A NOTE ON $a$-OPEN SETS AND $e^*$-OPEN SETS

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Abstract

The aim of this paper is to investigate some properties of $a$-open sets and $e^*$-open sets in topological spaces.

1 Introduction

Some types of sets play an important role in the study of various properties in topological spaces. Many authors introduced and studied various generalized properties and conditions containing some forms of sets in topological spaces. In this paper, we investigate some properties of $a$-open sets and $e^*$-open sets. Moreover, the relationships among $a$-open sets, $e^*$-open sets and the related classes of sets are investigated.

In this paper, spaces $X$ and $Y$ mean topological spaces. For a subset $A$ of a space $X$, $Cl(A)$ and $Int(A)$ represent the closure of $A$ and the interior of $A$, respectively. A subset $A$ of a space $X$ is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$) [7]. The $δ$-interior of a subset $A$ of $X$ is the union of all regular open sets of $X$ contained in $A$ and it is denoted by $δ-Int(A)$ [8]. A subset $A$ is called $δ$-open if $A = δ-Int(A)$. The complement of a $δ$-open set is called $δ$-closed. The $δ$-closure of a set $A$ in a space $(X, τ)$ is defined by $\{x ∈ X : A ∩ Int(Cl(B)) ≠ ∅, B ∈ τ and x ∈ B\}$ and it is denoted by $δ-Cl(A)$.

Definition 1 A subset $A$ of a space $(X, τ)$ is called

(3) $e$-open [2] if $A ⊂ Cl(δ-Int(A)) ∪ Int(δ-Cl(A))$ and $e$-closed [2] if $Cl(δ-Int(A)) ∩ Int(δ-Cl(A)) ⊂ A$,

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(5) $a$-open [4] if $A \subset \text{Int}(\text{Cl}(\delta-\text{Int}(A)))$ and $a$-closed [4] if $\text{Cl}(\text{Int}(\delta-\text{Cl}(A))) \subset A$.

We denote the $\delta$-boundary $\delta-\text{Cl}(A) \backslash \delta-\text{Int}(A)$ of $A$ by $\delta-\text{Fr}(A)$. A subset $A$ of a space $X$ is said to be a $\delta$-dense set if $\delta-\text{Cl}(A) = X$. The family of all $\delta$-semiopen (resp. $\delta$-preopen, $e^*\text{-open}, a\text{-open}$) sets of $X$ is denoted by $\delta\text{SO}(X)$ (resp. $\delta\text{PO}(X), e^*\text{O}(X), a\text{O}(X)$).

**Remark 2** ([4]) The following diagram holds for a subset $A$ of a space $X$:

\[
\begin{array}{cc}
e^*-\text{open} & \uparrow & e\text{-open} \\
\delta\text{-semiopen} & \nearrow & a\text{-open} & \searrow & \delta\text{-preopen} \\
\delta\text{-open} & \uparrow & \text{regular open}
\end{array}
\]

2 $a$-open sets and $e^*$-open sets

**Theorem 3** Let $N$ be a subset of a topological space $X$. The following are equivalent:

1. $N$ is regular open,
2. $N$ is $a$-open and $e^*$-closed,
3. $N$ is $\delta$-preopen and $\delta$-semiclosed.

**Proof.** (1) $\Rightarrow$ (2) : Obvious.

(2) $\Rightarrow$ (1) : Let $N$ be $a$-open and $e^*$-closed. We have $N \subset \text{Int}(\text{Cl}(\delta-\text{Int}(N)))$ and $\text{Int}(\text{Cl}(\delta-\text{Int}(N))) \subset N$ and hence $N = \text{Int}(\text{Cl}(\delta-\text{Int}(N)))$. Thus, $N$ is regular open.

(1) $\Leftrightarrow$ (3) : Let $N$ be $\delta$-preopen and $\delta$-semiclosed. Then $N \subset \text{Int}(\delta-\text{Cl}(N))$ and $\text{Int}(\delta-\text{Cl}(N)) \subset N$. Thus, $N = \text{Int}(\delta-\text{Cl}(N)) = \text{Int}(\text{Cl}(N))$ and hence $N$ is regular open. The converse is similar. $\blacksquare$

**Theorem 4** Let $N$ be a subset of a topological space $X$. The following are equivalent:

1. $N$ is $\delta$-semiopen,
2. $N$ is $e^*$-open and $\delta$-Int$(\delta-\text{Fr}(N)) = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2) : Let $N$ be $\delta$-semiopen. We have $\text{Int}(\delta-\text{Cl}(N)) \subset \delta-\text{Cl}(N) \subset \text{Cl}(\delta-\text{Int}(N))$. Since

$\delta-\text{Int}(\delta-\text{Fr}(N)) = \delta-\text{Int}(\delta-\text{Cl}(N) \cap (X \backslash \delta-\text{Int}(N))) = \delta-\text{Int}(\delta-\text{Cl}(N)) \backslash \text{Cl}(\delta-\text{Int}(N))$,
A note on $a$-open sets and $e^*$-open sets

then $\delta\text{-}\text{Int}(\delta\text{-Fr}(N)) = \varnothing$. 

(2) $\Rightarrow$ (1): Let $N$ be $e^*$-open and $\delta\text{-}\text{Int}(\delta\text{-Fr}(N)) = \varnothing$. Then $N \subset \text{Cl}(\text{Int}(\delta\text{-Cl}(N))) \subset \text{Cl}(\delta\text{-Int}(N))$. Thus, $N$ is $\delta$-semiopen. 

**Theorem 5** For a topological space $(X, \tau)$, the family of all $a$-open sets of $X$ forms a topology, denoted by $\tau_a$, for $X$.

**Proof.** It is obvious that $\varnothing, X \in aO(X)$ and any union of $a$-open sets is $a$-open. Let $A, B \in aO(X)$. This implies that $A \subset \text{Int}(\text{Cl}(\delta\text{-Int}(A)))$ and $B \subset \text{Int}(\text{Cl}(\delta\text{-Int}(B)))$ and hence

$$A \cap B \subset \text{Int}(\text{Cl}(\delta\text{-Int}(A))) \cap \text{Int}(\text{Cl}(\delta\text{-Int}(B)))$$
$$\subset \text{Int}(\text{Cl}(\text{Int}(\delta\text{-Int}(A))) \cap \text{Cl}(\delta\text{-Int}(B)))$$
$$\subset \text{Int}(\text{Cl}(\delta\text{-Int}(\text{Cl}(\delta\text{-Int}(A)) \cap \delta\text{-Int}(B))))$$
$$\subset \text{Int}(\text{Cl}(\delta\text{-Int}(\text{Cl}(\delta\text{-Int}(A \cap B)))))$$
$$= \text{Int}(\text{Cl}(\delta\text{-Int}(A \cap B))).$$

Thus, $A \cap B \in aO(X)$. 

**Theorem 6** Let $X$ be a topological space. Then $aO(X) = \delta SO(X) \cap \delta PO(X)$.

**Proof.** Let $N \in aO(X)$. Then $N \in \delta SO(X)$ and $N \in \delta PO(X)$. Thus, $aO(X) \subset \delta SO(X) \cap \delta PO(X)$.

Conversely, let $N \in \delta SO(X) \cap \delta PO(X)$. Then $N \in \delta SO(X)$ and $N \in \delta PO(X)$. Since $N \in \delta SO(X)$, then by Theorem 4, $\delta\text{-}\text{Int}(\delta\text{-Fr}(N)) = \varnothing$. Since

$$\delta\text{-}\text{Int}(\delta\text{-Fr}(N)) = \delta\text{-}\text{Int}(\delta\text{-Cl}(N) \cap (X \setminus \delta\text{-Int}(N))) = \delta\text{-}\text{Int}(\delta\text{-Cl}(N) \setminus \delta\text{-Cl}(\delta\text{-Int}(N)))$$

then $\text{Int}(\delta\text{-Cl}(N)) \subset \text{Cl}(\delta\text{-Int}(N))$. Since $N \in \delta PO(X)$, we have

$$N \subset \text{Int}(\delta\text{-Cl}(N)) \subset \text{Int}(\text{Cl}(\delta\text{-Int}(N))).$$

Thus, $N \in aO(X)$. 

**Theorem 7** Let $N$ be a subset of a topological space $X$. The following are equivalent:

(1) $N$ is $\delta$-clopen,
(2) $N$ is $a$-open and $\delta$-preclosed,
(3) $N$ is clopen.

**Proof.** (1)$\Rightarrow$(2): Obvious.

(2)$\Rightarrow$(1): Let $N$ be $a$-open and $\delta$-preclosed. We have $N \subset \text{Int}(\text{Cl}(\delta\text{-Int}(N)))$ and $\text{Cl}(\delta\text{-Int}(N)) \subset N$ and hence $N \subset \text{Int}(\text{Cl}(\delta\text{-Int}(N))) \subset \text{Cl}(\delta\text{-Int}(N)) \subset N$. Thus, $N = \text{Int}(\text{Cl}(\delta\text{-Int}(N))) = \text{Cl}(\delta\text{-Int}(N))$ and hence $N$ is $\delta$-open and $\delta$-closed.

(1)$\iff$(3): Obvious. 

Theorem 8 Let $N$ be a subset of a topological space $X$. The following are equivalent:
(1) $N$ is $\delta$-preopen,
(2) There exists a regular open set $U \subset X$ such that $N \subset U$ and $\delta-\text{Cl}(N) = \delta-\text{Cl}(U)$,
(3) $N$ is the intersection of a regular open set and a $\delta$-dense set,
(4) $N$ is the intersection of a $\delta$-open set and a $\delta$-dense set.

Proof. (1) $\Rightarrow$ (2): Let $N$ be $\delta$-preopen. We have $N \subset \text{Int}(\delta-\text{Cl}(N))$. Take $U = \text{Int}(\delta-\text{Cl}(N))$. Then $U$ is regular open such that $N \subset U$ and $\delta-\text{Cl}(N) = \delta-\text{Cl}(U)$.

(2) $\Rightarrow$ (3): Suppose that there exists a regular open set $U \subset X$ such that $N \subset U$ and $\delta-\text{Cl}(N) = \delta-\text{Cl}(U)$. Put $M = N \cup (X \setminus U)$. Thus, $M$ is $\delta$-dense and $N = U \cap M$.

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (1): Let $N = U \cap M$ such that $U$ is $\delta$-open and $M$ is $\delta$-dense. We have $\delta-\text{Cl}(N) = \delta-\text{Cl}(U)$. Thus,
\[ N \subset U \subset \delta-\text{Cl}(U) = \delta-\text{Cl}(N). \]

Thus, $N \subset \text{Int}(\delta-\text{Cl}(N))$ and hence $N$ is $\delta$-preopen.

Theorem 9 Let $X$ be a topological space. If $N \in \delta\text{SO}(X)$ and $M \in \delta\text{PO}(X)$, then $N \cap M \in e^*O(X)$.

Proof. Let $N \in \delta\text{SO}(X)$ and $M \in \delta\text{PO}(X)$. Then
\[ N \cap M \subset \text{Cl}(\delta-\text{Int}(N)) \cap \text{Int}(\delta-\text{Cl}(M)) \subset \text{Cl}(\delta-\text{Int}(N) \cap \text{Int}(\delta-\text{Cl}(M))) \subset \text{Cl}(\delta-\text{Int}(\delta-\text{Cl}(\delta-\text{Int}(N) \cap M))) \subset \text{Cl}(\text{Int}(\delta-\text{Cl}(N \cap M))). \]

Thus, $N \cap M \in e^*O(X)$.

Theorem 10 Let $N$ be a subset of a topological space $X$. The following are equivalent:
(1) $N \in e^*O(X)$,
(2) $N$ is the intersection of a $\delta$-semiopen and a $\delta$-dense set,
(3) $N$ is the intersection of a $\delta$-semiopen and a $\delta$-preopen set.

Proof. (1) $\Rightarrow$ (2): Let $N \in e^*O(X)$. Then $\delta-\text{Cl}(N)$ is regular closed and so $\delta$-semiopen. Take $M = N \cup (X \setminus \delta-\text{Cl}(N))$. This implies that $M$ is $\delta$-dense and $N = \delta-\text{Cl}(N) \cap M$.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): It follows from Theorem 9.
Theorem 11 ([3]) Let \( N \) be a subset of a topological space \( X \). The following are equivalent:

1. \( N \in \varepsilon^*O(X) \),
2. there exists \( U \in \delta PO(X) \) such that \( U \subset N \subset \delta Cl(U) \),
3. \( \delta Cl(N) \) is regular closed.

Theorem 12 Let \( X \) be a topological space. The following are equivalent:

1. \( \delta PO(X) \subset \delta SO(X) \),
2. \( \alpha O(X) = \delta PO(X) \),
3. \( \varepsilon^*O(X) = \delta SO(X) \).

Proof. (1) \( \Leftrightarrow \) (2) : It follows from Theorem 6.
(1) \( \Rightarrow \) (3) : We have \( \varepsilon^*O(X) \supset \delta SO(X) \).
Let \( N \in \varepsilon^*O(X) \). By Theorem 11, we have \( U \subset N \subset \delta Cl(U) \) for a \( U \in \delta PO(X) \). Since \( U \in \delta SO(X) \), then \( Cl(\delta Int(U)) = \delta Cl(U) \). Hence
\[
N \subset \delta Cl(U) = Cl(\delta Int(U)) \subset Cl(\delta Int(N)).
\]
Thus, \( N \in \delta SO(X) \). This implies that \( \varepsilon^*O(X) = \delta SO(X) \).
(3) \( \Rightarrow \) (1) : Obvious. \( \blacksquare \)

Remark 13 Let \( X \) be a topological space. If \( X \) is not a disjoint union of two nonempty \( \delta \)-preopen subsets, then \( X \) is connected. The following example shows that this implication is not reversible.

Example 14 Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). Then \( (X, \tau) \) is connected but it is a disjoint union of two nonempty \( \delta \)-preopen subsets.

Definition 15 A topological space \( X \) is said to be \( \delta p \)-connected [1] if \( X \) cannot be expressed as the union of two nonempty disjoint \( \delta \)-preopen sets of \( X \).

Theorem 16 Let \( X \) be a topological space. The following are equivalent:

1. \( X \) is connected and \( \delta PO(X) \subset \delta SO(X) \),
2. \( X \) is not a disjoint union of two nonempty \( \delta \)-preopen subsets.

Proof. (1) \( \Rightarrow \) (2) : Suppose that \( X = N \cup M \) such that \( N \) and \( M \) are disjoint \( \delta \)-preopen sets. Since \( \delta PO(X) \subset \delta SO(X) \), then \( N \) and \( M \) are \( \alpha \)-open. By Theorem 7, \( M \) and \( N \) are \( \delta \)-clopen. Since \( X \) is connected, then \( N = \emptyset \) or \( M = \emptyset \).
(2) \( \Rightarrow \) (1) : Suppose that \( X \) is not a disjoint union of two nonempty \( \delta \)-preopen subsets. Then \( X \) is connected.

Let \( N \) be a \( \delta \)-preopen set and \( M = N \setminus Cl(\delta Int(N)) \). Since an intersection of a \( \delta \)-preopen set and a \( \delta \)-open set is \( \delta \)-preopen, then \( M \) is \( \delta \)-preopen. We have
\[
\delta Int(M) = \delta Int(N) \cap (X \setminus Cl(\delta Int(N))) = \emptyset.
\]
Also, \( M \) is \( \delta \)-preclosed. By (2), either \( M = \emptyset \) or \( M = X \) and hence either \( N \subset Cl(\delta Int(N)) \) or \( N = X \). Hence, every \( \delta \)-preopen subset is \( \delta \)-semiopen. \( \blacksquare \)
3 Further properties

**Theorem 17** Let $X$ be a topological space. The following are equivalent:

1. The $\delta$-closure of every $\delta$-open subset of $X$ is $\delta$-open,
2. $\text{Cl}(\delta\text{-Int}(N)) \subseteq \text{Int}(\delta\text{-Cl}(N))$ for every subset $N$ of $X$,
3. $\delta\text{SO}(X) \subseteq \delta\text{PO}(X)$,
4. The $\delta$-closure of every $e^*$-open subset is $\delta$-open,
5. $e^*\text{O}(X) \subseteq \delta\text{PO}(X)$.

**Proof.** (1) $\Rightarrow$ (2) : Suppose that the $\delta$-closure of every $\delta$-open subset of $X$ is $\delta$-open. Then the set $\text{Cl}(\delta\text{-Int}(N))$ is $\delta$-open. Hence, $\text{Cl}(\delta\text{-Int}(N)) = \text{Int}(\text{Cl}(\delta\text{-Int}(N))) \subseteq \text{Int}(\delta\text{-Cl}(N))$.

(2) $\Rightarrow$ (3) : Let $N$ be $\delta$-semiopen. By (2), we have $N \subseteq \text{Cl}(\delta\text{-Int}(N)) \subseteq \text{Int}(\delta\text{-Cl}(N))$. Thus, $N$ is $\delta$-preopen.

(3) $\Rightarrow$ (4) : Let $N$ be $e^*$-open. Then $\delta\text{-Cl}(N)$ is $\delta$-semiopen. By (3), $\delta\text{-Cl}(N)$ is $\delta$-preopen. Thus, $\delta\text{-Cl}(N) \subseteq \text{Int}(\delta\text{-Cl}(N))$ and hence $\delta\text{-Cl}(N)$ is $\delta$-open.

(4) $\Rightarrow$ (5) : Let $N$ be $e^*$-open. By (4), $\delta\text{-Cl}(N) = \text{Int}(\delta\text{-Cl}(N))$. Thus, $N \subseteq \delta\text{-Cl}(N) = \text{Int}(\delta\text{-Cl}(N))$ and hence $N$ is $\delta$-preopen.

(5) $\Rightarrow$ (1) : Let $U$ be $\delta$-open. Then $\delta\text{-Cl}(U)$ is $e^*$-open. By (5), $\delta\text{-Cl}(U)$ is $\delta$-preopen. Thus, $\delta\text{-Cl}(U) \subseteq \text{Int}(\delta\text{-Cl}(U))$ and hence $\delta\text{-Cl}(U)$ is $\delta$-open.

**Theorem 18** Let $X$ be a topological space. The following are equivalent:

1. $\delta\text{PO}(X) \subseteq \delta\text{SO}(X)$,
2. every $\delta$-dense subset is $\delta$-semiopen,
3. $\delta\text{-Int}(N)$ is $\delta$-dense for every $\delta$-dense subset $N$,
4. $\delta\text{-Int}(\delta\text{-Fr}(N)) = \emptyset$ for every subset $N$,
5. $e^*\text{O}(X) \subseteq \delta\text{SO}(X)$,
6. $\delta\text{-Int}(\delta\text{-Fr}(N)) = \emptyset$ for every $\delta$-dense subset $N$.

**Proof.** (1) $\Rightarrow$ (2) : It follows from the fact that every $\delta$-dense set is $\delta$-preopen.

(2) $\Rightarrow$ (3) : Let $N$ be $\delta$-dense. Then $N$ is $\delta$-semiopen. Thus, $\text{Cl}(\delta\text{-Int}(N)) \supseteq \delta\text{-Cl}(N)$ and hence $\delta\text{-Int}(N)$ is $\delta$-dense.

(3) $\Rightarrow$ (4) : Let $N \subseteq X$. We have

$$X = \delta\text{-Cl}(N) \cup (X \setminus \delta\text{-Cl}(N)) = \delta\text{-Cl}(N) \cup \delta\text{-Int}(X \setminus N).$$

This implies that $N \cup \delta\text{-Int}(X \setminus N)$ is $\delta$-dense. Hence, $\delta\text{-Int}(N \cup \delta\text{-Int}(X \setminus N))$ is $\delta$-dense. We have

$$\delta\text{-Int}[N \cup \delta\text{-Int}(X \setminus N)] \cap \delta\text{-Int}[(X \setminus N) \cup \delta\text{-Int}(N)] = X \setminus \delta\text{-Fr}(N).$$

Since $X \setminus \delta\text{-Fr}(N)$ is an intersection of two $\delta$-dense $\delta$-open sets, then $X \setminus \delta\text{-Fr}(N)$ is $\delta$-dense.

(4) $\Rightarrow$ (6) : Obvious.

(6) $\Rightarrow$ (3) : Let $N$ be $\delta$-dense. By (6), $\delta\text{-Int}(\delta\text{-Fr}(N)) = \delta\text{-Int}(X \setminus \delta\text{-Int}(N)) = X \setminus \text{Cl}(\delta\text{-Int}(N)) = \emptyset$. Hence, $\delta\text{-Int}(N)$ is $\delta$-dense.

(4) $\Rightarrow$ (5) : Let $N$ be $e^*$-open. By (4) and Theorem 4, $N$ is $\delta$-semiopen.

(5) $\Rightarrow$ (1) : Obvious.
Theorem 19 Let $X$ be a topological space. The following are equivalent:

1. $\delta \text{PO}(X) \subset \delta \text{SO}(X)$,
2. $\text{Int}(\delta \text{-Cl}(N \cap M)) = \text{Int}(\delta \text{-Cl}(N)) \cap \text{Int}(\delta \text{-Cl}(M))$ for every $N \subset X$ and $M \subset X$.
3. $\text{Cl}(\delta \text{-Int}(N \cup M)) = \text{Cl}(\delta \text{-Int}(N)) \cup \text{Cl}(\delta \text{-Int}(M))$ for every $N \subset X$ and $M \subset X$.

Proof. (1) $\Rightarrow$ (2): Let $\delta \text{PO}(X) \subset \delta \text{SO}(X)$ and $N, M \subset X$. By Theorem 18, $\delta \text{-Int}(\delta \text{-Fr}(A)) = \emptyset$ for every subset $A$. Since $\delta \text{-Int}(\delta \text{-Fr}(A)) = \delta \text{-Int}(\delta \text{-Cl}(A) \cap (X \setminus \delta \text{-Int}(A))) = \delta \text{-Int}(\delta \text{-Cl}(A)) \setminus \delta \text{-Cl}(\delta \text{-Int}(A))$, then $\delta \text{-Int}(\delta \text{-Cl}(A)) \subset \delta \text{-Cl}(\delta \text{-Int}(A))$ and hence $\delta \text{-Int}(\delta \text{-Cl}(A)) = \delta \text{-Int}(\delta \text{-Cl}(\delta \text{-Int}(A)))$. This implies that

$$\text{Int}(\delta \text{-Cl}(N)) \cap \text{Int}(\delta \text{-Cl}(M)) = \text{Int}(\delta \text{-Int}(N)) \cap \text{Int}(\delta \text{-Cl}(M)) \subset \text{Cl}(\delta \text{-Int}(N)) \cap \text{Int}(\delta \text{-Cl}(M)).$$

On the other we have

$$\text{Cl}(\delta \text{-Int}(N)) \cap \text{Int}(\delta \text{-Cl}(M)) \subset \text{Cl}(\delta \text{-Int}(N)) \cap \text{Int}(\delta \text{-Cl}(M)) \subset \text{Cl}(\delta \text{-Int}(N)) \cap \delta \text{-Cl}(N \cap M).$$

Since $\text{Int}(\delta \text{-Cl}(N \cap M)) \subset \text{Int}(\delta \text{-Cl}(N)) \cap \text{Int}(\delta \text{-Cl}(M))$, we have $\text{Int}(\delta \text{-Cl}(N \cap M)) = \text{Int}(\delta \text{-Cl}(N)) \cap \text{Int}(\delta \text{-Cl}(M))$.

(2) $\Rightarrow$ (1): Suppose that (2) holds. This implies that

$$\delta \text{-Int}(\delta \text{-Fr}(N)) = \delta \text{-Int}(\delta \text{-Cl}(N) \cap \delta \text{-Cl}(X \setminus N)) = \text{Int}(\delta \text{-Cl}(N)) \cap \text{Int}(\delta \text{-Cl}(X \setminus N)) \subset \text{Cl}(\delta \text{-Cl}(N \cap X \setminus N)) = \emptyset.$$

By Theorem 18, we have $\delta \text{PO}(X) \subset \delta \text{SO}(X)$.

(2) $\Leftrightarrow$ (3): Obvious. $lacksquare$

Theorem 20 Let $X$ be a topological space. The following are equivalent:

1. $\delta \text{PO}(X) \subset \delta \text{SO}(X)$ and the $\delta$-closure of every $\delta$-open subset of $X$ is $\delta$-open,
2. $\text{Int}(\delta \text{-Cl}(N)) = \text{Cl}(\delta \text{-Int}(N))$ for every $N$ in $X$,
3. $\text{Cl}(\text{Int}(\delta \text{-Cl}(N)) = \text{Int}(\text{Cl}(\delta \text{-Int}(N)))$ for every $N$ in $X$,
4. $e^* \text{O}(X) \subset a \text{O}(X),$
5. $\delta \text{SO}(X) \subset a \text{O}(X)$ and $\delta \text{PO}(X) \subset a \text{O}(X),$
6. $\delta \text{PO}(X) = \delta \text{SO}(X),$
7. $N$ is $\delta$-semiopen if and only if $\delta \text{-Cl}(N)$ is $\delta$-open.

Proof. (1) $\Rightarrow$ (2): It follows from Theorem 17 and 18.

(2) $\Rightarrow$ (3): Let $N \subset X$. Since $\text{Int}(\delta \text{-Cl}(N)) = \text{Cl}(\delta \text{-Int}(N))$ is $\delta$-clopen, then $\text{Cl}(\text{Int}(\delta \text{-Cl}(N)) = \text{Int}(\text{Cl}(\delta \text{-Int}(N)))$. 


(3) ⇒ (4) : Let $N$ be $e^*$-open. We have $N \subset Cl(\text{Int}(\delta-Cl(N))) = \text{Int}(Cl(\delta-\text{Int}(N)))$. Hence, $N$ is $a$-open.

(4) ⇒ (5), (5) ⇒ (6) : Obvious.

(6) ⇒ (7) : Let $N$ be $\delta$-semiopen. This implies that $\delta-Cl(N)$ is $\delta$-semiopen and hence $\delta$-preopen. Thus, $\delta-Cl(N) \subset \text{Int}(\delta-Cl(N))$ and hence $\delta-Cl(N)$ is $\delta$-open.

Conversely, let $\delta-Cl(N)$ be $\delta$-open. This implies that $N \subset \delta-Cl(N) = \text{Int}(\delta-Cl(N))$ and $N$ is $\delta$-preopen and hence $\delta$-semiopen.

(7) ⇒ (1) : Let $N$ be $\delta$-open. Then $\delta-Cl(N)$ is $\delta$-open.

Let $N$ be a $\delta$-dense set. Then $\delta-Cl(N)$ is $\delta$-open. By 7, $N$ is $\delta$-semiopen. Thus, by Theorem 18, $\delta PO(X) \subset \delta SO(X)$. ■

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References

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