

ON WEAKLY β -CONTINUOUS FUNCTIONS IN BITOPOLOGICAL SPACES

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Abstract. As a generalization of β -continuous functions, we introduce and study several properties of weakly β -continuous functions in bitopological spaces and we obtain its several characterizations .

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1. Introduction

The notion of β -open sets due to Mashhour et al.[1] or semi-preopen sets due to Andrijević [2] plays a significant role in general topology. In [1] the concept of β -continuous functions is introduced and further Popa and Noiri[13] studied the concept of weakly β -continuous functions.In 1992,Khedr et al.[10] introduced and studied semi-precontinuity or β -continuity in bitopological spaces .In this paper,we introduce and study the notion of weakly β -continuous functions in bitopological spaces further and investigate the properties of these functions.

2. Preliminaries

Throughout the present paper, (X,τ_1,τ_2) (resp. (X,τ)) denotes a bitopological (resp.topological) space.Let (X,τ) be a topological space and A be a subset of X .The closure and interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively.

Let (X,τ_1,τ_2) be a bitopological space and let A be a subset of X .The closure and interior of A with respect to τ_i are denoted by $iCl(A)$ and $iInt(A)$,respectively,for $i=1,2$.

Definition 2.1. A subset A of a bitopological space (X,τ_1,τ_2) is said to be

- (i). (i,j) -regular open[4] if $A=iInt(jCl(A))$ where $i \neq j$, $i,j=1,2$.
- (ii). (i,j) -regular closed[5] if $A=iCl(jInt(A))$ where $i \neq j$, $i,j=1,2$.
- (iii). (i,j) -preopen[7] if $A \subseteq iInt(jCl(A))$ where $i \neq j$, $i,j=1,2$.

Definition 2.2. A subset A of a bitopological space (X,τ_1,τ_2) is said to be (τ_i,τ_j) -semi-preopen(briefly (i,j) -semi-preopen)[10] if there exists a (i,j) -preopen set U such that $U \subseteq A \subseteq jCl(U)$ or it is said to be (i,j) - β -open if $A \subseteq jCl(iInt(jCl(A)))$,where $i \neq j$, $i,j=1,2$.The complement of (i,j) -semi-preopen set is said to be (i,j) -semi-preclosed[10] or it is said to be (i,j) - β -closed if $iInt(jCl(iInt(A))) \subseteq A$, where $i \neq j$, $i,j=1,2$.

Lemma 2.1.Let (X,τ_1,τ_2) be a topological space and $\{A_\lambda:\lambda \in \Delta\}$ be a family of subsets of X .Then

- (i). If A_λ is (i,j) - β -open for each $\lambda \in \Delta$, then $\bigcup_{\lambda \in \Delta} A_\lambda$ is (i,j) - β -open.
- (ii). If A_λ is (i,j) - β -closed for each $\lambda \in \Delta$, then $\bigcap_{\lambda \in \Delta} A_\lambda$ is (i,j) - β -closed.

Proof. The proof follows from Theorem 3.2 of [10].

(ii). This is an immediate consequence of (i).

Definition 2.3. Let A be subset of a bitopological space (X, τ_1, τ_2) .

- (i). The (i,j) β -closure [10] of A , denoted by (i,j) - β Cl(A), is defined to be the intersection of all (i,j) - β -closed sets containing A .
- (ii). The (i,j) β -interior of A , denoted by (i,j) - β Int(A), is defined to be the union of all (i,j) - β -open sets contained in A .

Lemma 2.2. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X . Then

- (i). (i,j) - β Int(A) is (i,j) - β -open.
- (ii). (i,j) - β Cl(A) is (i,j) - β -closed.
- (iii). A is (i,j) - β -open if and only if $A = (i,j)$ - β Int(A).
- (iv). A is (i,j) - β -closed if and only if $A = (i,j)$ - β Cl(A).

Proof. (i) and (ii) are obvious from Lemma 2.1.

(iii) and (iv) are obvious from (i) and (ii).

Lemma 2.3. For any subset A of a bitopological space (X, τ_1, τ_2) , $x \in (i,j)$ - β Cl(A) if and only if $U \cap A \neq \emptyset$ for every (i,j) - β -open set U containing x .

Lemma 2.4. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X . Then

- (i). $X \sim (i,j)$ - β Int(A) = (i,j) - β Cl($X \sim A$).
- (ii). $X \sim (i,j)$ - β Cl(A) = (i,j) - β Int($X \sim A$).

Proof. (i). By Lemma 2.2, (i,j) - β Cl(A) is (i,j) - β -closed. Then $X \sim (i,j)$ - β Cl(A) is (i,j) - β -open. On the other hand, $X \sim (i,j)$ - β Cl($X \sim A$) $\subseteq A$ and hence $X \sim (i,j)$ - β Cl($X \sim A$) $\subseteq (i,j)$ - β Int(A). Conversely, let $x \in (i,j)$ - β Int(A). Then there exists (i,j) - β -open set G such that $x \in G \subseteq A$. Then $X \sim G$ is (i,j) - β -closed and $X \sim A \subseteq X \sim G$. Since $x \notin X \sim G$, $x \notin (i,j)$ - β Cl($X \sim A$) and hence (i,j) - β Int(A) $\subseteq X \sim (i,j)$ - β Cl($X \sim A$). Therefore $X \sim (i,j)$ - β Int(A) = (i,j) - β Cl($X \sim A$).

(ii). This follows immediately from (i).

Definition 2.4. Let (X, τ_1, τ_2) be a bitopological space and A be a subset of X . A point x of X is said to be in the (i,j) - θ -closure [8] of A , denoted by (i,j) -Cl $_\theta$ (A), if $A \cap_j$ Cl(U) $\neq \emptyset$ for every τ_i -open set U containing x , where $i, j = 1, 2$ and $i \neq j$.

A subset A of X is said to be (i,j) - θ -closed if $A = (i,j)$ -Cl $_\theta$ (A). A subset A of X is said to be (i,j) - θ -open if $X \sim A$ is (i,j) - θ -closed. The (i,j) - θ -interior of A , denoted by (i,j) -Int $_\theta$ (A), is defined as the union of all (i,j) - θ -open sets contained in A . Hence $x \in (i,j)$ -Int $_\theta$ (A) if and only if there exists a τ_i -open set U containing x such that $x \in U \subseteq_j$ Cl(U) $\subseteq A$.

Lemma 2.5. For a subset A of a bitopological space (X, τ_1, τ_2) , the following properties hold:

- (i). $X \sim (i,j)\text{-Int}_0(A) = (i,j)\text{-Cl}_0(X \sim A)$.
(ii). $X \sim (i,j)\text{-Cl}_0(A) = (i,j)\text{-Int}_0(X \sim A)$.

Lemma 2.6. [8]. Let (X, τ_1, τ_2) be a bitopological space. If U is a τ_j -open set of X , then $(i,j)\text{-Cl}_0(U) = i\text{Cl}(U)$

Definition 2.5. A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i,j) - β -continuous [10] if $f^{-1}(V)$ is (i,j) - β -open in X for each σ_i -open set V of Y .

Definition 2.6. (i). A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i,j) -weakly precontinuous [12] if for each $x \in X$ and each σ_i -open set V of Y containing $f(x)$, there exists (i,j) -preopen set U containing x such that $f(U) \subseteq_j \text{Cl}(V)$.

(ii). A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i,j) -weakly- β -continuous if for each $x \in X$ and each σ_i -open set V of Y containing $f(x)$, there exists (i,j) - β -open set U containing x such that $f(U) \subseteq_j \text{Cl}(V)$.

A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise weakly precontinuous [12] (resp. pairwise weakly β -continuous) if f is weakly $(1,2)$ -precontinuous and weakly $(2,1)$ -precontinuous (resp. if f is weakly $(1,2)$ - β -continuous and weakly $(2,1)$ - β -continuous).

Remark 2.1. Since every (i,j) -preopen set is (i,j) - β -open ([10], Remark 3.1), every (i,j) weakly precontinuous function is (i,j) weakly β -continuous for $i, j = 1, 2$ and $i \neq j$. The converse is not true.

3. Characterizations

Theorem 3.1. For a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (i). f is (i,j) -weakly β -continuous.
(ii). $(i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Int}(i\text{Cl}(B)))) \subseteq f^{-1}(i\text{Cl}(B))$ for every subset B of Y .
(iii). $(i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Int}(F))) \subseteq f^{-1}(F)$ for every (i,j) -regular closed set F of Y .
(iv). $(i,j)\text{-}\beta\text{Cl}(f^{-1}(i\text{Cl}(V))) \subseteq f^{-1}(i\text{Cl}(V))$ for every σ_j -open set V of Y .
(v). $f^{-1}(V) \subseteq (i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V)))$ for every σ_i -open set V of Y .

Proof. (i) \rightarrow (ii). Let B be any subset of Y . Assume that $x \in X \sim f^{-1}(i\text{Cl}(B))$. Then $f(x) \in Y \sim i\text{Cl}(B)$ and so there exists a σ_i -open set V of Y containing $f(x)$ such that $V \cap B = \emptyset$, so $V \cap j\text{Int}(i\text{Cl}(B)) = \emptyset$ and hence $j\text{Cl}(V) \cap j\text{Int}(i\text{Cl}(B)) = \emptyset$. Therefore, there exists (i,j) - β -open set U containing x such that $f(U) \subseteq_j \text{Cl}(V)$. Hence we have $U \cap f^{-1}(j\text{Int}(i\text{Cl}(B))) = \emptyset$ and $x \in X \sim (i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Int}(i\text{Cl}(B))))$ by Lemma 2.3. Thus we obtain $(i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Int}(i\text{Cl}(B)))) \subseteq f^{-1}(i\text{Cl}(B))$.

(ii) \rightarrow (iii). Let F be any (i,j) -regular closed set of Y . Then $F = i\text{Cl}(j\text{Int}(F))$ and we have $(i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Int}(F))) = (i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Int}(i\text{Cl}(j\text{Int}(F)))) \subseteq f^{-1}(i\text{Cl}(j\text{Int}(F))) = f^{-1}(F)$.

(iii) \rightarrow (iv). For any σ_j -open set V of Y , Then $iCl(V)$ is (i,j) -regular closed. Then we have (i,j) - $\beta Cl(f^{-1}(V)) \subseteq (i,j)$ - $\beta Cl(f^{-1}(jInt(iCl(V)))) \subseteq f^{-1}(iCl(V))$.

(iv) \rightarrow (v). Let V be any σ_i -open set of Y . Then $Y \sim jCl(V)$ is σ_j -open set in Y and we have (i,j) - $\beta Cl(f^{-1}(Y \sim jCl(V))) \subseteq f^{-1}(iCl(Y \sim jCl(V)))$ and hence $X \sim (i,j)$ - $\beta Int(f^{-1}(jCl(V))) \subseteq X \sim f^{-1}(iInt(jCl(V))) \subseteq X \sim f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subseteq (i,j)$ - $\beta Int(f^{-1}(jCl(V)))$.

(v) \rightarrow (i). Let $x \in X$ and let V be a σ_i -open set containing $f(x)$. We have $x \in f^{-1}(V) \subseteq (i,j)$ - $\beta Int(f^{-1}(jCl(V)))$. Put $U = (i,j)$ - $\beta Int(f^{-1}(jCl(V)))$. By Lemma 2.2, U is (i,j) - β -open set containing x and $f(U) \subseteq jCl(V)$. This shows that f is (i,j) -weakly β -continuous.

Remark 3.1. Let $\tau = \tau_1 = \tau_2$ and $\sigma = \sigma_1 = \sigma_2$. Then by Theorem 3.1 we obtain the results for a function $f: (X, \tau) \rightarrow (Y, \sigma)$ established in Theorem 2 of [13].

Theorem 3.2. For a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (i). f is (i,j) -weakly β -continuous.
- (ii). $f((i,j)$ - $\beta Cl(A)) \subseteq ((i,j)$ - $Cl_\theta(f(A))$ for every subset A of X .
- (iii). (i,j) - $\beta Cl(f^{-1}(B)) \subseteq f^{-1}((i,j)$ - $Cl_\theta(B))$ for every subset B of Y .
- (iv). (i,j) - $\beta Cl(f^{-1}(jInt((i,j)$ - $Cl_\theta(B)))) \subseteq f^{-1}((i,j)$ - $Cl_\theta(B))$ for every subset B of Y .

Proof. (i) \rightarrow (ii). Assume that f is (i,j) -weakly β -continuous. Let A be any subset of X , $x \in (i,j)$ - $\beta Cl(A)$ and V be a σ_i -open set of Y containing $f(x)$. Then there exists (i,j) - β -open set U containing x such that $f(U) \subseteq jCl(V)$. Since $x \in (i,j)$ - $\beta Cl(A)$, by Lemma 2.3, we obtain $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subseteq jCl(V) \cap f(A)$. Therefore, we obtain $f(x) \in (i,j)$ - $Cl_\theta(f(A))$.

(ii) \rightarrow (iii). Let B be any subset of Y . Then we have $f((i,j)$ - $\beta Cl(f^{-1}(B))) \subseteq (i,j)$ - $Cl_\theta(f(f^{-1}(B))) \subseteq (i,j)$ - $Cl_\theta(B)$ and hence (i,j) - $\beta Cl(f^{-1}(B)) \subseteq f^{-1}((i,j)$ - $Cl_\theta(B))$.

(iii) \rightarrow (iv). Let B be any subset of Y . Since (i,j) - $Cl_\theta(B)$ is σ_i -closed in Y , by Lemma 2.6, (i,j) - $\beta Cl(f^{-1}(jInt((i,j)$ - $Cl_\theta(B)))) \subseteq f^{-1}((i,j)$ - $Cl_\theta(jInt((i,j)$ - $Cl_\theta(B)))) = f^{-1}(iCl(jInt((i,j)$ - $Cl_\theta(B)))) \subseteq f^{-1}(iCl((i,j)$ - $Cl_\theta(B))) = f^{-1}((i,j)$ - $Cl_\theta(B))$.

(iv) \rightarrow (i). Let V be any σ_j -open set of Y . Then by Lemma 2.6, $V \subseteq jInt(iCl(V)) = jInt((i,j)$ - $Cl_\theta(V))$ and we have (i,j) - $\beta Cl(f^{-1}(V)) \subseteq (i,j)$ - $\beta Cl(f^{-1}(jInt((i,j)$ - $Cl_\theta(V)))) \subseteq f^{-1}((i,j)$ - $Cl_\theta(V)) = f^{-1}(iCl(V))$. Thus we have (i,j) - $\beta Cl(f^{-1}(V)) \subseteq f^{-1}(iCl(V))$. It follows from Theorem 3.1 that f is (i,j) -weakly β -continuous.

Remark 3.2. By above Theorem, we obtain the results established in Theorem 4 of [13].

Definition 3.1. A bitopological space (X, τ_1, τ_2) is said to be (i,j) -regular [9] if for each $x \in X$ and each τ_i -open set U containing x , there exists a τ_i -open set V such that $x \in V \subseteq jCl(V) \subseteq U$.

Lemma 3.1. [14]. If a bitopological space (X, τ_1, τ_2) is (i,j) -regular, then (i,j) - $Cl_\theta(F) = F$ for every τ_i -closed set F .

Theorem 3.3. Let (Y, σ_1, σ_2) be an (i, j) -regular bitopological space. For a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:

- (i). f is (i, j) - β -continuous.
- (ii). $f^{-1}((i, j)\text{-Cl}_\theta(B))$ is (i, j) - β -closed in X for every subset B of Y .
- (iii). f is (i, j) -weakly β -continuous.
- (iv). $f^{-1}(F)$ is (i, j) - β -closed in X for every (i, j) - θ -closed set F of Y .
- (v). $f^{-1}(F)$ is (i, j) - β -open in X for every (i, j) - θ -open set V of Y .

Proof. (i) \rightarrow (ii). Let B be any subset of Y . Since $(i, j)\text{-Cl}_\theta(B)$ is σ_i -closed in Y , it follows from Theorem 5.1 of [10] that $f^{-1}((i, j)\text{-Cl}_\theta(B))$ is (i, j) - β -closed in X .

(ii) \rightarrow (iii). Let B be any subset of Y . Then we have $(i, j)\text{-}\beta\text{Cl}(f^{-1}(B)) \subseteq (i, j)\text{-}\beta\text{Cl}(f^{-1}((i, j)\text{-Cl}_\theta(B))) = f^{-1}((i, j)\text{-Cl}_\theta(B))$. By Theorem 3.2, f is (i, j) -weakly β -continuous.

(iii) \rightarrow (iv). Let F be any (i, j) - θ -closed set of Y . Then by Theorem 3.2, $(i, j)\text{-}\beta\text{Cl}(f^{-1}(F)) \subseteq f^{-1}((i, j)\text{-Cl}_\theta(F)) = f^{-1}(F)$. Therefore, by Lemma 2.2, $f^{-1}(F)$ is (i, j) - β -closed in X .

(iv) \rightarrow (v). Let V be any (i, j) - θ -open set of Y . By (iv), $f^{-1}(Y \sim V) = X \sim f^{-1}(V)$ is (i, j) - β -closed in X and hence $f^{-1}(V)$ is (i, j) - β -open in X .

(v) \rightarrow (i). Since Y is (i, j) -regular, by Lemma 3.1, $(i, j)\text{-Cl}_\theta(B) = B$ for every σ_i -closed set B of Y and hence σ_i -open set is (i, j) - θ -open. Therefore $f^{-1}(V)$ is (i, j) - β -open for every σ_i -open set V of Y . By Theorem 5.1 of [10], f is (i, j) - β -continuous.

4. Weak β -continuity and β -continuity

Definition 4.1. A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) -weakly* quasi continuous (briefly, $w^*.q.c$) [14] if for every σ_i -open set V of Y , $f^{-1}(j\text{Cl}(V) \sim V)$ is biclosed in X .

Theorem 4.1. If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) -weakly β -continuous and (i, j) - $w^*.q.c$, then f is (i, j) - β -continuous.

Proof. Let $x \in X$ and V be any σ_i -open set of Y containing $f(x)$. Since f is (i, j) -weakly β -continuous, there exists an (i, j) - β -open set U of X containing x such that $f(U) \subseteq j\text{Cl}(V)$. Hence $x \notin f^{-1}(j\text{Cl}(V) \sim V)$. Therefore, $x \in U \sim f^{-1}(j\text{Cl}(V) \sim V) = U \cap (X \sim (f^{-1}(j\text{Cl}(V) \sim V)))$. Since U is (i, j) - β -open and $X \sim (f^{-1}(j\text{Cl}(V) \sim V))$ is biopen, by Theorem 3.3 of [10], $G = U \cap (X \sim (f^{-1}(j\text{Cl}(V) \sim V)))$ is (i, j) - β -open. Then $x \in G$ and $f(G) \subseteq V$. For if $y \in G$, then $f(y) \notin (j\text{Cl}(V) \sim V)$ and hence $f(y) \in V$. Therefore, f is (i, j) - β -continuous.

Definition 4.2. A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to have a (i, j) - β -interiority condition if $(i, j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V))) \subseteq f^{-1}(V)$ for every σ_i -open set V of Y .

Theorem 4.2. If a function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is (i,j) -weakly β -continuous and satisfies the (i,j) - β -interiority condition, then f is (i,j) - β -continuous.

Proof. Let V be any σ_i -open set of Y . Since f is (i,j) -weakly β -continuous, by Theorem 3.1, $f^{-1}(V) \subseteq (i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V)))$. By the (i,j) - β -interiority condition of f , we have $(i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V))) \subseteq f^{-1}(V)$ and hence $(i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V))) = f^{-1}(V)$. By Lemma 2.2, $f^{-1}(V)$ is (i,j) - β -open in X and thus f is (i,j) - β -continuous.

Definition 4.3. Let (X,τ_1,τ_2) be a bitopological space and let A be a subset of X . The (i,j) - β -frontier of A is defined as follows: $(i,j)\text{-}\beta\text{Fr}(A) = (i,j)\text{-}\beta\text{Cl}(A) \cap (i,j)\text{-}\beta\text{Cl}(X \sim A) = (i,j)\text{-}\beta\text{Cl}(A) \sim (i,j)\text{-}\beta\text{Int}(A)$.

Theorem 4.3. The set of all points x of X for which a function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is not (i,j) -weakly β -continuous is identical with the union of the (i,j) - β -frontier of the inverse images of the σ_j -closure of σ_i -open sets of Y containing $f(x)$.

Proof. Let x be a point of X at which $f(x)$ is not (i,j) -weakly β -continuous. Then there exists a σ_i -open set V of Y containing $f(x)$ such that $U \cap (X \sim f^{-1}(j\text{Cl}(V))) \neq \emptyset$ for every (i,j) - β -open set U of X containing x . By Lemma 2.3, $x \in (i,j)\text{-}\beta\text{Cl}(X \sim f^{-1}(j\text{Cl}(V)))$. Since $x \in f^{-1}(j\text{Cl}(V))$, we have $x \in (i,j)\text{-}\beta\text{Cl}(f^{-1}(j\text{Cl}(V)))$ and hence $x \in (i,j)\text{-}\beta\text{Fr}(f^{-1}(j\text{Cl}(V)))$. Conversely, if f is (i,j) -weakly β -continuous at x , then for each σ_i -open set V of Y containing $f(x)$, there exists an (i,j) - β -open set U containing x such that $f(U) \subseteq j\text{Cl}(V)$ and hence $x \in U \subseteq f^{-1}(j\text{Cl}(V))$. Therefore we obtain that $x \in (i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V)))$. This contradicts that $x \in (i,j)\text{-}\beta\text{Fr}(f^{-1}(j\text{Cl}(V)))$.

Remark 4.1. By above Theorem, we obtain the results established in Theorem 4.7 of [3].

5. Weak β -continuity and almost β -continuity

Definition 5.1. A function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j) -almost β -continuous if for each $x \in X$ and each σ_i -open set V containing $f(x)$, there exists an (i,j) - β -open set U of X containing x such that $f(U) \subseteq i\text{Int}(j\text{Cl}(V))$.

Lemma 5.1. A function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is (i,j) -almost β -continuous if and only if $f^{-1}(V)$ is (i,j) - β -open for each (i,j) -regular open set V of Y .

Definition 5.2. A bitopological space (X,τ_1,τ_2) is said to be (i,j) -almost regular [16] if for each $x \in X$ and each (i,j) -regular open set U containing x , there exists an (i,j) -regular open set V of X such that $x \in V \subseteq j\text{Cl}(V) \subseteq U$.

Theorem 5.1. Let a bitopological space (Y,σ_1,σ_2) be (i,j) -almost regular. Then a function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is (i,j) -almost β -continuous if and only if it is (i,j) -weakly β -continuous.

Proof. Necessity. This is obvious.

Sufficiency. Suppose that f is (i,j) -weakly- β -continuous. Let V be any (i,j) -regular open set of Y and $x \in f^{-1}(V)$. Then we have $f(x) \in V$. By the almost (i,j) -regularity of Y , there exists an (i,j) -regular open set V_0 of Y such that $f(x) \in V_0 \subseteq jCl(V_0) \subseteq V$. Since f is (i,j) -weakly- β -continuous, there exists an (i,j) - β -open set U of X containing x such that $f(U) \subseteq jCl(V_0) \subseteq V$. This implies that $x \in U \subseteq f^{-1}(V)$. Therefore we have $f^{-1}(V) \subseteq (i,j)$ - β Int($f^{-1}(V)$) and hence $f^{-1}(V) = (i,j)$ - β Int($f^{-1}(V)$). By Lemma 2.2, $f^{-1}(V)$ is (i,j) - β -open and by Lemma 5.1, f is (i,j) -almost β -continuous.

Remark 5.1. By above theorem, we obtain the result established in Theorem 6 of [11].

Definition 5.3. A bitopological space (X, τ_1, τ_2) is said to be pairwise Hausdroff or pairwise- T_2 [9] if for each pair of distinct points x and y of X , there exists τ_i -open set U containing x and a τ_j -open set V containing y such that $U \cap V = \emptyset$ for $i \neq j, i, j = 1, 2$.

Definition 5.4. A bitopological space (X, τ_1, τ_2) is said to be pairwise β -Hausdroff or pairwise β - T_2 if for each pair of distinct points x and y of X , there exists a (i,j) - β -open set U containing x and a τ_j -open set V containing y such that $U \cap V = \emptyset$ for $i \neq j, i, j = 1, 2$.

Theorem 5.2. Let (X, τ_1, τ_2) be a bitopological space. If for each pair of distinct points x and y in X , there exists a function f of (X, τ_1, τ_2) into pairwise T_2 bitopological space (Y, σ_1, σ_2) such that

- (i). $f(x) \neq f(y)$.
 - (ii). f is (i,j) -weakly- β -continuous at x ,
 - (iii). f is (j,i) -almost- β -continuous at y ,
- then (X, τ_1, τ_2) is pairwise β - T_2 .

Proof. Let x and y be a pair of distinct points of X . Since Y is pairwise T_2 , there exists a σ_i -open set U containing $f(x)$ and a σ_j -open set V containing $f(y)$ such that $U \cap V = \emptyset$. Since U and V are disjoint, we have $jCl(U) \cap iCl(V) = \emptyset$. Since f is (i,j) -weakly- β -continuous at x , there exists an (i,j) - β -open set U_x of X containing x such that $f(U_x) \subseteq jCl(U)$. Since f is (j,i) -almost- β -continuous at y , there exists (j,i) - β -open set U_y of X containing y such that $f(U_y) \subseteq iCl(V)$. Hence we have $U_x \cap U_y = \emptyset$. This shows that (X, τ_1, τ_2) is pairwise β - T_2 .

Remark 5.2. By above theorem, we obtain the result established in Theorem 13 of [11].

6. Some Properties

Definition 6.1. A bitopological space (X, τ_1, τ_2) is said to be pairwise Urysohn[6] if for each distinct points x, y of X , there exists τ_i open set U and τ_j open set V such that $x \in U, y \in V$ and $jCl(U) \cap iCl(V) = \emptyset$ for $i \neq j, i, j = 1, 2$.

Theorem 6.1. If (Y, σ_1, σ_2) is pairwise Urysohn and $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise weakly- β -continuous injection, then (Y, σ_1, σ_2) is pairwise β - T_2 .

Proof. Let x and y be two distinct points of X . Then $f(x) \neq f(y)$. Since Y is pairwise Urysohn, there exist τ_i -open set U and τ_j -open set V such that $f(x) \in U, f(y) \in V$ and $jCl(U) \cap iCl(V) = \emptyset$. Hence $f^{-1}(jCl(U)) \cap f^{-1}(iCl(V)) =$

\emptyset . Therefore, $(i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(U))) \cap (j,i)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V))) = \emptyset$. Since f is pairwise weakly β -continuous, by Theorem 3.1, $x \in f^{-1}(U) \subseteq (i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(U)))$ and $y \in f^{-1}(V) \subseteq (j,i)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V)))$. This implies that (X, τ_1, τ_2) is pairwise β - T_2 .

Remark 6.1. By above theorem, we obtain the result established in Theorem 4.4 of [3].

Definition 6.2. A bitopological space (X, τ_1, τ_2) is said to be pairwise connected [15] (resp. pairwise β -connected) if it cannot be expressed as the union of two non empty disjoint sets U and V such that U is τ_i open and V is τ_j open (resp. U is $(i,j)\text{-}\beta$ -open and V is $(j,i)\text{-}\beta$ -open).

Theorem 6.2. If a function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise weakly- β -continuous surjection and (X, τ_1, τ_2) is pairwise β -connected, then (Y, σ_1, σ_2) is pairwise connected.

Proof. Suppose that (Y, σ_1, σ_2) is not pairwise connected. Then there exists a τ_i -open set U and τ_j -open set V such that $U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset$ and $U \cup V = X$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are non empty. Moreover $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Since f is pairwise weakly β -continuous, by Theorem 3.1, we have $f^{-1}(U) \subseteq (i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(U)))$ and $f^{-1}(V) \subseteq (j,i)\text{-}\beta\text{Int}(f^{-1}(i\text{Cl}(V)))$. Since U and V are σ_j -closed and σ_i -closed, respectively, we have $f^{-1}(U) \subseteq (i,j)\text{-}\beta\text{Int}(f^{-1}(U))$ and $f^{-1}(V) \subseteq (j,i)\text{-}\beta\text{Int}(f^{-1}(V))$. Hence $f^{-1}(U) = (i,j)\text{-}\beta\text{Int}(f^{-1}(U))$ and $f^{-1}(V) = (j,i)\text{-}\beta\text{Int}(f^{-1}(V))$. By Lemma 2.2, $f^{-1}(U)$ is $(i,j)\text{-}\beta$ -open and $f^{-1}(V)$ is $(j,i)\text{-}\beta$ -open in (X, τ_1, τ_2) . This shows that (X, τ_1, τ_2) is not pairwise connected.

Remark 6.2. By above theorem, we obtain the result established in Theorem 13 of [13].

Definition 6.3. A subset K of a bitopological space (X, τ_1, τ_2) is said to be (i,j) -quasi H-closed relative to X [4] if for each cover $\{U_\alpha: \alpha \in \Delta\}$ of K by τ_i -open sets of X , there exists a finite subset Δ_0 of Δ such that $K \subseteq \bigcup \{j\text{Cl}(U_\alpha): \alpha \in \Delta_0\}$.

Definition 6.4. A subset K of a bitopological space (X, τ_1, τ_2) is said to be $(i,j)\text{-}\beta$ -compact relative to X if every cover of K by $(i,j)\text{-}\beta$ -open sets of X has a finite subcover.

Theorem 6.3. If $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise weakly- β -continuous and K is $(i,j)\text{-}\beta$ -compact relative to X , then $f(K)$ is (i,j) -quasi H-closed relative to Y .

Proof. Let K be $(i,j)\text{-}\beta$ -compact relative to X and $\{V_\alpha: \alpha \in \Delta\}$ be any cover of $f(K)$ by σ_i -open sets of (Y, σ_1, σ_2) . Then $f(K) \subseteq \bigcup \{V_\alpha: \alpha \in \Delta\}$ and so $K \subseteq \bigcup \{f^{-1}(V_\alpha): \alpha \in \Delta\}$. Since f is (i,j) -weakly- β -continuous, by Theorem 3.1, we have $f^{-1}(V_\alpha) \subseteq (i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))$ for each $\alpha \in \Delta$. Therefore, $K \subseteq \bigcup \{(i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V_\alpha))): \alpha \in \Delta\}$. Since K is $(i,j)\text{-}\beta$ -compact relative to X and $(i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))$ is $(i,j)\text{-}\beta$ -open for each $\alpha \in \Delta$, there exists a finite subset Δ_0 of Δ such that $K \subseteq \bigcup \{(i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V_\alpha))): \alpha \in \Delta_0\}$. This implies that $f(K) \subseteq \bigcup \{f((i,j)\text{-}\beta\text{Int}(f^{-1}(j\text{Cl}(V_\alpha)))): \alpha \in \Delta_0\} \subseteq \bigcup \{f(f^{-1}(j\text{Cl}(V_\alpha))): \alpha \in \Delta_0\} \subseteq \bigcup \{j\text{Cl}(V_\alpha): \alpha \in \Delta_0\}$. Hence $f(K)$ is (i,j) -quasi H-closed relative to Y .

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