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ON CERTAIN CLASSES OF HARMONIC P-VALENT FUNCTIONS BY APPLYING THE RUSCHEWEYH DERIVATIVES

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Abstract

In this paper we have introduced two new classes $\mathcal{HR}_p(\beta, \lambda, k, v), \overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$ of complex valued harmonic multivalent functions of the form $f = h + \overline{g}$, where h and g are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and $f(z)$ satisfying the condition

$$Re \quad (1-\lambda)\frac{D^v f}{z^p} + \lambda(1-k)\frac{(D^v f)'}{(z^p)'} + \lambda k\frac{(D^v f)''}{(z^p)''} \quad > \frac{\beta}{p}.$$

A sufficient coefficient condition for this function in the class $\mathcal{HR}_p(\beta, \lambda, k, v)$ and a necessary and sufficient coefficient condition for the function f in the class
$\overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$ are determined. We investigate inclusion relations, distortion theorem, extreme points, convex combination and other interesting properties for these families.

1 Introduction

A continuous complex-valued function $f = u + iv$ in a simply connected complex domain Ω is said to be harmonic if both u and v are real harmonic functions in Ω. For u and v in $f = u + iv$, there exists analytic functions L and k such that $u = Re(L)$ and $v = Im(k)$. So we can write

$$f = u + iv = \frac{L + k}{2} + \frac{\overline{L} - \overline{k}}{2} = h + \overline{g}.$$

In this case the Jacobin of f is given by

$$\mathcal{J}_f(z) = \ |h'(z)|^2 - |g'(z)|^2. \tag{1.1}$$

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Then the mapping $z \to f(z)$ is orientation preserving in Ω if and only if $\mathcal{J}_f(z) > 0$ in Ω. See Clunie and Sheil [4] and Jahangiri [5], [6]. Without loss of generality, we let Ω be the open unit disk Δ because it is a simply connected domain. Let $\mathcal{H}_p$ denote the family of functions $f = h + \overline{g}$ which are multivalent harmonic in Δ and of the form

$$h(z) = z^p + \sum_{n=m+p}^{\infty} a_n z^n, \quad g(z) = \sum_{n=m+p-1}^{\infty} b_n z^n, \quad |b_{m+p-1}| < 1. \tag{1.2}$$

Let $\overline{\mathcal{H}}_p$ be the subclass of $\mathcal{H}_p$ consisting of function $f = h + \overline{g}$ such that

$$h(z) = z^p - \sum_{n=m+p}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=m+p-1}^{\infty} |b_n| z^n \quad |b_{m+p-1}| < 1. \tag{1.3}$$

For $f = h + \overline{g}$, given by (1.2) and $v > -1$ in Jahangiri [5], Jahangiri and et. al. [7] have defined the Ruscheweyh Derivatives of harmonic functions $f = h + \overline{g}$ in $\mathcal{H}_p$ by

$$D^v f(z) = D^v h(z) + D^v g(z) \tag{1.4}$$

where the Ruscheweyh Derivative of p-valent function (see [8]), $Q(z) = z^p + \sum_{n=1}^{\infty} t_n z^n$ is given by

$$D^v Q(z) = Q(z) * \frac{z^p}{(1-z)^{v+1}} = z^p + \sum_{n=1}^{\infty} C^p(n,v) t_n z^n \tag{1.5}$$

where $v = \xi + p - 1, \xi > -p$ and

$$C^p(n,v) = \frac{\Gamma(v - p + 1 + n)}{\Gamma(v+1)(n-p)!}\ , \quad v > -1 \tag{1.6}$$

and $*$ stands for the Hadamard product or convolution of two power series $Q(z) = z^p + \sum_{n=1}^{\infty} t_n z^n$ and $\psi(z) = z^p + \sum_{n=1}^{\infty} s_n z^n$ defined by

$$(Q * \psi)(z) = z^p + \sum_{n=1}^{\infty} (t_n s_n) z^n.$$

For fixed values of $v (v > -1)$, we let $\mathcal{HR}_p(\beta, \lambda, k, v)$ consist of harmonic functions $f = h + \overline{g}$ in $\mathcal{H}_p$ so that

$$Re\left\{(1-\lambda)\frac{D^v f}{z^p} + \lambda(1-k)\frac{(D^v f)'}{(z^p)'} + \lambda k \frac{(D^v f)''}{(z^p)''}\right\} > \frac{\beta}{p} \tag{1.7}$$

$(\lambda \geq 0, 0 \leq k \leq 1, v > -1, 0 < \beta \leq p, p \in \mathcal{N})$
where $z = re^{i\theta} \in \Delta$. We also let $\overline{\mathcal{HR}_p}(\beta, \lambda, k, v) = \mathcal{HR}_p(\beta, \lambda, k, v) \cap \overline{\mathcal{H}}_p$.

As λ changes from 0 to 1, the family $\mathcal{HR}_p(\beta,\lambda,k,v)$ produces a passage from the class of harmonic functions $\mathcal{HA}_p(\beta,k,v) \equiv \mathcal{HR}_p(\beta,0,k,v)$ consisting of function f where

$$Re\left\{\frac{D^v f}{pz^p}\right\} \geq \beta \tag{1.8}$$

to the class of harmonic functions $\mathcal{HB}_p(\beta,k,v) \equiv \mathcal{HR}_p(\beta,1,k,v)$ consisting of functions f where

$$Re\left\{(1-k)\frac{(D^v f)'}{p(z^p)'} + k\frac{(D^v f)''}{p(z^p)''}\right\} \geq \beta. \tag{1.9}$$

Note that if $v=0$ and the co-analytic part $g \equiv 0$ and $k=p-1=0$ then the class $\mathcal{HR}_1(\beta,\lambda,0,0) = F_\lambda(\alpha)$ as studied by Booshnurmath and Swamay [3]. Further if $v=k=p-1=0, \mathcal{HB}_1(\beta,0,0) = N_H(\alpha)$ as studied by Ahuja and Jahangiri [2] and if $v=k=p-1=0, \overline{\mathcal{HB}_1}(\beta,0,0) = N_{\overline{H}}(\alpha)$ as studied by Ahuja and Jahangiri [2]. If $v=k=0$ the class $\mathcal{HR}_p(\beta,\lambda,0,0) = H_pk(n;\lambda,\alpha)$ studied by Ahuja and Jahangiri in [1], and also if $k=p-1=0$ the class $\mathcal{HR}_1(\beta,\lambda,0,v) = F_H(n,\lambda,\alpha)$ studied by Murugusundaramoorthy and Vijaya in [7].

In this paper we obtain the coefficient conditions for the classes $\mathcal{HR}_p(\beta,\lambda,k,v)$ and $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$. A representation theorem and inclusion properties for the class $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$ are established.

2 Coefficient Bounds

In the first theorem we give the sufficient condition for $f(z)$ to be in the class $\mathcal{HR}_p(\beta,\lambda,k,v)$.

Theorem 1 : Let $f = h + \overline{g}$ be such that h and g are given by the form (1.2), if

$$\begin{aligned}\sum_{n=m+p}^{\infty} & C^p(n,v)|p^2 - \lambda(n-p)(kn-p)||a_n| \\ & + \sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2 + \lambda(n+p)(kn-p)||b_n| < p(p-\beta)\end{aligned} \tag{2.1}$$

where $\lambda \geq 0, v > -1, 0 \leq \beta < 1, 0 \leq k \leq 1, p \in \mathcal{N}, |b_{m+p-1}| < 1$. Then $f \in \mathcal{HR}_p(\beta,\lambda,k,v)$.

Proof : Suppose

$$A(z) = (1-\lambda)\frac{D^v f}{z^p} + \lambda(1-k)\frac{(D^v f)'}{(z^p)'} + \lambda k\frac{(D^v f)''}{(z^p)''}.$$

It suffices to show that $|p - \beta + pA(z)| \geq |p + \beta - pA(z)|$. Substituting for h and g

in $A(z)$, we obtain

$$\begin{aligned}A(z) &= 1+\sum_{n=p+m}^{\infty} C^p(n,v)a_n(1-(\frac{1}{p}n-1)(\frac{1}{p}kn-1)\lambda)\frac{z^n}{z^p}\\ &\quad+\sum_{n=p+m-1}^{\infty} C^p(n,v)b_n(1+(\frac{1}{p}n+1)(\frac{1}{p}nk-1)\lambda)\frac{\overline{z^n}}{z^p}\end{aligned}$$

and then we have

$$\begin{aligned}&|p-\beta+pA(z)|-|p+\beta-pA(z)|\\ &=|2p-\beta+\sum_{n=p+m}^{\infty} C^p(n,v)a_n(p-(n-p)(\frac{1}{p}kn-1)\lambda)\frac{z^n}{z^p}\\ &+\sum_{n=p+m-1}^{\infty} C^p(n,v)b_n\frac{\overline{z^n}}{z^p}(p+(n+p)(\frac{1}{p}nk-1)\lambda)|\\ &-|\beta-\sum_{n=p+m}^{\infty} C^p(n,v)a_n(p-(n-p)(\frac{1}{p}kn-1)\lambda)\frac{z^n}{z^p}\\ &-\sum_{n=p+m-1}^{\infty} C^p(n,v)b_n\frac{\overline{z^n}}{z^p}(p+(n+p)(\frac{1}{p}nk-1)\lambda)|\\ &\geq 2p-\sum_{n=p+m}^{\infty} C^p(n,v)|a_n||p-(n-p)(\frac{1}{p}kn-1)\lambda|\frac{|z|^n}{|z|^p}\\ &-\sum_{n=p+m-1}^{\infty} C^p(n,v)|b_n||p+(n+p)(\frac{1}{p}nk-1)\lambda|\frac{|z|^n}{|z|^p}\\ &-\sum_{n=p+1}^{\infty} C^p(n,v)|a_n||p-(n-p)(\frac{1}{p}kn-1)\lambda|\frac{|z|^n}{|z|^p}\\ &-\sum_{n=p+m-1}^{\infty} C^p(n,v)|b_n||p+(n+p)(\frac{1}{p}nk-1)\lambda|\frac{|z|^n}{|z|^p}\\ &\geq 2(p-\beta)-\sum_{n=p+m}^{\infty} C^p(n,v)|a_n||p-(n-p)(\frac{1}{p}kn-1)\lambda|\\ &-\sum_{n=p+m-1}^{\infty} C^p(n,v)|b_n||p+(n+p)(\frac{1}{p}nk-1)\lambda|\geq 0\end{aligned}$$

when $z=r\to 1$ and by (2.1)

The coefficient bound (2.1) given in Theorem 1 is sharp for the function

$$\begin{aligned} f(z) &= z^p + \sum_{n=m+p}^{\infty} \frac{x_n \cdot z^n}{C^p(n,v)|p-\lambda(n-p)(\frac{1}{p}kn-1|)} \\ &\quad + \sum_{n=m+p-1}^{\infty} \frac{\overline{y}_n \cdot \overline{z^n}}{C^p(n,v)|p+\lambda(n+p)(\frac{1}{p}kn-p)|} \end{aligned}$$

where $\sum\limits_{n=m+p}^{\infty}|x_n| + \sum\limits_{n=m+p-1}^{\infty}|y_n| = p-\beta.$

Our next theorem establishes that such coefficient bounds cannot be improved.

Theorem 2 : Let $f = h + \overline{g}$ be so that h and g are given by (1.2). Then $f \in \overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$ if and only if

$$\begin{aligned} &\sum_{n=m+p}^{\infty} C^p(n,v)|p^2-\lambda(n-p)(kn-p)|\ |a_n| \\ &\sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+\lambda(n+p)(kn-p)|\ |b_n| \leq (p-\beta)p. \end{aligned} \tag{2.2}$$

Proof : The "if" part follows from Theorem 1, upon noting that $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v) \subset \mathcal{HR}_p(\beta,\lambda,k,v)$. For the "only if" part assume that $f \in \overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$. Then for $z = re^{i\theta}$ in Δ we obtain

$$\begin{aligned} &Re\left\{(1-\lambda)\frac{D^v f}{z^p} + \lambda(1-k)\frac{(D^v f)'}{(z^p)'} + \lambda k\frac{(D^v f)''}{(z^p)''}\right\} \\ &= Re\left\{(1-\lambda)\frac{D^v f + \overline{D^v g}}{z^p} + \lambda(1-k)\frac{z(D^v h)' - z\overline{(D^v g)}'}{pz^p}\right. \\ &\left. +\lambda k\frac{z^2(D^v h)'' + z(D^v h)' + \overline{z^2(D^v g)}'' + \overline{z(D^v g)'}}{p^2 z^p}\right\} \\ &\geq 1 - \sum_{n=m+p}^{\infty} C^p(n,v)|a_n|\ |1-\lambda(\frac{1}{p}n-1)(\frac{1}{p}kn-1)|\ |z|^{n-p} \\ &- \sum_{n=m+p-1}^{\infty} C^p(n,v)|1+\lambda(\frac{1}{p}n+1)(\frac{1}{p}kn-1)|\ |z|^{n-p} \geq \frac{\beta}{p}. \end{aligned}$$

The above inequality must hold for all $z \in \Delta$. In particular, letting $z = r \to 1$ yields the required condition (2.2). □

As a special case of Theorem 2, we obtain the following two corollaries.

Corollary 1 : For class (1.8) we can write $f = h+\overline{g} \in \overline{\mathcal{HA}_p}(\beta,k,v) = \mathcal{HA}_p(\beta,k,v) \cap \overline{\mathcal{H}_p}$ if and only if

$$\sum_{n=m+p}^{\infty} C^p(n,v)\frac{p}{p-\beta}|a_n| + \sum_{n=m+p-1}^{\infty} C^p(n,v)\frac{p}{p-\beta}|b_n| \leq 1$$

Corollary 2 : For class (1.9) we can write, $f = h + \overline{g} \in \overline{\mathcal{HB}_p}(\beta, k, v) \equiv \mathcal{HB}_p(\beta, k, v) \cap \overline{\mathcal{H}_p}$ if and only if

$$\begin{aligned}&\sum_{n=m+p}^{\infty} C^p(n,v)\left(\frac{|p^2-(n-p)(kn-p)|}{p(p-\beta)}\right)|a_n|\\&+\sum_{n=m+p-1}^{\infty} C^p(n,v)\left(\frac{|p^2+(n+p)(kn-p)|}{p(p-\beta)}\right)|b_n| \leq 1.\end{aligned}$$

3 Extreme Points

Theorem 3 : The function $f(z) = h(z) + \overline{g}(z) \in \overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$ if and only if

$$f(z) = X_p h_p(z) + \sum_{n=m+p}^{\infty} X_n h_n(z) + \sum_{n=m+p-1}^{\infty} Y_n g_n(z), \quad z \in \Delta \tag{3.1}$$

where

$$h_p(z) = z^p, h_n(z) = z^p - \frac{p(p-\beta)}{C^p(n,v)(p^2-\lambda(n-p)(nk-p))}z^n, \quad (n = m+p+1, \cdots)$$

$$g_n(z) = z^p + \frac{p(p-\beta)}{C^p(n,v)(p^2+\lambda(n+p)(nk-p))}\overline{z^n}, \quad (n = m+p-1, m+p, \cdots)$$

$X_p \geq 0$ and $Y_{m+p-1} \geq 0, X_p + \sum\limits_{n=m+p}^{\infty} X_n + \sum\limits_{n=m+p-1}^{\infty} Y_n = 1$ and $X_n \geq 0, Y_n \geq 0$, for $n = m+p, m+p+1, \cdots$.
Proof : For functions f of the form (3.1) we have

$$\begin{aligned}f(z) &= X_p h_p(z) + \sum_{n=m+p}^{\infty} X_n h_n(z) + \sum_{n=m+p-1}^{\infty} Y_n g_n(z), \quad z \in \Delta\\&= X_p z^p + \sum_{n=m+p}^{\infty}\left(z^p - \frac{p(p-\beta)}{C^p(n,v)|p-\lambda(n-p)(nk-p)|}\right)X_n z^n\\&\quad+\sum_{n=m+p-1}^{\infty}\left(z^p + \frac{p(p-\beta)}{C^p(n,v)|p^2+\lambda(n+p)(nk-p)|}\right)Y_n\overline{z^n}\\&= z^p + \sum_{n=m+p}^{\infty}\frac{p(p-\beta)}{C^p(n,v)|p^2-\lambda(n-p)(nk-p)|}X_n z^n\\&\quad+\sum_{n=m+p}^{\infty}\frac{p(p-\beta)}{C^p(n,v)|p^2+\lambda(n+p)(nk-p)|}Y_n\overline{z^n}\end{aligned}$$

So we have

$$\sum_{n=m+p}^{\infty} C^p(n,v)|p^2-\lambda(n-p)(kn-p)||a_n|$$
$$+\sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+\lambda(n+p)(kn-p)||b_n|$$
$$=\sum_{n=m+p}^{\infty} C^p(n,v)|p^2-\lambda(n-p)(kn-p)|\frac{p(p-\beta)|X_n|}{C^p(n,v)|p^2-\lambda(n-p)(nk-p)|}$$
$$+\sum_{n=m+p-1}^{\infty} C^p(n,v)(p^2+\lambda(n+p)(kn-p))\frac{p(p-\beta)|Y_n|}{C^p(n,v)|p^2+\lambda(n+p)(nk-p)|}$$
$$=p(p-\beta)\left(\sum_{n=m+p}^{\infty}|X_n|+\sum_{n=m+p-1}^{\infty}|Y_n|\right)=p(p-\beta)(1-X_p)\leq p(p-\beta).$$

Consequently, $f\in\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$ by (2.1).

Conversely, suppose $f\in\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$. Letting

$$X_p=1-\sum_{n=m+p}^{\infty}X_n-\sum_{n=m+p-1}^{\infty}Y_n$$

where

$$X_n=\frac{C^p(n,v)|p^2-\lambda(n-p)(kn-p)|}{p(p-\beta)}|a_n|,\ \ Y_n=\frac{C^p(n,v)|p^2+\lambda(n+p)(kn-p)|}{p(p-\beta)}|b_n|.$$

We obtain the required representation, since

$$\begin{aligned}
f(z) &= z^p-\sum_{n=m+p}^{\infty}|a_n|z^n+\sum_{n=m+p-1}^{\infty}|b_n|\overline{z^n}\\
&= z^p-\sum_{n=m+p}^{\infty}\frac{p(p-\beta)X_nz^n}{C^p(n,v)|p^2-\lambda(n-p)(kn-p)|}\\
&\quad+\sum_{n=m+p-1}^{\infty}\frac{p(p-\beta)Y_n\overline{z^n}}{C^p(n,v)|p^2+\lambda(n+p)(kn-p)|}\\
&= z^p-\sum_{n=m+p}^{\infty}(z^p-h_n(z))X_n-\sum_{n=m+p-1}^{\infty}(z^p-g_n(z))Y_n\\
&= \left(1-\sum_{n=m+p}^{\infty}X_n-\sum_{n=m+p-1}^{\infty}Y_n\right)z^p+\sum_{n=m+p}^{\infty}h_n(z)X_n+\sum_{n=m+p-1}^{\infty}g_n(z)Y_n\\
&= X_ph_p(z)+\sum_{n=m+p}^{\infty}X_nh_n(z)+\sum_{n=m+p-1}^{\infty}Y_ng_n(z).
\end{aligned}$$

$\square$

4 Inclusion Relations

The inclusion relation between the classes $\overline{\mathcal{HB}_p}(\beta,k,v)$ and $\overline{\mathcal{HA}_p}(\beta,k,v)$ for different values of λ are not so obvious. In this section we discuss the inclusion relations between the above mentioned classes.

Theorem 7 : For $n \in \{1,2,3,\cdots\}$ and $0 \le \beta < p$, we have

(i) $\overline{\mathcal{HB}_p}(\beta,k,v) \subset \overline{\mathcal{HA}_p}(\beta,k,v)$

(ii) $\overline{\mathcal{HB}_p}(\beta,k,v) \subset \overline{\mathcal{HR}_p}(\beta,\lambda,k,v), \quad 0 < \lambda \le 1.$

(iii) $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v) \subset \overline{\mathcal{HB}_p}(\beta,k,v), \quad \lambda \ge 1.$

Proof : (i) Let $f(z) \in \overline{\mathcal{HB}_p}(\beta,k,v)$. In view of corollaries 1 and 2, we have

$$\begin{aligned}
&\sum_{n=m+p}^{\infty} C^p(n,v)p^2|a_n| + \sum_{n=m+p-1}^{\infty} C^p(n,v)p^2|b_n| \\
&\le \sum_{n=m+p}^{\infty} C^p(n,v)|p^2-(n-p)(kn-p)||a_n| + \sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+(n+p)(kn-p)||b_n| \\
&\le p(p-\beta).
\end{aligned}$$

(ii) Let $f(z) \in \overline{\mathcal{HB}_p}(\beta,k,v)$. For $0 \le \lambda < 1$, we can write

$$\begin{aligned}
&\sum_{n=m+p}^{\infty} C^p(n,v)|\lambda(n-p)(kn-p)-p^2||a_n| \\
&+ \sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+\lambda(n+p)(kn-p)||b_n| \\
&\le \sum_{n=m+p}^{\infty} C^p(n,v)|(n-p)(kn-p)-p^2||a_n| \\
&+ \sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+(n+p)(kn-p)||b_n| \\
&\le p(p-\beta)
\end{aligned}$$

by Corollary 2 and so (ii) follows from Theorem 2.

(iii) By Theorem 2, if $\lambda \ge 1$, then we have

$$\begin{aligned}
&\sum_{n=m+p}^{\infty} C^p(n,v)|p^2-(n-p)(kn-p)||a_n| \\
&+ \sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+(n+p)(kn-p)||b_n| \\
&\le \sum_{n=m+p}^{\infty} C^p(n,v)|\lambda(n-p)(kn-p)-p^2|\ |a_n| \\
&+ \sum_{n=m+p-1}^{\infty} C^p(n,v)|p^2+\lambda(n-p)(kn-p)|\ |b_n| \le p(p-\beta).
\end{aligned}$$

Therefore, the result follows from Corollary 2.

5 Convolution and Convex Combinations

In the next theorem we examine the convolution properties of the class $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$.

Define the convolution of two harmonic functions

$$\begin{aligned} f(z) &= z^p - \sum_{n=m+p}^{\infty} |a_n| z^n + \sum_{n=m+p-1}^{\infty} |b_n| \overline{z^n} \\ F(z) &= z^p - \sum_{n=m+p}^{\infty} |c_n| z^n + \sum_{n=m+p-1}^{\infty} |d_n| \overline{z^n} \end{aligned}$$

by

$$(f * F)z = z^p - \sum_{m+p}^{\infty} |a_n c_n| z^n + \sum_{m+p-1}^{\infty} |b_n d_n| \overline{z^n}. \tag{5.1}$$

Theorem 8 : For $0 \le \beta < \alpha < 1$, let $f, F \in \overline{\mathcal{HR}_p}(\alpha,\lambda,k,v)$. Then

$$f * F \in \overline{\mathcal{HR}_p}(\alpha,\lambda,k,v) \subset \mathcal{HR}_p(\beta,\lambda,k,v).$$

Proof : For $f \in \overline{\mathcal{HR}_p}(\alpha,\lambda,k,v)$ and $F \in \overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$, let $f * F$ be given in (5.1). Since $|c_n| < 1$ and $|d_n| < 1$ we can write

$$\begin{aligned} &\sum_{n=m+p}^{\infty} C^p(n,v) |p^2 - \lambda(n-p)(kn-p)| \; |a_n c_n| \\ &\quad + \sum_{n=m+p-1}^{\infty} C^p(n,v) |p^2 + \lambda(n+p)(kn-p)| \; |b_n d_n| \\ &\le \sum_{n=m+p}^{\infty} C^p(n,v) |p^2 - \lambda(n-p)(kn-p)| \; |a_n| \\ &\quad + \sum_{n=m+p-1}^{\infty} C^p(n,v) |p^2 + \lambda(n+p)(kn-p)| \; |b_n| \end{aligned}$$

The right hand side of the above inequality is bounded by $p(p-\alpha)$ because $f \in \overline{\mathcal{HR}_p}(\alpha,\lambda,k,v)$. Therefore $f * F \in \overline{\mathcal{HR}_p}(\alpha,\lambda,k,v) \subset \mathcal{HR}_p(\beta,\lambda,k,v)$.

Finally, we determine the convex combination properties of the members of $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$.

Theorem 9 : The class $\overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$ is closed under convex combination.

Proof : For $i = 1, 2, 3, \cdots$ suppose that $f_i(z) \in \overline{\mathcal{HR}_p}(\beta,\lambda,k,v)$ where $f_i(z)$ is given by

$$f_i(z) = z^p - \sum_{n=m+p}^{\infty} |a_{n,i}| z^n + \sum_{n=m+p-1}^{\infty} |b_{n,i}| \overline{z^n}.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{n=m+p}^{\infty} (\sum_{i=1}^{\infty} t_i |a_{n,i}| z^n) + \sum_{n=m+p-1}^{\infty} (\sum_{i=1}^{\infty} t_i |b_{n,i}|) \overline{z^n}).$$

Since

$$\left\{ \sum_{n=p+m}^{\infty} \frac{|p^2 - (n-p)(kn-p)| \; |a_n|}{p(p-\beta)} + \sum_{m=n+p-1}^{\infty} \frac{|p^2 + (n+p)(kn-p)| \; |b_n|}{p(p-\beta)} \right\} \times$$
$$\times \, C^p(n,v) \leq 1.$$

So we obtain

$$\begin{aligned}
&\sum_{n=m+p}^{\infty} C^p(n,v) \frac{|1 - p^2 - (n-p)(kn-p)|}{p(p-\beta)} \left(\sum_{i=1}^{\infty} t_i |a_{n,i}| \right) \\
&+ \sum_{n=m+p-1}^{\infty} C^p(n,v) \frac{|p^2 + (n+p)(kn-p)|}{p(p-\beta)} \left(\sum_{i=1}^{\infty} t_i |b_{n,i}| \right) \\
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=m+p}^{\infty} C^p(n,v) \frac{|p^2 + (n+p)(kn-p)|}{p(p-\beta)} |a_{n,i}| \right. \\
&\left. + \sum_{n=m+p-1}^{\infty} C^p(n,v) \frac{|p^2 + (n+p)(kn-p)}{p(p-\beta)} |b_{n,i}| \right\} \\
&\leq \sum_{i=1}^{\infty} t_i = 1
\end{aligned}$$

and so $\sum\limits_{i=1}^{\infty} t_i f_i \in \overline{\mathcal{HR}_p}(\beta, \lambda, k, v)$. $\square$

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