matrix are obtained. The unknown parameters are estimated and their asymptotic properties are obtained.

1. INTRODUCTION

A minification process of the first-order is given by

$$X_n = K\min(X_{n-1}, \varepsilon_n), n \geq 1,$$

where $K > 1$ and $\{\varepsilon_n,\ n \geq 1\}$ is an innovation process of independent and identically distributed (i.i.d.) random variables. Several authors have introduced minification processes with given marginals. Tavares [13] introduced the minification process with exponential marginal distribution. Sim [10] introduced the minification process with Weibull marginal distribution. Yeh, Arnold and Robertson [14] introduced a Pareto minification process. Arnold and Robertson [1] introduced a logistic minification process. Pillai [7] and Pillai, Jose and Jayakumar [8] introduced semi-Pareto minification processes. Balakrishna [2] considered some properties of

2000 *Mathematics Subject Classifications.* 62M10, 60G10.
Key words and Phrases. Minification process, Bivariate Marshall and Olkin Exponential Distribution, Estimation, Ergodic, Uniformly mixing.
Received: February 8, 2007
Communicated by Dragan S. Djordjević

the semi-Pareto minification process of Pillai [7] and estimated the unknown parameters of the model. Lewis and McKenzie [5] introduced the minification process with marginal distribution function $F_{X_0}(x)$. Some bivariate and multivariate minification processes are introduced by Balakrishna and Jayakumar [3], Thomas and Jose [11], [12] and Ristic [9].

In this paper we consider a stationary bivariate minification process of Ristić [9] with the bivariate Marshall and Olkin exponential distributions BVE$(\lambda_1, \lambda_2, \lambda_{12})$ and $K = L$. Motivated by situations that arise in reliability theory such as the failure of paired jet engines or the registration of an event by two adjacent geiger counters, Marshall and Olkin [6] introduced the bivariate exponential distribution with survival function

$$P\{X > x, Y > y\} = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)}, \quad x, y > 0,$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$. The random variables are constructed such that X and Y are dependent exponentially distributed random variables with parameters $\lambda_1 + \lambda_{12}$ and $\lambda_2 + \lambda_{12}$, respectively. The important property of this distribution is that it is not absolutely continuous distribution, since the probability $P\{X = Y\} = \lambda_{12}/(\lambda_1 + \lambda_2 + \lambda_2)$ is non-negative. The density function $f(x, y)$ of the BVE$(\lambda_1, \lambda_2, \lambda_{12})$ distribution is given by

$$f(x,y) = \begin{cases} \lambda_1(\lambda_2 + \lambda_{12}) e^{-\lambda_1 x - (\lambda_2 + \lambda_{12}) y}, & y > x > 0, \\ \lambda_2(\lambda_1 + \lambda_{12}) e^{-(\lambda_1 + \lambda_{12}) x - \lambda_2 y}, & x > y > 0, \\ \lambda_{12} e^{-(\lambda_1 + \lambda_2 + \lambda_{12}) x}, & x = y > 0. \end{cases}$$

This paper is organized as follows. The properties of the process are considered in Section 2. In Section 3 we give the estimates of the parameters of the process.

2. PROPERTIES OF THE PROCESS

In this section we consider a stationary bivariate minification process with bivariate Marshall and Olkin exponential distribution BVE$(\lambda_1, \lambda_2, \lambda_{12})$. The process is given by

$$\begin{aligned} X_n &= K \min(X_{n-1}, Y_{n-1}, \eta_{n1}), \\ Y_n &= K \min(X_{n-1}, Y_{n-1}, \eta_{n2}), \end{aligned} \tag{1}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} > 0$, $K > \lambda/\lambda_{12}$, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, $\{(\eta_{n1}, \eta_{n2}),\ n \geq 1\}$ is a sequence of i.i.d. random vectors and the random vectors (X_m, Y_m) and (η_{n1}, η_{n2}) are independent for $m < n$.

Ristić [9] derived the innovation distribution of the random vector (η_{n1}, η_{n2}). The random vector (η_{n1}, η_{n2}) has the bivariate Marshall and Olkin exponential distribution BVE$(\lambda_1 K, \lambda_2 K, \lambda_{12} K - \lambda)$. The marginal distributions of the random variables η_{n1} and η_{n2} are $\varepsilon(c_1)$ and $\varepsilon(c_2)$, respectively, where $c_1 = (\lambda_1 + \lambda_{12})K - \lambda$ and $c_2 = (\lambda_2 + \lambda_{12})K - \lambda$. Following Ristić [9], we obtain the joint survival function of the random vectors (X_n, Y_n) and (X_{n-j}, Y_{n-j}), $j > 0$. Denote the joint survival function of (X_n, Y_n) and (X_{n-j}, Y_{n-j}) by

$$S_j(x_1, y_1, x_2, y_2; K) = P\{X_n > x_1, Y_n > y_1, X_{n-j} > x_2, Y_{n-j} > y_2\}. \quad (2)$$

The joint survival function $S_j(x_1, y_1, x_2, y_2; K)$, $j \geq 1$, can be obtained recursively as

$$\begin{aligned} S_j(x_1, y_1, x_2, y_2; K) &= \frac{\overline{F}\left(\max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}, x_2\right), \max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}, y_2\right)\right) \cdot \overline{F}(x_1, y_1)}{\overline{F}\left(\max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}\right), \max\left(\frac{x_1}{K^j}, \frac{y_1}{K^j}\right)\right)}, \\ &= S_1(x_1, y_1, x_2, y_2; K^j). \end{aligned}$$

It is obvious that the joint distribution and the properties of the random vector $(X_n, Y_n,\ X_{n-j}, Y_{n-j})$ can be derived from the joint distribution and the properties of the random vector $(X_n, Y_n, X_{n-1}, Y_{n-1})$ replacing K by K^j.

Now we discuss the autocovariance structure of the bivariate Marshall and Olkin exponential minification process. We define the autocovariance matrix of a bivariate process $\{(X_n, Y_n),\ n \geq 0\}$ by

$$\Gamma(j) = \begin{bmatrix} Cov(X_n, X_{n-j}) & Cov(X_n, Y_{n-j}) \\ Cov(Y_n, X_{n-j}) & Cov(Y_n, Y_{n-j}) \end{bmatrix}.$$

To derive the autocovariance matrix $\Gamma(j)$ it suffices to derive the autocovariance matrix $\Gamma(1)$.

In order to compute the moment $E(X_n X_{n-1})$, we consider the conditional expectation $E(X_n | X_{n-1}, Y_{n-1})$. From the definition of the process $\{(X_n, Y_n),\ n \geq 0\}$, we have that conditional distribution for X_n, given $X_{n-1} = x$ and $Y_{n-1} = y$, is given by

$$P\{X_n \leq z | X_{n-1} = x, Y_{n-1} = y\} = \begin{cases} 1 - e^{-\frac{c_1 z}{K}} & ,\quad z < K\min(x, y), \\ 1 & ,\quad z \geq K\min(x, y). \end{cases}$$

Note that this is not an absolutely continuous distribution, since the probability

$$\begin{aligned} P\{X_n = K\min(X_{n-1}, Y_{n-1}) | X_{n-1} = x, Y_{n-1} = y\} &= P\{\eta_{n1} > \min(x, y)\} \\ &= e^{-c_1 \min(x,y)} \end{aligned}$$

is non-negative. Now, the conditional expectation is

$$\begin{aligned} E(X_n | X_{n-1} &= x, Y_{n-1} = y) = \frac{c_1}{K} \int_0^{K\min(x,y)} z e^{-\frac{c_1 z}{K}} dz \\ &+ K\min(x, y) e^{-c_1 \min(x,y)} = \frac{K}{c_1}\left(1 - e^{-c_1 \min(x,y)}\right). \end{aligned}$$

Using this it is easy to verify that

$$E(X_n X_{n-1}) = \frac{K}{c_1} \cdot E\left[X_{n-1}\left(1 - e^{-c_1 \min(X_{n-1}, Y_{n-1})}\right)\right]. \quad (3)$$

In order to compute the moment $E(X_nX_{n-1})$ we will need the following lemma.

Lemma 1. Let (X,Y) be a random vector with bivariate Marshall and Olkin exponential distribution $BVE(\lambda_1,\lambda_2,\lambda_{12})$. Let $U=X$, $W=Y$ and $V=\min(X,Y)$. Then:

(i) the random vector (U,V) has the survival function

$$P\{U>x,V>y\}=e^{-(\lambda_1+\lambda_{12})\max(x,y)-\lambda_2 y},$$

and

$$P\{U=V\}=\frac{\lambda_1+\lambda_{12}}{\lambda},$$

(ii) the random vector (W,V) has the survival function

$$P\{W>x,V>y\}=e^{-\lambda_1 y-(\lambda_2+\lambda_{12})\max(x,y)},$$

and

$$P\{W=V\}=\frac{\lambda_2+\lambda_{12}}{\lambda}.$$

Proof. (i) From the definition of the random variables U and V, we have that

$$\begin{aligned}P\{U>x,V>y\} &= P\{X>\max(x,y),Y>y\}\\ &= e^{-\lambda_1\max(x,y)-\lambda_2 y-\lambda_{12}\max(\max(x,y),y)}\\ &= e^{-(\lambda_1+\lambda_{12})\max(x,y)-\lambda_2 y},\end{aligned}$$

and

$$P\{U=V\}=P\{X\le Y\}=\frac{\lambda_1+\lambda_{12}}{\lambda}.$$

(ii) The proof is very similar to the proof of (i). □

Now, setting $U=X_{n-1}$ and $V=\min(X_{n-1},Y_{n-1})$ in (3) and using Lemma 1, we have that

$$\begin{aligned}E(X_nX_{n-1}) &= \frac{K}{c_1}\cdot E\left[U\left(1-e^{-c_1V}\right)\right]\\ &= \frac{K}{c_1}\lambda_2(\lambda_1+\lambda_{12})\int_0^\infty\int_0^u u(1-e^{-c_1v})e^{-(\lambda_1+\lambda_{12})u-\lambda_2 v}dvdu\\ &+ \frac{K}{c_1}(\lambda_1+\lambda_{12})\int_0^\infty u(1-e^{-c_1u})e^{-\lambda u}du\\ &= \frac{K+1}{K(\lambda_1+\lambda_{12})^2}.\end{aligned}$$

Using this result, we conclude that

$$Cov(X_n,X_{n-1})=\frac{1}{K(\lambda_1+\lambda_{12})^2}.$$

Similarly, we can verify that

$$E(X_nY_{n-1}) = \frac{K}{c_1} \cdot E\left[W\left(1 - e^{-c_1 V}\right)\right] = \frac{K(\lambda_1 + \lambda_{12}) + \lambda_2 + \lambda_{12}}{K(\lambda_1 + \lambda_{12})^2(\lambda_2 + \lambda_{12})},$$

and

$$Cov(X_n, Y_{n-1}) = \frac{1}{K(\lambda_1 + \lambda_{12})^2}.$$

Let us consider now $Cov(Y_n, X_{n-1})$ and $Cov(Y_n, Y_{n-1})$. The conditional distribution for Y_n, given $X_{n-1} = x$ and $Y_{n-1} = y$, is given by

$$P\{Y_n \le z | X_{n-1} = x, Y_{n-1} = y\} = \begin{cases} 1 - e^{-\frac{c_2 z}{K}} & , \quad z < K\min(x, y), \\ 1 & , \quad z \ge K\min(x, y). \end{cases}$$

Also, we have $P\{Y_n = K\min(X_{n-1}, Y_{n-1}) | X_{n-1} = x, Y_{n-1} = y\} = e^{-c_2 \min(x,y)}$. Finally, we obtain

$$Cov(Y_n, X_{n-1}) = Cov(Y_n, Y_{n-1}) = \frac{1}{K(\lambda_2 + \lambda_{12})^2}$$

in a similar way as we have obtained $Cov(X_n, X_{n-1})$ and $Cov(X_n, Y_{n-1})$.

Thus we obtain the autocovariance matrix $\Gamma(1)$ as

$$\Gamma(1) = \frac{1}{K}\begin{bmatrix} \frac{1}{(\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} \\ \frac{1}{(\lambda_2+\lambda_{12})^2} & \frac{1}{(\lambda_2+\lambda_{12})^2} \end{bmatrix}.$$

If we replace K by K^j in $\Gamma(1)$, then we will obtain the autocovariance matrix $\Gamma(j)$ as

$$\Gamma(j) = \frac{1}{K^j}\begin{bmatrix} \frac{1}{(\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} \\ \frac{1}{(\lambda_2+\lambda_{12})^2} & \frac{1}{(\lambda_2+\lambda_{12})^2} \end{bmatrix}.$$

We will now discuss the autocorrelation structure of the bivariate minification process with bivariate Marshall and Olkin exponential distribution. We define the autocorrelation matrix by

$$R(j) = \begin{bmatrix} Corr(X_n, X_{n-j}) & Corr(X_n, Y_{n-j}) \\ Corr(Y_n, X_{n-j}) & Corr(Y_n, Y_{n-j}) \end{bmatrix}.$$

After elementary calculation we get

$$R(j) = \frac{1}{K^j}\begin{bmatrix} 1 & \frac{\lambda_2+\lambda_{12}}{\lambda_1+\lambda_{12}} \\ \frac{\lambda_1+\lambda_{12}}{\lambda_2+\lambda_{12}} & 1 \end{bmatrix}.$$

Now, we will derive the range of the correlations $Corr(X_n, X_{n-1})$, $Corr(X_n, Y_{n-1})$, $Corr(Y_n, X_{n-1})$ and $Corr(Y_n, Y_{n-1})$. Since $K > \lambda/\lambda_{12}$, it follows that

$$0 < Corr(X_n, X_{n-1}) = Corr(Y_n, Y_{n-1}) < \frac{\lambda_{12}}{\lambda} < 1,$$

$$0 < Corr(X_n, Y_{n-1}) < \frac{\lambda_{12}(\lambda_2 + \lambda_{12})}{(\lambda_1 + \lambda_{12})\lambda} < 1,$$

$$0 < Corr(Y_n, X_{n-1}) < \frac{\lambda_{12}(\lambda_1 + \lambda_{12})}{(\lambda_2 + \lambda_{12})\lambda} < 1.$$

3. ESTIMATION OF THE PARAMETERS

In this section we will estimate the unknown parameters K, λ_1, λ_2 and λ_{12}. Ristić [9] showed that our bivariate minification process is ergodic and uniformly mixing. Let us consider the estimation of the unknown parameters. Let $\{(X_0, Y_0), (X_1, Y_1), \ldots, (X_{N-1}, Y_{N-1})\}$ be a sample of size N. First, we estimate the parameter K. Ristić [9] used the estimate

$$\hat{K}_N = \max_{1 \le n \le N-1} \left\{ \frac{X_n}{\min(X_{n-1}, Y_{n-1})} \right\}.$$

He showed that the estimate $\hat{K}_N$ is strongly consistent estimate and is not asymptotically normal. As an alternative strongly consistent estimator of K, we can consider

$$\widetilde{K}_N = \max_{1 \le n \le N-1} \left\{ \frac{Y_n}{\min(X_{n-1}, Y_{n-1})} \right\}.$$

Both estimators $\widehat{K}_N$ and $\widetilde{K}_N$ can be used in practical situation, since the true values of the parameters can be obtained for small N. Now we consider the estimation of the parameters λ_1, λ_2 and λ_{12}. We will use the estimates

$$\begin{aligned} \overline{X}_N &= \frac{1}{N} \sum_{i=0}^{N-1} X_i, \\ \overline{Y}_N &= \frac{1}{N} \sum_{i=0}^{N-1} Y_i, \\ \overline{I}_{N-1} &= \frac{1}{N-1} \sum_{i=1}^{N-1} I(X_i > \min(X_{i-1}, Y_{i-1})), \end{aligned}$$

where

$$I(X_i > \min(X_{i-1}, Y_{i-1})) = \begin{cases} 1, & X_i > \min(X_{i-1}, Y_{i-1}), \\ 0, & X_i \le \min(X_{i-1}, Y_{i-1}). \end{cases}$$

Since the bivariate minification process with bivariate Marshall and Olkin exponential distribution is ergodic, it follows that the estimates $\overline{X}_N$, $\overline{Y}_N$ and $\overline{I}_{N-1}$ are strongly consistent estimates of the parameters $1/(\lambda_1 + \lambda_{12})$, $1/(\lambda_2 + \lambda_{12})$ and $K\lambda/(K\lambda + K(\lambda_1 + \lambda_{12}) - \lambda)$. Also, since the bivariate minification process is stationary and uniformly mixing and $\sum_{i=1}^{\infty} \phi^{1/2}(h) < \infty$, it follows from Theorem 20.1 (Billingsley [4]) that

$$\sqrt{N} \begin{bmatrix} \overline{X}_N - \frac{1}{\lambda_1 + \lambda_{12}} \\ \overline{Y}_N - \frac{1}{\lambda_2 + \lambda_{12}} \end{bmatrix}$$

has asymptotically bivariate normal distribution $\mathcal{N}_2(\mathbf{0}, \Sigma)$, as $N \to \infty$, where

$$\Sigma = \begin{bmatrix} \frac{K+1}{(K-1)(\lambda_1 + \lambda_{12})^2} & \sigma_{xy} \\ \sigma_{xy} & \frac{K+1}{(K-1)(\lambda_2 + \lambda_{12})^2} \end{bmatrix}$$

and

$$\sigma_{xy} = \frac{\lambda_{12}}{\lambda(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} + \frac{1}{K-1}\left[\frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{1}{(\lambda_2+\lambda_{12})^2}\right].$$

Following Balakrishna and Jacob (2003), we can show that

$$\sqrt{N-1}\left(\overline{I}_{N-1} - \frac{\lambda K}{c_1+\lambda K}\right)$$

has asymptotically normal distribution with zero mean and variance

$$\sigma^2 = \frac{c_1\lambda K}{(c_1+\lambda K)^2} + 2c_1^2\sum_{h=1}^{\infty}\left[\frac{K^h+1}{(c_1+\lambda)((c_1+\lambda)K^h+c_1)} - \frac{1}{(c_1+\lambda K)^2}\right] > 0\,.$$

So, we can take the estimates of the parameters λ_1, λ_2 and λ_{12} as the solutions of the system of the equations

$$\begin{aligned}\overline{X}_N &= \frac{1}{\lambda_1+\lambda_{12}}\,,\\ \overline{Y}_N &= \frac{1}{\lambda_2+\lambda_{12}}\,,\\ \overline{I}_{N-1} &= \frac{K\lambda}{K\lambda+K(\lambda_1+\lambda_{12})-\lambda}\,.\end{aligned}$$

Now we present some numerical results. We simulated 10000 realizations of our process for the true values: a) $K=2$, $\lambda_1=0.2$, $\lambda_2=0.5$, $\lambda_{12}=0.8$, b) $K=3$, $\lambda_1=0.5$, $\lambda_2=1.5$, $\lambda_{12}=2.5$. The simulation was replicated 100 times and for each data set we computed sample means of the estimates $\hat{K}_N$, $\widetilde{K}_N$, $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_{12}$ and the standard errors (SE). The results are summarized in Table 1.

4. ACKNOWLEDGEMENTS

The authors would like to thank Professor N. Balakrishna and Professor K.K. Jose for sending copies of their published works on this subject. Also, the authors wish to thank to a referee for the suggestions which have improved the article. This work was partially supported by Grant of MNTR 144025.

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N	$\widehat{K}_N$	$\widetilde{K}_N$	λ_1	SE(λ_1)	λ_2	SE(λ_2)	λ_{12}	SE(λ_{12})
100	2	2	0.2164	0.1152	0.5214	0.1314	0.8316	0.1592
500	2	2	0.2309	0.0742	0.5145	0.0708	0.8073	0.1163
1000	2	2	0.2319	0.0745	0.5179	0.0713	0.8114	0.1139
5000	2	2	0.2308	0.0659	0.5163	0.0649	0.8083	0.1049
10000	2	2	0.2268	0.0577	0.5139	0.0575	0.8069	0.0958
N	$\widehat{K}_N$	$\widetilde{K}_N$	λ_1	SE(λ_1)	λ_2	SE(λ_2)	λ_{12}	SE(λ_{12})
100	3	3	0.5627	0.3330	1.6093	0.4489	2.4995	0.4645
500	3	3	0.5734	0.2314	1.5891	0.2457	2.4878	0.3293
1000	3	3	0.5675	0.2278	1.5844	0.2425	2.4938	0.3285
5000	3	3	0.5739	0.2033	1.5891	0.2087	2.4737	0.3117
10000	3	3	0.5812	0.1580	1.6014	0.1748	2.4782	0.2989

Table 1: Some numerical results of the estimations (a) $K = 2$, $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, $\lambda_{12} = 0.8$, b) $K = 3$, $\lambda_1 = 0.5$, $\lambda_2 = 1.5$, $\lambda_{12} = 2.5$).

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Address

Department of Mathematics and Informatics, Faculty of Sciences and Mathematics, University of Niš, Serbia
E-mail:
Miroslav Ristić: miristic@ptt.yu
Biljana Popović: biljanap@ban.junis.ni.ac.yu