VECTOR INEQUALITIES FOR POWERS OF SOME OPERATORS IN HILBERT SPACES

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Abstract

Vector inequalities for powers of some operators in Hilbert spaces with applications for operator norm, numerical radius, commutators and self-commutators are given.

1 Introduction

Let \((H; \langle \cdot, \cdot \rangle)\) be a complex Hilbert space. The numerical range of an operator \(T\) is the subset of the complex numbers \(\mathbb{C}\) given by [13, p. 1]:

\[
W(T) = \{ \langle Tx, x \rangle, \; x \in H, \; \|x\| = 1 \}.
\]

The numerical radius \(w(T)\) of an operator \(T\) on \(H\) is given by [13, p. 8]:

\[
w(T) = \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \}.
\]

It is well known that \(w(\cdot)\) is a norm on the Banach algebra \(B(H)\) of all bounded linear operators \(T : H \to H\). This norm is equivalent to the operator norm. In fact, the following more precise result holds [13, p. 9]:

\[
w(T) \leq \|T\| \leq 2w(T),
\]

for any \(T \in B(H)\).

For more results on numerical radii, see [14], Chapter 11.

For other results and historical comments on the above see [13, p. 39–41]. For recent inequalities involving the numerical radius, see [2]-[10], [15], [19]-[21] and [22].

The Schwarz inequality for positive operators asserts that if \(T\) is a positive operator in \(B(H)\), then

\[
|\langle Tx, y \rangle|^2 \leq \langle Tx, x \rangle \langle Ty, y \rangle \; \text{for all} \; x, y \in H.
\]

2000 Mathematics Subject Classifications. 47A12 (47A30, 47A63, 47B15).

Key words and Phrases. Hilbert space, Bounded linear operators, Vector inequalities, Operator norm and numerical radius, Commutator, Self-commutator.

Received: September 9, 2008
Communicated by Vladimir Rakočević
For an arbitrary operator $T$ in $B(H)$ the following "mixed Schwarz" inequality has been established by Kato in [18] (see also [12] and [14, p. 265]):

$$|\langle Tx, y \rangle|^2 \leq \langle (T^*T)^\alpha x, x \rangle \left\langle (TT^*)^{1-\alpha} y, y \right\rangle$$ for all $x, y \in H$  \hspace{1cm} (1.4)

and for $\alpha \in [0,1].$

An important consequence of Kato’s inequality (1.4) is the famous Heinz inequality (see [1], [16], [17], [18]) which says that if $T, A$ and $B$ are operators in $B(H)$ such that $A$ and $B$ are positive and $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y$ in $H$ then

$$|\langle Tx, y \rangle| \leq \|A^\alpha x\| \|B^{1-\alpha}y\|$$

for all $x, y \in H$ and for $\alpha \in [0,1].$

In this paper we establish some vector inequalities for powers of various operators in Hilbert spaces. Applications for norm and numerical radius inequalities are provided. Particular cases for commutators and self-commutators are also given.

## 2 Vector Inequalities for Two Operators

The first results concerning powers of two operators is incorporated in:

**Theorem 1.** For any $A, B \in B(H)$ and $r \geq 1$ we have the vector inequality:

$$|\langle Ax, By \rangle|^r \leq \frac{1}{2} \left[ \langle (A^*A)^r x, x \rangle + \langle (B^*B)^r y, y \rangle \right],$$ \hspace{1cm} (2.1)

where $x, y \in H$, $\|x\| = \|y\| = 1.$

In particular, we have the norm inequality

$$\|B^*A\|^r \leq \frac{1}{2} (\|A^*A\|^r + \|B^*B\|^r)$$ \hspace{1cm} (2.2)

and the numerical radius inequality

$$w^r (B^*A) \leq \frac{1}{2} \| (A^*A)^r + (B^*B)^r \|,$$ \hspace{1cm} (2.3)

respectively.

The constant $\frac{1}{2}$ is best possible in all inequalities (2.1), (2.2) and (2.3).

**Proof.** By the Schwarz inequality in the Hilbert space $(H; \langle \cdot, \cdot \rangle)$ we have:

$$|\langle B^*Ax, y \rangle| = |\langle Ax, By \rangle| \leq \|Ax\| \cdot \|By\|$$ \hspace{1cm} (2.4)

$$= \langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*By, y \rangle^{1/2}, \quad x, y \in H.$$ Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*By, y \rangle^{1/2} \leq \frac{\langle A^*Ax, x \rangle + \langle B^*By, y \rangle}{2}$$ \hspace{1cm} (2.5)

$$\leq \left( \frac{\langle A^*Ax, x \rangle^r + \langle B^*B^r y, y \rangle^r}{2} \right)^{\frac{1}{r}}.$$
for any $x, y \in H$.

It is known that if $P$ is a positive operator then for any $r \geq 1$ and $z \in H$ with $\|z\| = 1$ we have the inequality (see for instance [20])

$$\langle Pz, z \rangle^r \leq \langle P^r z, z \rangle.$$ (2.6)

Applying this property to the positive operators $A^*A$ and $B^*B$, we deduce that

$$\left( \frac{\langle A^*Ax, x \rangle^r + \langle B^*By, y \rangle^r}{2} \right)^{\frac{1}{r}} \leq \left( \frac{\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r y, y \rangle}{2} \right)^{\frac{1}{r}}$$ (2.7)

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, on making use of the inequalities (2.4), (2.5) and (2.7), we get the inequality:

$$|\langle (B^*A)x, y \rangle|^r \leq \frac{1}{2} \left[ |\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r y, y \rangle| \right]$$ (2.8)

for any $x, y \in H$, $\|x\| = \|y\| = 1$, which proves (2.1).

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.8) and since the operators $(A^*A)^r$ and $(B^*B)^r$ are self-adjoint, we deduce the desired inequality (2.2).

Now, if we take $y = x$ in (2.1), then we get

$$|\langle (B^*A)x, x \rangle|^r \leq \frac{1}{2} \left[ |\langle (A^*A)^r + (B^*B)^r \rangle x, x \rangle| \right]$$ (2.9)

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.9) we get (2.3).

The sharpness of the constant follows by taking $r = 1$ and $B = A$ in all inequalities (2.1), (2.2) and (2.3). The details are omitted. ⊡

**Corollary 1.** For any $A \in B(H)$ and $r \geq 1$ we have the vector inequalities:

$$|\langle Ax, y \rangle|^r \leq \frac{1}{2} \left[ |\langle (A^*A)^r x, x \rangle + 1| \right],$$ (2.10)

and

$$|\langle A^2 x, y \rangle|^r \leq \frac{1}{2} \left[ |\langle (A^*A)^r x, x \rangle + |\langle AA^*\rangle^r y, y \rangle| \right],$$ (2.11)

where $x, y \in H$, $\|x\| = \|y\| = 1$.

In particular, we have the norm inequalities

$$\|A\|^r \leq \frac{1}{2} \left( \| (A^*A)^r \| + 1 \right)$$ (2.12)

and

$$\|A^2\|^r \leq \frac{1}{2} \left( \| (A^*A)^r \| + \| (AA^*)^r \| \right),$$ (2.13)

respectively.
We also have the numerical radius inequalities
\[ w^r(A) \leq \frac{1}{2} \| (A^*A)^r + I \| \]  
(2.14)
and
\[ w^r(A^2) \leq \frac{1}{2} \| (A^*A)^r + (AA^*)^r \|, \]  
(2.15)
respectively.

A different approach is considered in the following result:

**Theorem 2.** For any \( A, B \in B(H) \), any \( \alpha \in (0,1) \) and \( r \geq 1 \), we have the vector inequality:
\[ |\langle Ax, By \rangle|^r \leq \alpha \langle (A^*A)^\frac{r}{\alpha} x, x \rangle + (1 - \alpha) \langle (B^*B)^\frac{r}{1-\alpha} y, y \rangle \]  
(2.16)
for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \).

In particular, we have the norm inequality
\[ \| B^*A \|^{2r} \leq \alpha \| (A^*A)^{\frac{r}{\alpha}} \| + (1 - \alpha) \| (B^*B)^{\frac{r}{1-\alpha}} \| \]  
(2.17)
and the numerical radius inequality
\[ w^{2r} (B^*A) \leq \| \alpha (A^*A)^{\frac{r}{\alpha}} + (1 - \alpha) (B^*B)^{\frac{r}{1-\alpha}} \|, \]  
(2.18)
respectively.

**Proof.** By Schwarz’s inequality, we have:
\[ |\langle (B^*A) x, y \rangle|^2 \leq \langle (A^*A) x, x \rangle \cdot \langle (B^*B) y, y \rangle \]  
(2.19)
\[ = \left[ \langle (A^*A)^{\frac{r}{\alpha}} x, x \rangle \cdot \left[ \langle (B^*B)^{\frac{1}{1-\alpha}} y, y \rangle \right]^{1-\alpha} \right], \]
for any \( x, y \in H \).

It is well known that (see for instance [20]) if \( P \) is a positive operator and \( q \in (0,1] \) then for any \( u \in H, \|u\| = 1 \), we have
\[ \langle Pu, u \rangle^q \leq \langle P^q u, u \rangle, \]  
(2.20)
Applying this property to the positive operators \( (A^*A)^{\frac{1}{r}} \) and \( (B^*B)^{\frac{1}{1-\alpha}} \) \( (\alpha \in (0,1)) \), we have
\[ \left[ \langle (A^*A)^{\frac{r}{\alpha}} x, x \rangle \cdot \left[ \langle (B^*B)^{\frac{1}{1-\alpha}} y, y \rangle \right]^{1-\alpha} \right] \]
\[ \leq \left[ \langle (A^*A)^{\frac{1}{r}} x, x \rangle \cdot \langle (B^*B)^{\frac{1}{1-\alpha}} y, y \rangle \right]^{1-\alpha}, \]  
(2.21)
for any \( x, y \in H, \|x\| = \|y\| = 1 \).

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e.,
\[ a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b, \alpha \in (0,1) , a, b \geq 0, \]
we get
\[
\left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle^\alpha \cdot \left\langle (B^*B)^{\frac{1}{2\alpha}} y, y \right\rangle^{1-\alpha} \leq \alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{2\alpha}} y, y \right\rangle
\]
(2.22)
for any \( x, y \in H, \|x\| = \|y\| = 1 \).

Moreover, by the elementary inequality following from the convexity of the function \( f(t) = t^r, r \geq 1 \), namely
\[
\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \quad \alpha \in (0,1) , a, b \geq 0, \]
we deduce that
\[
\alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{2\alpha}} y, y \right\rangle \leq \alpha \left[ \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle^r + (1-\alpha) \left\langle (B^*B)^{\frac{1}{2\alpha}} y, y \right\rangle^r \right]^{\frac{1}{r}} \]
(2.23)
\[
\leq \alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{2\alpha}} y, y \right\rangle \]
for any \( x, y \in H, \|x\| = \|y\| = 1 \), where, for the last inequality we used the inequality (2.6) for the positive operators \((A^*A)^{\frac{1}{2}}\) and \((B^*B)^{\frac{1}{2\alpha}}\).

Now, on making use of the inequalities (2.19), (2.21), (2.22) and (2.23), we get
\[
\|\left\langle (B^*A) x, y \right\rangle \|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{2\alpha}} y, y \right\rangle
\]
(2.24)
for any \( x, y \in H, \|x\| = \|y\| = 1 \), and the inequality (2.16) is proved.

Taking the supremum over \( x, y \in H, \|x\| = \|y\| = 1 \) in (2.24) produces the desired inequality (2.17).

The numerical radius inequality follows from (2.24) written for \( y = x \). The details are omitted. \( \square \)

The following particular instances are of interest:

**Corollary 2.** For any \( A \in B(H) \) and \( \alpha \in (0,1) , r \geq 1 \), we have the vector inequalities
\[
\|\langle Ax, y \rangle \|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + 1-\alpha,
\]
(2.25)
\[
\|\langle A^2 x, y \rangle \|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + (1-\alpha) \left\langle (AA^*)^{\frac{1}{2\alpha}} y, y \right\rangle
\]
(2.26)
and
\[
\|\langle Ax, Ay \rangle \|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{1}{2}} x, x \right\rangle + (1-\alpha) \left\langle (A^*A)^{\frac{1}{2\alpha}} y, y \right\rangle,
\]
(2.27)
respectively, where \( x, y \in H, \|x\| = \|y\| = 1 \).
We have the norm inequalities
\[ \| A \|^2r \leq \alpha \| (A^*A)^{\frac{r}{2}} \| + 1 - \alpha \] (2.28)
and
\[ \| A^2 \|^2r \leq \alpha \| (A^*A)^{\frac{r}{2}} \| + (1 - \alpha) \| (AA^*)^{\frac{r}{2}} \|, \] (2.29)
respectively.

We have the numerical radius inequalities
\[ w^2r (A) \leq \alpha \| (A^*A)^{\frac{r}{2}} \| + (1 - \alpha) I \| \] (2.30)
and
\[ w^2r (A^2) \leq \alpha \| (A^*A)^{\frac{r}{2}} \| + (1 - \alpha) \| (AA^*)^{\frac{r}{2}} \|, \] (2.31)
respectively.

Moreover, we have the norm inequality
\[ \| A \|^4r \leq \alpha \| (A^*A)^{\frac{r}{2}} \| + (1 - \alpha) \| (A^*A)^{\frac{r}{2}} \|. \] (2.32)

3 Vector Inequalities for the Sum of Two Products

The following result concerning four operators may be stated:

**Theorem 3.** For any \( A, B, C, D \in B (H) \) and \( r, s \geq 1 \) we have:

\[ \left| \left\langle \left[ \frac{B^*A + D^*C}{2} \right] x, y \right\rangle \right|^2 \]
\[ \leq \left\langle \left[ \frac{(A^*A)^{\frac{r}{2}} + (C^*C)^{\frac{r}{2}}}{2} \right] x, x \right\rangle^{\frac{1}{2}} \cdot \left\langle \left[ \frac{(B^*B)^{\frac{s}{2}} + (D^*D)^{\frac{s}{2}}}{2} \right] y, y \right\rangle^{\frac{1}{2}} \] (3.1)

for any \( x, y \in H \) with \( \| x \| = \| y \| = 1 \).

Moreover, we have the norm inequality
\[ \left\| \frac{B^*A + D^*C}{2} \right\| \leq \left\| \frac{(A^*A)^{\frac{r}{2}} + (C^*C)^{\frac{r}{2}}}{2} \right\|^{\frac{1}{2}} \cdot \left\| \frac{(B^*B)^{\frac{s}{2}} + (D^*D)^{\frac{s}{2}}}{2} \right\|^{\frac{1}{2}}. \] (3.2)

**Proof.** By the Schwarz inequality in the Hilbert space \( (H; \langle \cdot, \cdot \rangle) \) we have:

\[ |\langle (B^*A + D^*C) x, y \rangle|^2 \]
\[ = |\langle B^*Ax, y \rangle + \langle D^*Cx, y \rangle|^2 \]
\[ \leq |\langle B^*Ax, y \rangle| + |\langle D^*Cx, y \rangle|^2 \]
\[ \leq \left[ \langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*By, y \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y \rangle^{\frac{1}{2}} \right]^2, \] (3.3)
for any $x, y \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2), \quad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$\langle A^* Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^* By, y \rangle^{\frac{1}{2}} + \langle C^* Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^* Dy, y \rangle^{\frac{1}{2}} \leq ((A^* Ax, x) + (C^* Cx, x)) \cdot ((B^* By, y) + (D^* Dy, y)), \quad (3.4)$$

for any $x, y \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r, s \geq 1$ that

$$\langle A^* Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^* By, y \rangle^{\frac{1}{2}} + \langle C^* Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^* Dy, y \rangle^{\frac{1}{2}} \leq 4 \cdot \left\langle \left( \frac{(A^* A)^r + (C^* C)^r}{2} \right) x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left( \frac{(B^* B)^s + (D^* D)^s}{2} \right) y, y \right\rangle^{\frac{1}{s}}, \quad (3.5)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Consequently, by (3.3) – (3.5) we have:

$$\left\langle \left( \frac{B^* A + D^* C}{2} \right) x, y \right\rangle^2 \leq \left\langle \left( \frac{(A^* A)^r + (C^* C)^r}{2} \right) x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left( \frac{(B^* B)^s + (D^* D)^s}{2} \right) y, y \right\rangle^{\frac{1}{s}}, \quad (3.6)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which provides the desired result (3.1).

Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ in (3.6) we deduce the desired inequality (3.2).

**Remark 1.** If we make $y = x$ in (3.6) and take the supremum over $\|x\| = 1$, then we get the inequality

$$w^2 \left( \frac{B^* A + D^* C}{2} \right) \leq \left\langle \left( \frac{(A^* A)^r + (C^* C)^r}{2} \right) x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left( \frac{(B^* B)^s + (D^* D)^s}{2} \right) y, y \right\rangle^{\frac{1}{s}},$$

which is not as good as (3.2) since we always have

$$w^2 \left( \frac{B^* A + D^* C}{2} \right) \leq \left\| \frac{B^* A + D^* C}{2} \right\|^2.$$

**Remark 2.** If $s = r$, then the inequality (3.1) becomes:

$$\left\langle \left( \frac{B^* A + D^* C}{2} \right) x, y \right\rangle^{2r} \leq \left\langle \left( \frac{(A^* A)^r + (C^* C)^r}{2} \right) x, x \right\rangle \cdot \left\langle \left( \frac{(B^* B)^s + (D^* D)^s}{2} \right) y, y \right\rangle \quad (3.7)$$
for any \( x, y \in H \) with \( \| x \| = \| y \| = 1 \) while (3.2) is equivalent with

\[
\left\| \frac{B^*A + D^*C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(B^*B) + (D^*D)^r}{2} \right\| .
\]  

(3.8)

**Corollary 3.** For any \( A, C \in B(H) \) we have:

\[
\left\langle \left( \frac{A + C}{2} \right)^r x, y \right\rangle^{2r} \leq \left\langle \left[ \frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(AA^*)^r + (CC^*)^r}{2} \right] y, y \right\rangle
\]

for any \( x, y \in H \) with \( \| x \| = \| y \| = 1 \). In particular, we have the norm inequality

\[
\left\| \frac{A + C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|,
\]

(3.10)

where \( r \geq 1 \).

The result is obvious by choosing \( B = D = I \) in Theorem 3.

**Corollary 4.** For any \( A, C \in B(H) \) we have:

\[
\left\langle \left( \frac{A^2 + C^2}{2} \right)^r x, y \right\rangle^{2r} \leq \left\langle \left[ \frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(AA^*)^r + (CC^*)^r}{2} \right] y, y \right\rangle
\]

for any \( x, y \in H \) with \( \| x \| = \| y \| = 1 \). Also, we have the norm inequality

\[
\left\| \frac{A^2 + C^2}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\|^{2r}
\]

(3.12)

for all \( r, s \geq 1 \).

If \( s = r \), then we have, in particular,

\[
\left\langle \left( \frac{A^2 + C^2}{2} \right)^r x, y \right\rangle^{2r} \leq \left\langle \left[ \frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(AA^*)^r + (CC^*)^r}{2} \right] y, y \right\rangle
\]

(3.13)

for any \( x, y \in H \) with \( \| x \| = \| y \| = 1 \) and the norm inequality

\[
\left\| \frac{A^2 + C^2}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\|
\]

(3.14)

for \( r \geq 1 \).
The result is obvious by choosing $B = A^*$ and $D = C^*$ in Theorem 3.

Another particular result of interest is the following one:

**Corollary 5.** For any $A, B \in B(H)$ we have:

$$\left\| \left[ \frac{B^*A + A^*B}{2} \right] x, y \right\|^2 \leq \left\langle \left[ \frac{(A^*A)^r + (B^*B)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[ \frac{(A^*A)^s + (B^*B)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

(3.15)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$\left\| \frac{B^*A + A^*B}{2} \right\|^2 \leq \left\langle \left[ \frac{(A^*A)^r + (B^*B)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(A^*A)^s + (B^*B)^s}{2} \right] y, y \right\rangle$$

(3.16)

for any $r, s \geq 1$.

In particular we have

$$\left\| \frac{B^*A + A^*B}{2} \right\|^{2r} \leq \left\langle \left[ \frac{(A^*A)^r + (B^*B)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(A^*A)^s + (B^*B)^s}{2} \right] y, y \right\rangle$$

(3.17)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\left\| \frac{B^*A + A^*B}{2} \right\|^r \leq \left\langle \left[ \frac{(A^*A)^r + (B^*B)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(A^*A)^s + (B^*B)^s}{2} \right] y, y \right\rangle$$

(3.18)

where $r \geq 1$.

The proof is obvious by choosing $D = A$ and $C = B$ in Theorem 3.

Another particular case that might be of interest is the following one.

**Corollary 6.** For any $A, D \in B(H)$ we have:

$$\left\| \left( \frac{A + D}{2} \right) x, y \right\|^2 \leq \left\langle \left[ \frac{(A^*A)^r + I}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[ \frac{(DD^*)^s + I}{2} \right] y, y \right\rangle^{\frac{1}{s}}$$

(3.19)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\left\| \frac{A + D}{2} \right\|^2 \leq \left\langle \left[ \frac{(A^*A)^r + I}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[ \frac{(DD^*)^s + I}{2} \right] y, y \right\rangle^{\frac{1}{s}},$$

(3.20)

where $r, s \geq 1$.
In particular we have
\[ |\langle Ax, y \rangle|^2 \leq \left\langle \left[ \frac{(A^*A)^r + I}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[ \frac{(AA^*)^s + I}{2} \right] y, y \right\rangle^{\frac{1}{s}} \] (3.21)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \) and the norm inequality
\[ \|A\|^2 \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + I}{2} \right\|^{\frac{1}{s}} . \] (3.22)

Moreover, for any \( r \geq 1 \) we have
\[ |\langle Ax, y \rangle|^{2r} \leq \left\langle \left[ \frac{(A^*A)^r + I}{2} \right] x, x \right\rangle \cdot \left\langle \left[ \frac{(AA^*)^s + I}{2} \right] y, y \right\rangle \] (3.23)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \) and
\[ \|A\|^{2r} \leq \left\| \frac{(A^*A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^s + I}{2} \right\| . \] (3.24)

The proof of (3.19) is obvious by the Theorem 3 on choosing \( B = I, C = I \) and writing the inequality for \( D^* \) instead of \( D \). The details are omitted.

**Remark 3.** If \( T \in B(H) \) and \( T = A + iC \), i.e., \( A \) and \( C \) are its Cartesian decomposition, then we get from (3.9)
\[ |\langle Tx, y \rangle|^{2r} \leq 2^{2r-1} \left\langle \left[ (A^*A)^r + (CC^*)^r \right] x, x \right\rangle \] (3.25)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \). In particular, we have the norm inequality
\[ \|T\|^{2r} \leq 2^{2r-1} \left\| (A^*A)^r + (CC^*)^r \right\|, \] (3.26)

where \( r \geq 1 \).

Now, if we use the inequality (3.19) for \( T, A \) and \( B \), then we get:
\[ |\langle Tx, y \rangle|^2 \leq 2^{2-r-\frac{1}{r}} \left\langle \left[ (A^*A)^r + I \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[ (CC^*)^s + I \right] y, y \right\rangle^{\frac{1}{s}} \] (3.27)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \) and the norm inequality
\[ \|T\|^2 \leq 2^{2-r-\frac{1}{r}} \left\| (A^*A)^r + I \right\|^{\frac{1}{r}} \cdot \left\| (CC^*)^s + I \right\|^{\frac{1}{s}}, \] (3.28)

where \( r, s \geq 1 \). In particular, we have
\[ |\langle Tx, y \rangle|^{2r} \leq 2^{2r-2} \left\langle \left[ (A^*A)^r + I \right] x, x \right\rangle \cdot \left\langle \left[ (CC^*)^s + I \right] y, y \right\rangle \] (3.29)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \) and the norm inequality
\[ \|T\|^{2r} \leq 2^{2r-2} \left\| (A^*A)^r + I \right\| \cdot \left\| (CC^*)^s + I \right\|, \] (3.30)

for any \( r \geq 1 \).
In terms of the Euclidean radius of two operators \( w_e (\cdot, \cdot) \), where, as in [2],
\[
    w_e (T, U) := \sup_{\|x\| = 1} \left( \|\langle T x, x \rangle\|^2 + \|\langle U x, x \rangle\|^2 \right)^{1/2},
\]
we have the following result as well.

**Theorem 4.** For any \( A, B, C, D \in B (H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have the vector inequality:
\[
    \| \langle Ax, By \rangle \|^2 + \| \langle Cx, Dy \rangle \|^2
    \leq (\| (A^* A)^p + (C^* C)^p \|_p)^{1/p} \cdot (\| (B^* B)^q + (D^* D)^q \|_q)^{1/q}
\]
for each \( x, y \in H \) with \( \|x\| = \|y\| = 1 \).

In particular, we have the inequality for the Euclidean radius:
\[
    w_e^2 (B^* A, D^* C) \leq \| (A^* A)^p + (C^* C)^p \|_p \cdot \| (B^* B)^q + (D^* D)^q \|_q.
\]

**Proof.** On utilising the elementary inequality
\[
    ac + bd \leq (a^p + b^p)^{1/p} \cdot (c^q + d^q)^{1/q},
\]
then for any \( x, y \in H \), \( \|x\| = \|y\| = 1 \) we have the inequalities:
\[
    \| \langle B^* Ax, y \rangle \|^2 + \| \langle D^* Cx, y \rangle \|^2
    \leq \langle A^* Ax, x \rangle \cdot \langle B^* By, y \rangle + \langle C^* Cx, x \rangle \cdot \langle D^* Dy, y \rangle
    \leq (\| (A^* A)^p + (C^* C)^p \|_p)^{1/p} \cdot (\| (B^* B)^q + (D^* D)^q \|_q)^{1/q}
    \leq (\| (A^* A)^p x \|_p \cdot (\| (C^* C)^p x \| x)^{1/p}) \cdot (\| (B^* B)^q y \|_q \cdot (\| (D^* D)^q y \| y)^{1/q}
    = (\| (A^* A)^p + (C^* C)^p \|_p \cdot (\| (B^* B)^q + (D^* D)^q \|_q \cdot (\| x \|_p \cdot (\| y \|_q).
\]

For the second inequality, let us make the choice \( y = x \) to get
\[
    \| \langle B^* Ax, x \rangle \|^2 + \| \langle D^* Cx, x \rangle \|^2
    \leq (\| (A^* A)^p + (C^* C)^p \|_p \cdot (\| (B^* B)^q + (D^* D)^q \|_q x, x) \|^{1/q},
\]
for any \( x \in H \), \( \|x\| = 1 \). Taking the supremum over \( x \in H \), \( \|x\| = 1 \) and noticing that the operators \( (A^* A)^p + (C^* C)^p \) and \( (B^* B)^q + (D^* D)^q \) are self-adjoint, we deduce the desired inequality (3.32).

The following particular case is of interest.

**Corollary 7.** For any \( A, C \in B (H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have:
\[
    \| \langle Ax, y \rangle \|^2 + \| \langle Cx, y \rangle \|^2 \leq 2^{1/q} \| (A^* A)^p + (C^* C)^p \|_p \cdot (\| x \|)
\]
for each \( x, y \in H \), with \( \|x\| = \|y\| = 1 \). In particular,
\[
    w_e^2 (A, C) \leq 2^{1/q} \| (A^* A)^p + (C^* C)^p \|_p.
\]
The proof follows from (3.31) and (3.32) for \( B = D = I \).

**Corollary 8.** For any \( A, D \in B(H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have:

\[
|\langle Ax, y \rangle|^2 + |\langle Dx, y \rangle|^2 \leq \left( |\left[ (A^*A)^p + I \right] x, x \right)^{1/p} \cdot \left( |\left[ (DD^*)^q + I \right] y, y \right)^{1/q}
\]

(3.34)

for each \( x, y \in H \), with \( \|x\| = \|y\| = 1 \). In particular,

\[
w_2^2(A, D) \leq \|(A^*A)^p + I\|^{1/p} \cdot \|(DD^*)^q + I\|^{1/q}.
\]

4 Inequalities for the Commutator

The commutator of two bounded linear operators \( T \) and \( U \) is the operator \( TU - UT \).

For the usual norm \( \|\cdot\| \) and for any two operators \( T \) and \( U \), by using the triangle inequality and the submultiplicivity of the norm, we can state the following inequality:

\[
\|TU - UT\| \leq 2\|T\|\|U\|.
\]

(4.1)

In [11], the following result has been obtained as well

\[
\|TU - UT\| \leq 2 \min \{\|T\|, \|U\|\} \min \{\|T - U\|, \|T + U\|\}.
\]

(4.2)

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator:

**Proposition 1.** For any \( T, U \in B(H) \) and \( r, s \geq 1 \) we have the vector inequality

\[
\|\left( (TU - UT) x, y \right) \|^2 \\
\leq 2^{2r - \frac{1}{2} - \frac{1}{2}} \left( \left( (U^*U)^r + (T^*T)^r \right) x, x \right)^{\frac{1}{2}} \cdot \left( \left( (UU^*)^s + (TT^*)^s \right) y, y \right)^{\frac{1}{2}},
\]

(4.3)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \). Moreover, we have the norm inequality

\[
\|TU - UT\|^2 \leq 2^{2r - \frac{1}{2} - \frac{1}{2}} \|(U^*U)^r + (T^*T)^r\|^\frac{1}{2} \cdot \|(UU^*)^s + (TT^*)^s\|^\frac{1}{2}.
\]

(4.4)

In particular, we have

\[
\|\left( (TU - UT) x, y \right) \|^{2r} \\
\leq 2^{2r - 2} \left( \left( (U^*U)^r + (T^*T)^r \right) x, x \right) \cdot \left( \left( (UU^*)^r + (TT^*)^r \right) y, y \right)
\]

(4.5)

for any \( x, y \in H \) with \( \|x\| = \|y\| = 1 \) and the norm inequality

\[
\|TU - UT\|^{2r} \leq 2^{2r - 2} \|(U^*U)^r + (T^*T)^r\| \cdot \|(UU^*)^r + (TT^*)^r\|,
\]

(4.6)

for any \( r \geq 1 \).

**Proof.** Follows by Theorem 3 on choosing \( B = T^* \), \( A = U \), \( D = -U^* \) and \( C = T \). \( \square \)
Now, for $U = T^*$ we can state the following corollary.

**Corollary 9.** For any $T \in B(H)$ we have the vector inequality for the self commutator:

$$\left| \langle (TT^* - T^*T)x, y \rangle \right|^2 \leq 2^{2 - \frac{1}{r} - \frac{1}{s}} \langle ((TT^*)^r + (T^*T)^r)x, x \rangle^{\frac{1}{r}} \cdot \langle ((TT^*)^s + (T^*T)^s)y, y \rangle^{\frac{1}{s}}$$ \hspace{1cm} (4.7)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$\|TT^* - T^*T\|^2 \leq 2^{2 - \frac{1}{r} - \frac{1}{s}} \|((TT^*)^r + (T^*T)^r)^{\frac{1}{r}} \cdot \|((TT^*)^s + (T^*T)^s)^{\frac{1}{s}} \| \cdot$$ \hspace{1cm} (4.8)

In particular we have

$$\left| \langle (TT^* - T^*T)x, y \rangle \right|^{2r} \leq 2^{2r - 2} \langle ((TT^*)^r + (T^*T)^r)x, x \rangle \cdot \langle ((TT^*)^r + (T^*T)^r)y, y \rangle$$ \hspace{1cm} (4.9)

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\|TT^* - T^*T\|^r \leq 2^{r - 1} \|(TT^*)^r + (T^*T)^r\|$$ \hspace{1cm} (4.10)

for any $r \geq 1$.

**References**


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